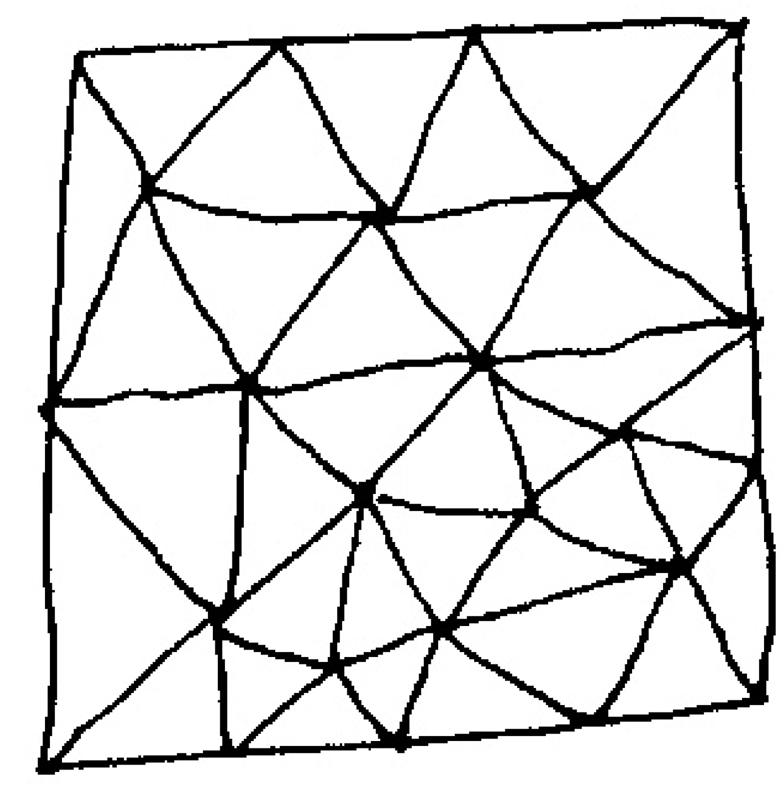
Mass-spring model of a thin sheet

Consider a triangular mesh of n discrete masses joined to their neighbors by springs. General equations of motion for node i of mass m are



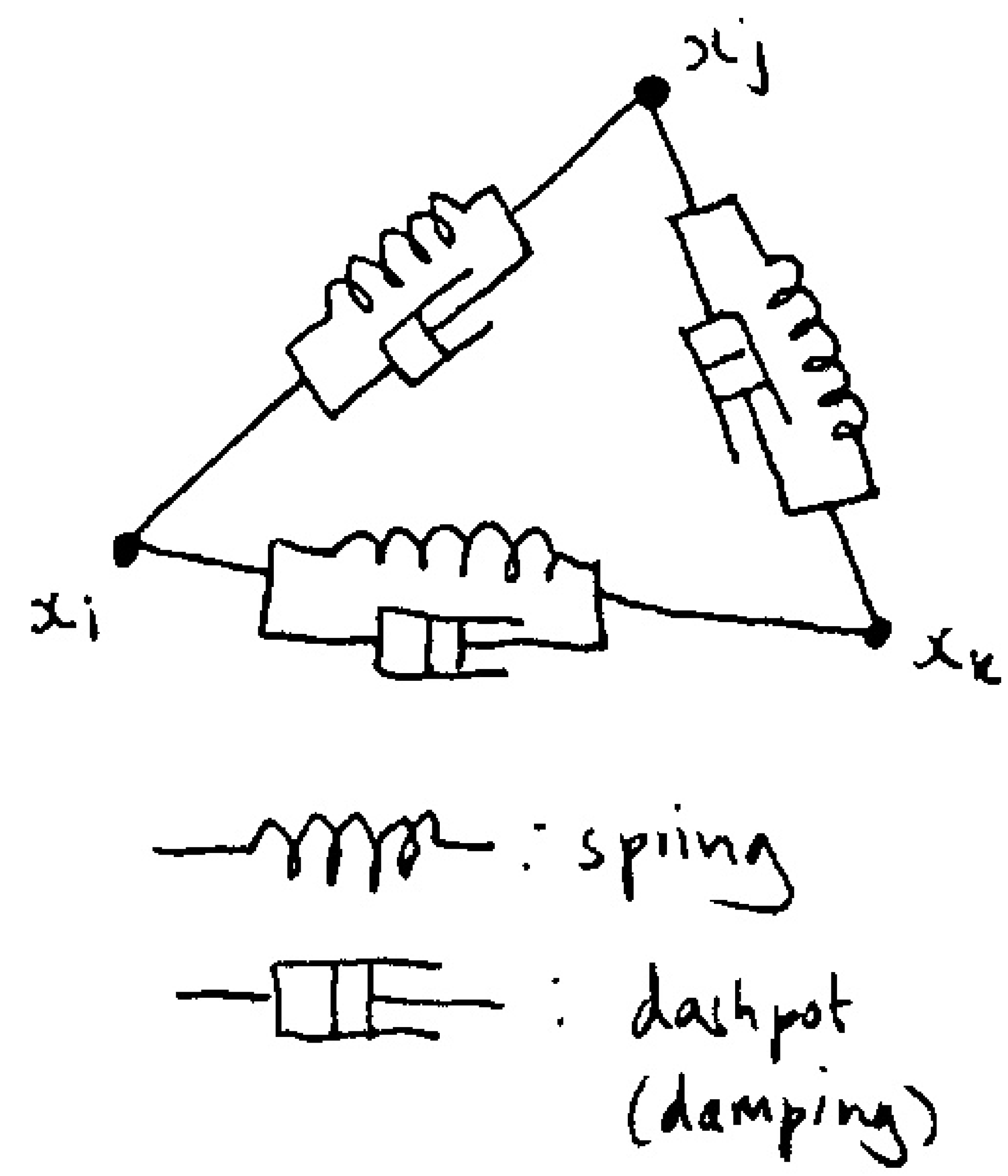
$$\dot{x}_i = v_i, \qquad m v_i = F_i,$$

where x_i is the three-dimensional position, v_i is the three-dimensional velocity, and F_i is the 3D force, which has multiple components:

- in-plane stretching
- out of plane bending
- drag/damping
- self-avoidance
- applied forces.

Elastic model for stretching and bending

Model is based on work by Seung and Nelson [1], with improvements by several authors [2-4].



Energy in spring between x_i and x_j is

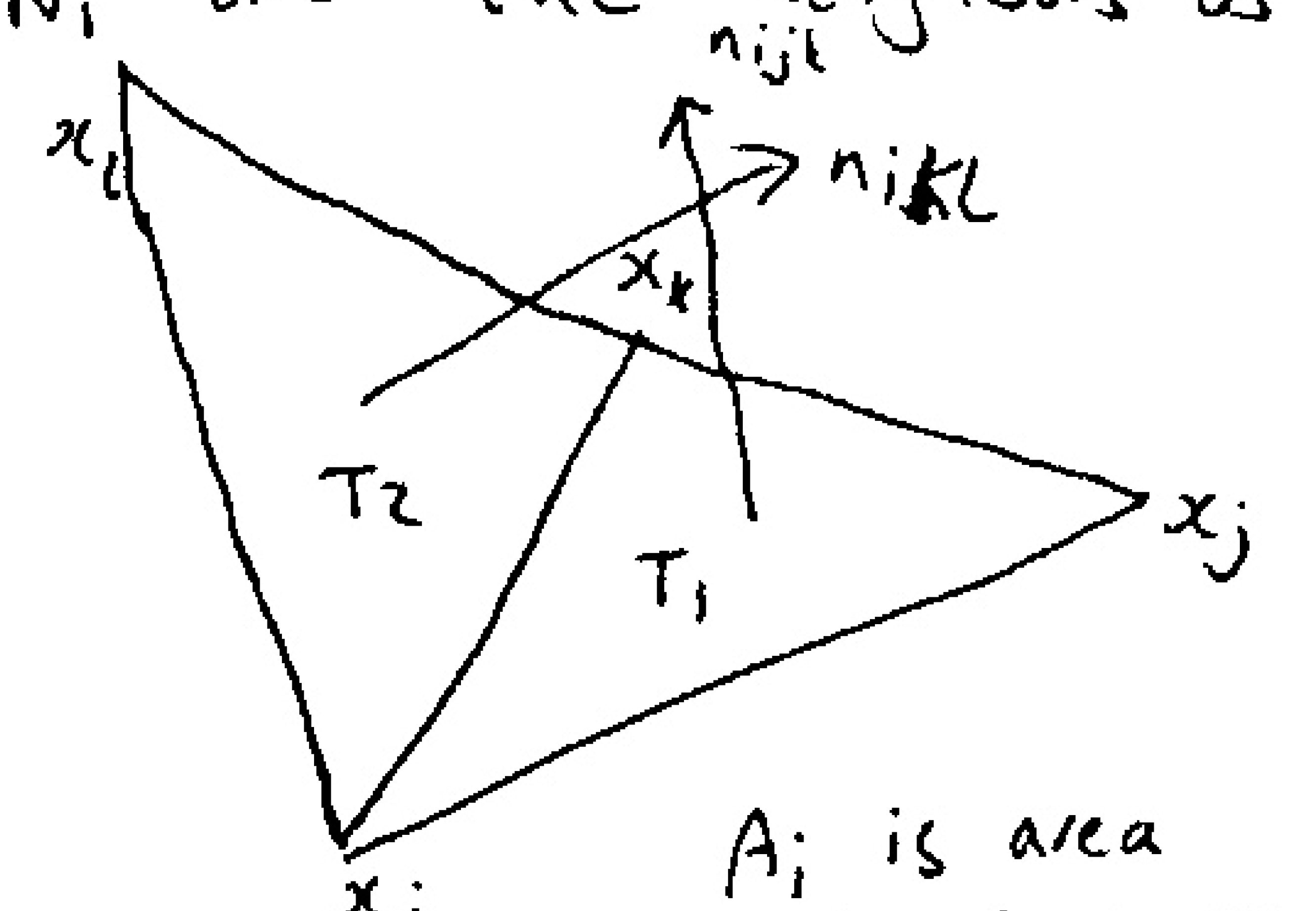
$$E_s(x_i, x_j) = \frac{k_s}{2}\left(\|r_{ij}\| - s_{ij}\right)^2$$

where $r_{ij} = x_i - x_j$ and s_{ij} is the rest length. k_s is the spring constant.

Resulting force contribution is

$$F_i^s = -\sum_{j \in N_i} \left(\nabla_{x_i} E_s(x_i, x_j)\right)^T$$

where the gradient creates a row-vector. N_i are the neighbors of x_i.



A_i is area of tringle T_i

Discrete bending energy at an edge shared by two triangles T_1 & T_2 is

$$E_b(\hat{n}_1, \hat{n}_2) = \frac{k_b}{2} \frac{\sqrt{3}}{2\bar{A}} \|e_0\|^2 \|\hat{n}_1 - \hat{n}_2\|^2,$$

where $\bar{A} = A_1 + A_2$ is the area of the two triangles, k_b is a bending constant, $\hat{n}_1$ & $\hat{n}_2$ are normal vectors on the triangles, and $\|e_0\|$ is the shared edge length. The corresponding force is

$$F_i^b = - \sum_{\substack{(i,j,k) \in T \\ (i,k,l) \in T}} \left(\nabla_{x_i} E_b(\hat{n}_{ijk}, \hat{n}_{ikl}) \right)$$

where the set of all triangles in the mesh is given by T.

Effective thickness

Consider a thin elastic sheet with thickness h and Young's modulus* Y. Let the Poisson ratio be ν. Then the bending rigidity is

$$K = \frac{Y h^3}{12(1-\nu^2)}$$

For a 2D model of a thin sheet, the in-plane and out-of-plane elasticity can be thought of as imparting an effective thickness on the sheet

* This is the three-dimensional modulus. Usually E is used, but we avoid that due to the energy terms.

$$h_{eff} = \sqrt{\frac{12(1-\nu^2)\kappa}{Y_{2D}}}$$

where $\frac{Y_{2D}}{h_{eff}} = Y$. For a periodic hexagonal lattice with

$Y_{2D} = \frac{2k_s}{\sqrt{3}}$, $\nu = 1/3$, and $\kappa = \sqrt{3}\frac{k_b}{2}$, we obtain

$$h_{eff} = \sqrt{\frac{8k_b}{k_s}}$$

Drag / damping

We consider two types of energy loss. The first is an isotropic drag force on each node

$$F_i^{d,iso} = -b_{iso} v_i .$$

This is simple to implement and can help with numerical stability. We also consider damping via dashpot elements in parallel with the in-plane springs. This gives

$$F_{ij}^{d,int} = -b_{int} \frac{d}{dt}(r_{ij} - s_{ij}\hat{r}_{ij})$$

$$= -b_{int} \left\{ \left[1 - \frac{s_{ij}}{\|r_{ij}\|}\right](v_i - v_j) + \frac{s_{ij}}{\|r_{ij}\|^3}\left[r_{ij} \cdot (v_i - v_j)\right] r_{ij} \right\},$$

where $r_{ij} = x_i - x_j$, $\hat{r}_{ij} = \frac{r_{ij}}{\|r_{ij}\|}$, and s_{ij} is the rest length. The total damping on node i is

$$F_i^{d,int} = \sum_{j \in N_i} F_{ij}^{d,int}(x_i, x_j)$$

Self-contact

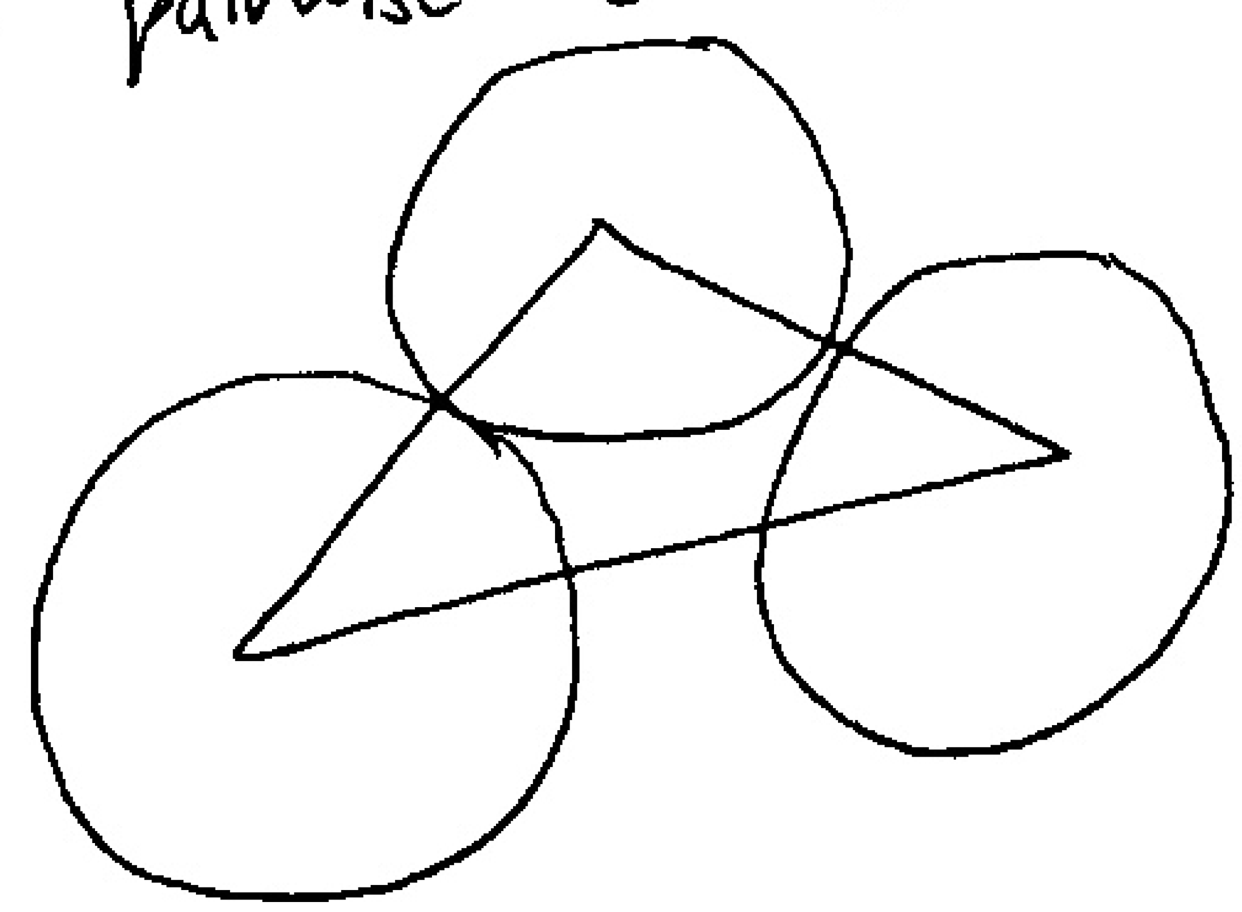
Two general approaches to contact:

(1) The penalty method: introduce stiff contact forces over a short range

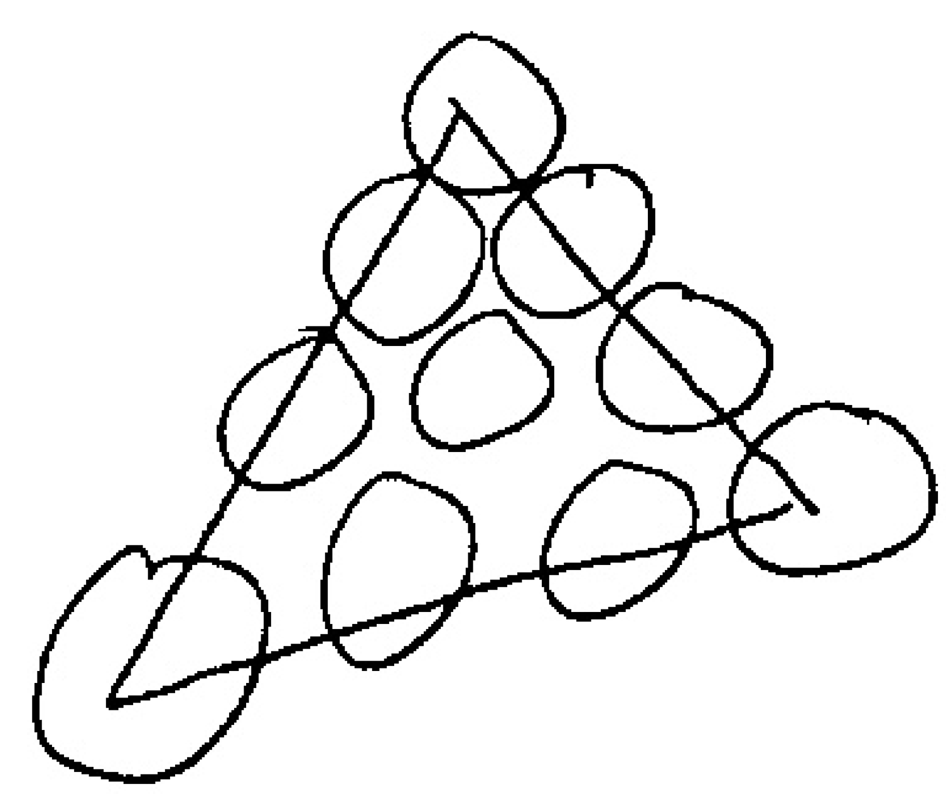
(2) The impulse method [5]: determine the precise time of contact and correct node positions and velocities consistent with conservation of momentum.

Method (2) can be complicated (e.g. resolving one contact creates another, multiple contacts can occur in a single timestep). We use method (1).

Can look at all edge-edge and point-triangle intersections. Possible, but can lead to discontinuities in applied forces that lower integration accuracy. Instead, use pairwise contacts between nodes. Visualize as spheres around each node.



Can also refine to smaller spheres in a "pool table starting setup"



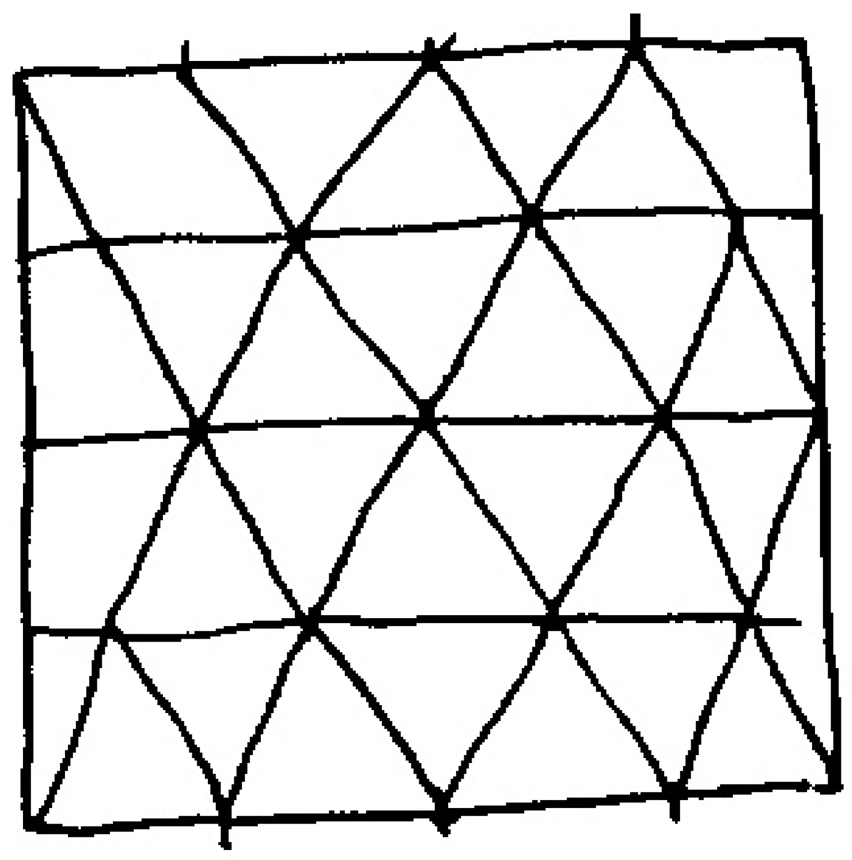
For interacting sites 1 and 2, contact force is

$$F_1 = \begin{cases} k_s \dfrac{\sigma}{\|r_{12}\|}\left(\dfrac{\sigma}{\|r_{12}\|-\sigma}\right)^2 \exp\left(\dfrac{\sigma}{\|r_{12}\|-\sigma}\right) r_{12} & \text{if } \|r_{12}\|<\sigma \\ 0 & \text{if } \|r_{12}\|\geq\sigma. \end{cases}$$

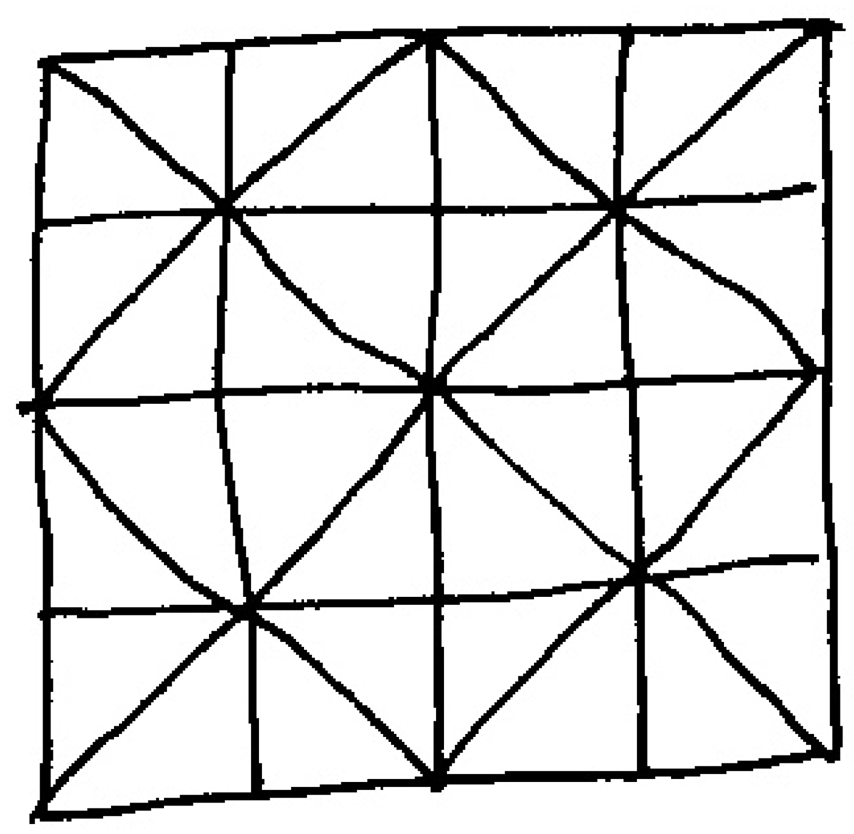
on particle 1.

Meshes

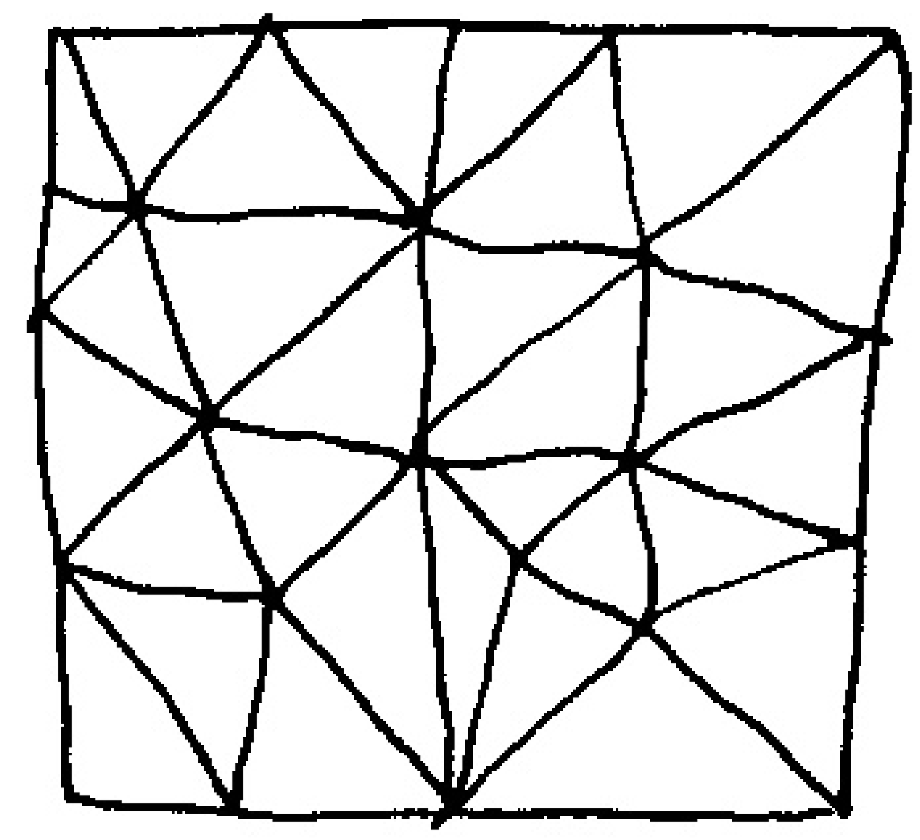
Can run using several different meshes:



Regular hexagonal mesh



Warren girder mesh



Random mesh

Each has pros and cons. We will look in detail at how to create random meshes.

References

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3. M. Kot, H. Nagahashi, & P. Szymczak, Vis. Comput. 31, 1339-1350 (2015)
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5. R. Bridson, R. Fedkiw, & J. Anderson, in Proceedings of the 29th Annual Conference on Computer Graphics and Interactive Techniques, pp. 594-603 (2002).