Boulder Summer School 2026

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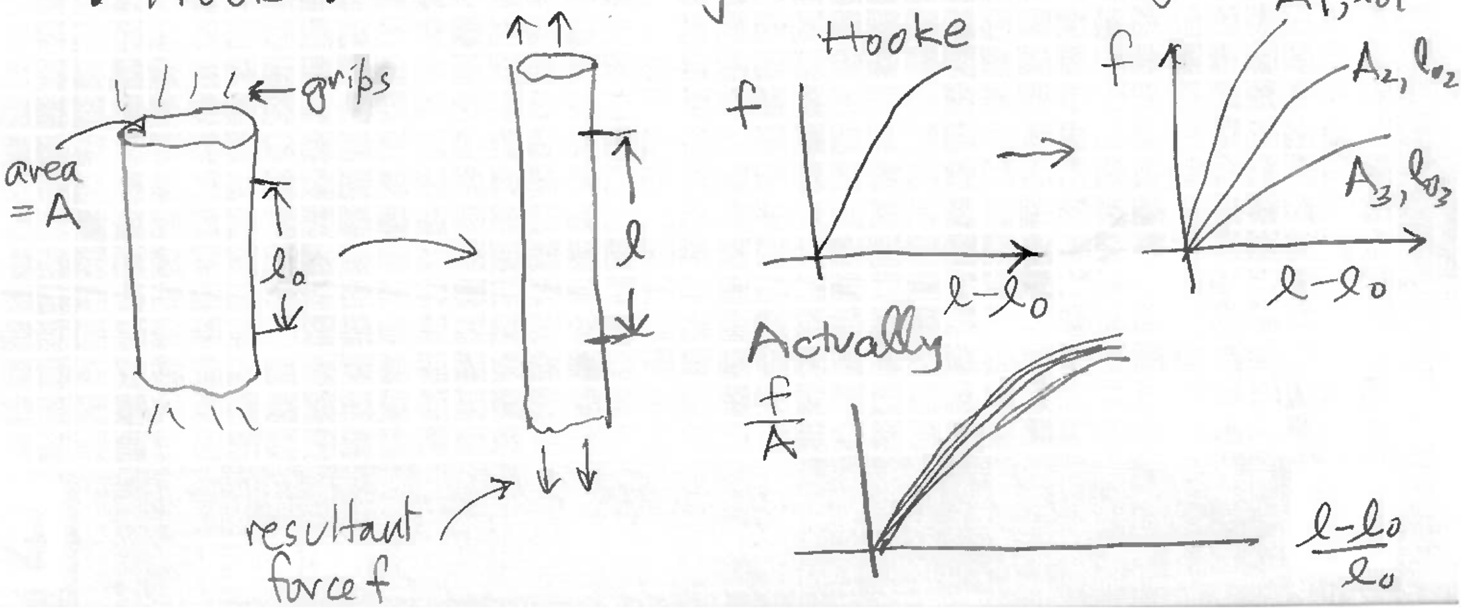
Handout (Available from the staff): Notes and Examples from Tensor Analysis for Continuum Mechanics. These notes set the notation for most papers that use continuum theories.

LECTURE I

What is elasticity?

- An elastic material returns to its original shape after having been deformed? That was an original idea behind elasticity (Hooke) but is no longer accepted: there are materials governed by elasticity theory that do not have that property. Rather, a key unifying idea behind elasticity is the assumption of a free energy having a certain form.
- Hooke: an elementary idea of elasticity



- Modern view of elasticity. Elasticity shares much in common with materials having free energy densities, $\varphi(\underset{\sim}{d}, \nabla \underset{\sim}{d}, \nabla^2 \underset{\sim}{d}, \ldots, \theta)$ θ = temperature.

- Elasticity theory includes a diversity of theories: strings, rods, plates, shells, 3D elasticity. These theories even have different kinematics. In undergraduate courses they are treated as completely different subjects. In fact, they are believed to be closely related. Although (in most cases) it only concerns absolute minimizers of energy, a mathematical method called Γ-convergence has revealed some explicit relations between these theories.

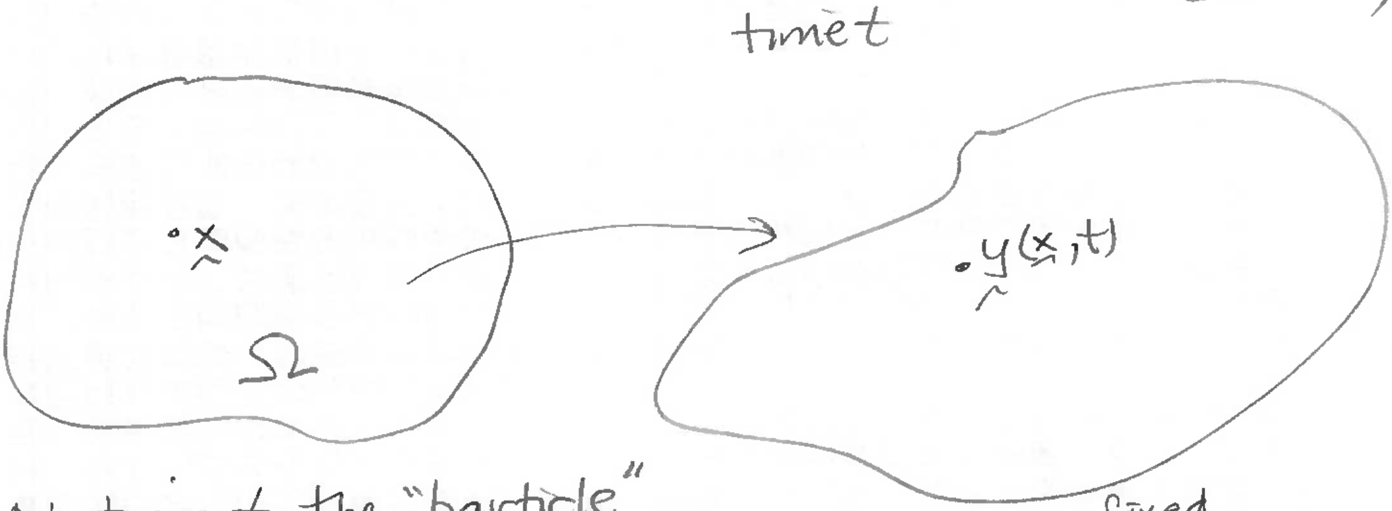
- Elasticity and Materials Science. More broadly, elasticity theory has much in common with theories of other materials, e.g., theories for ferroelectric, ferromagnetic and superconducting materials. Among these materials, with few exceptions (e.g., for ferroelectric materials, diamond), elastic materials are able to store the greatest amount of energy.

Below, we use the notation of the handout

Kinematics of elasticity

Elasticity is conventionally presented in the Lagrangian description of motion.

Reference configuration Ω (connected, open set in $\mathbb{R}^3$)



At time t the "particle" $\underset{\sim}{x}$ occupies the position $\underset{\sim}{y}(\underset{\sim}{x},t)$. $\underset{\sim}{x}\to\underset{\sim}{y}(\underset{\sim}{x},\overset{\text{fixed}}{t})$ is assumed invertible

$\underset{\sim}{F}=\nabla\underset{\sim}{y}$ is the deformation gradient. We always assume that any set of positive volume in Ω cannot be shrunk to zero volume, i.e.,

$$\det\underset{\sim}{F}>0. \qquad \text{RCC } F_{ij}=\frac{\partial y_i}{\partial x_j}$$

i.e.,

$$\text{vol. } \underset{\sim}{y}(P,t)=\int_{y(P,t)} d\underset{\sim}{y}=\int_{P}|\det\nabla\underset{\sim}{y}|\,d\underset{\sim}{x}$$

Some basic definitions ($\underset{\sim}{U}$ is a tensor from $\mathbb{R}^n$ to $\mathbb{R}^n$)

1) $SO(n)=\{\underset{\sim}{R}: \mathbb{R}^n\to\mathbb{R}^n : \underset{\sim}{R}^T\underset{\sim}{R}=\underset{\sim}{I},\ \det\underset{\sim}{R}=+1\}$
2) $\underset{\sim}{y}(\underset{\sim}{x},t)=\underset{\sim}{R}(t)\underset{\sim}{x}+\underset{\sim}{c}(t)$ is a <u>rigid motion</u>
3) $\underset{\sim}{U}=\underset{\sim}{U}^T$ means $\underset{\sim}{U}$ is symmetric. $\underset{\sim}{U}>0$ means $\underset{\sim}{U}$ is positive-definite

A basic kinematic result for elasticity is:

Polar Decomposition: Let $\underset{\sim}{F}$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ (i.e., a tensor) with $\det \underset{\sim}{F} > 0$. There exists $\underset{\sim}{U} = \underset{\sim}{U}^T > 0$ and $\underset{\sim}{R} \in SO(n)$ such that

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$$

The decomposition is unique (if $\underset{\sim}{F} = \underset{\sim}{R}'\underset{\sim}{U}'$, $\underset{\sim}{R}' \in SO(n)$, $\underset{\sim}{U}' = \underset{\sim}{U}'^T > 0$, then $\underset{\sim}{R}' = \underset{\sim}{R}$ and $\underset{\sim}{U}' = \underset{\sim}{U}$)

Proof. Let $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F} = \underset{\sim}{C}^T > 0$. There is a unique positive-definite $\underset{\sim}{U} = \underset{\sim}{U}^T$ such that

$$\underset{\sim}{C} = \underset{\sim}{U}^2.$$

That is, write $\underset{\sim}{C} = \lambda_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_i$ ($\lambda_i > 0$, $\underset{\sim}{e}_1, .., e_n$ orthonormal) and $\underset{\sim}{U} = \sqrt{\lambda_i}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_i$ is its unique positive-definite, symmetric square root. Let

$$\underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}$$

Then $\underset{\sim}{R}^T\underset{\sim}{R} = \underset{\sim}{U}^{-1}\underset{\sim}{F}^T\underset{\sim}{F}\,\underset{\sim}{U}^{-1} = \underset{\sim}{U}^{-1}\underset{\sim}{C}\,\underset{\sim}{U}^{-1} = \underset{\sim}{U}^{-1}\underset{\sim}{U}^{2}\underset{\sim}{U}^{-1} = \underset{\sim}{I}$

and $\det \underset{\sim}{R} = (\det \underset{\sim}{F})(\det \underset{\sim}{U}^{-1}) > 0$. Uniqueness of $\underset{\sim}{U}$ $\Rightarrow$ uniqueness of $\underset{\sim}{R}$.

$\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$ is the right Cauchy-Green tensor

$\underset{\sim}{C}$ is positive definite, $\underset{\sim}{x} \cdot \underset{\sim}{F}^T\underset{\sim}{F}\underset{\sim}{x} = |\underset{\sim}{F}\underset{\sim}{x}|^2 > 0$ ($\det \underset{\sim}{F} > 0$)

$\underset{\sim}{B} = \underset{\sim}{F}\,\underset{\sim}{F}^T$ is the left Cauchy-Green tensor, $\underset{\sim}{B} > 0$

The above is the right polar decomposition; there is also a left polar decomposition $\underset{\sim}{F} = \underset{\sim}{V}\,\underset{\sim}{R}$ (same $\underset{\sim}{R}$ as above)

Geometric interpretation of $\underset{\sim}{C}$ and $\underset{\sim}{B}$. Take the example of $\underset{\sim}{B}$. Consider a small ball (radius ε) centered at $\underset{\wedge}{x}_0 \in \Omega$. By the definition of the gradient ((14) in the notes on tensor analysis)

t fixed

$$\nabla \underset{\sim}{y}(\underset{\wedge}{x}_0)\underset{\sim}{v} = \underset{\sim}{y}(\underset{\wedge}{x}+\underset{\sim}{v}) - \underset{\sim}{y}(\underset{\wedge}{x}) + o(\varepsilon)$$

$$|\underset{\sim}{v}| = \varepsilon \qquad \frac{o(\varepsilon)}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0.$$

We are interested in the geometric figure $\mathcal{E}$:

$$\mathcal{E} = \{\nabla \underset{\sim}{y}(\underset{\wedge}{x}_0)\underset{\sim}{v} : |\underset{\sim}{v}| = \varepsilon\}$$

$$= \{\underset{\sim}{z} : |\underset{\sim}{F}^{-1}\underset{\sim}{z}| = \varepsilon\} \qquad \underset{\sim}{F} = \nabla \underset{\sim}{y}(\underset{\wedge}{x}_0)$$

$$\rightarrow \underset{\sim}{F}^{-1}\underset{\sim}{z} \cdot \underset{\sim}{F}^{-1}\underset{\sim}{z} = \varepsilon^2$$

$$\rightarrow \underset{\sim}{z} \cdot \underset{\sim}{F}^{-T}\underset{\sim}{F}^{-1}\underset{\sim}{z} = \varepsilon^2$$

$$\rightarrow \underset{\sim}{z} \cdot (\underset{\sim}{F}\underset{\sim}{F}^T)^{-1}\underset{\sim}{z} = \varepsilon^2 \qquad \underset{\sim}{x} \cdot \underset{\sim}{B}\underset{\sim}{x} = \underset{\wedge}{x} \cdot \underset{\sim}{F}\underset{\sim}{F}^T\underset{\wedge}{x} = |\underset{\sim}{F}^T\underset{\sim}{x}|^2 \geq$$

$$\rightarrow \underset{\sim}{z} \cdot \underset{\sim}{B}^{-1}\underset{\sim}{z} = \varepsilon^2 \quad * \qquad \underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T = (\underset{\sim}{F}\underset{\sim}{F}^T)^T = \underset{\sim}{B}^T$$

$\underset{\sim}{B} = \underset{\sim}{B}^T$ so we can write $\underset{\sim}{B}$ in spectral form

$\underset{\wedge}{e}_1, \underset{\wedge}{e}_2, \underset{\sim}{e}_3$ orthonorm

$$\underset{\sim}{B} = \lambda_1^2 \underset{\wedge}{e}_1 \otimes \underset{\wedge}{e}_1 + \lambda_2^2 \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + \lambda_3^2 \underset{\sim}{e}_3 \otimes \underset{\wedge}{e}_3$$

$\lambda_1^{>0}, \lambda_2^{>0}, \lambda_3^{>0}$ are called the principal stretches. Also,

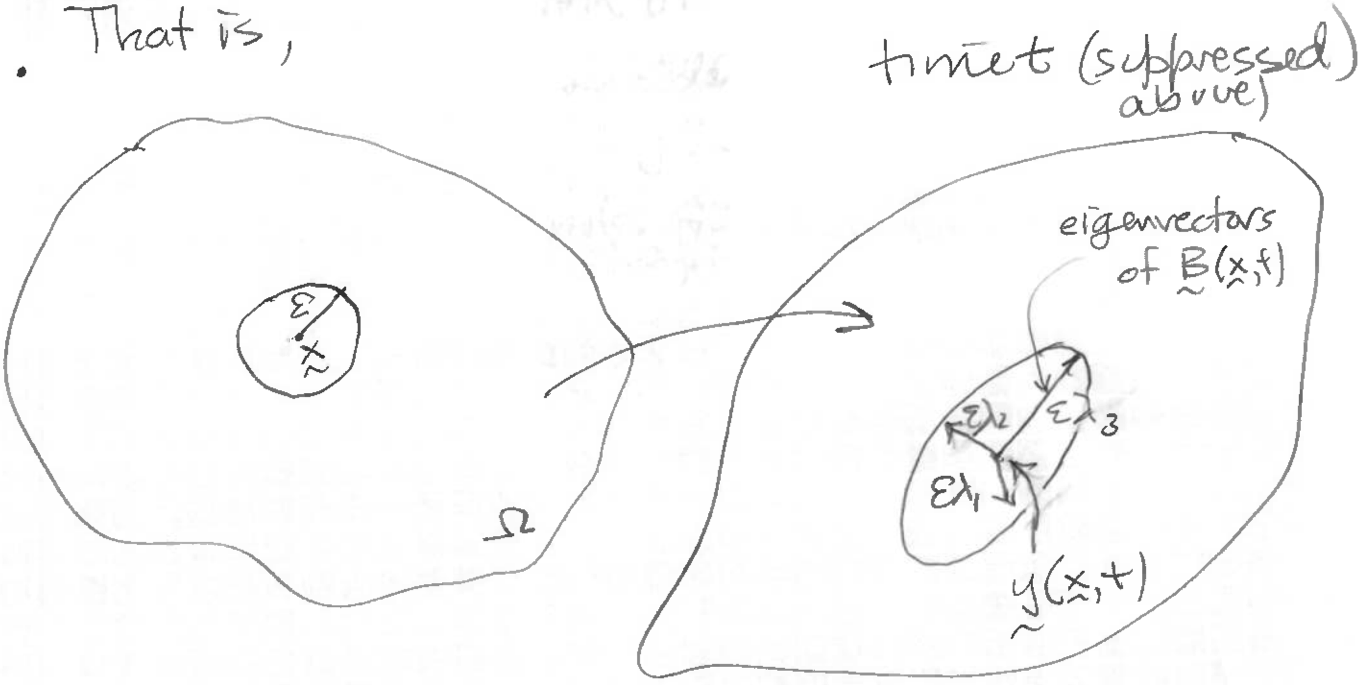
$$\underset{\sim}{B}^{-1} = \frac{1}{\lambda_1^2}\underset{\wedge}{e}_1 \otimes \underset{\wedge}{e}_1 + \frac{1}{\lambda_2^2}\underset{\wedge}{e}_2 \otimes \underset{\wedge}{e}_2 + \frac{1}{\lambda_3^2}\underset{\sim}{e}_3 \otimes \underset{\wedge}{e}_3$$

Thus, we can write $*$ in this basis ($\underset{\sim}{z} = z_1\underset{\sim}{e}_1 + z_2\underset{\sim}{e}_2 + z_3\underset{\sim}{e}_3$)

as

$$\frac{z_1^2}{(\varepsilon\lambda_1)^2} + \frac{z_2^2}{(\varepsilon\lambda_2)^2} + \frac{z_3^2}{(\varepsilon\lambda_3)^2} = 1$$

• That is,



Similarly, $\underset{\sim}{C}=\underset{\sim}{F}^T\underset{\sim}{F}$ defines an ellipsoid at $\underset{\sim}{x}$ that is mapped to a ball at $\underset{\sim}{y}(\underset{\sim}{x},t)$, by a similar argument.

* * *

Elasticity theory is based on a free energy function of the form

$$\varphi(\underset{\sim}{F},\underset{\uparrow\text{ absolute temperature}}{\theta}) \qquad \det \underset{\sim}{F}>0.$$

By Galilean invariance (rotate rigidly the deformed configuration... or the observer)

$$\varphi(\underset{\sim}{Q}\underset{\sim}{F},\theta)=\varphi(\underset{\sim}{F},\theta) \qquad \text{for all } \underset{\sim}{F}, \det\underset{\sim}{F}> \quad \text{all } \underset{\sim}{Q}\in SO(3$$

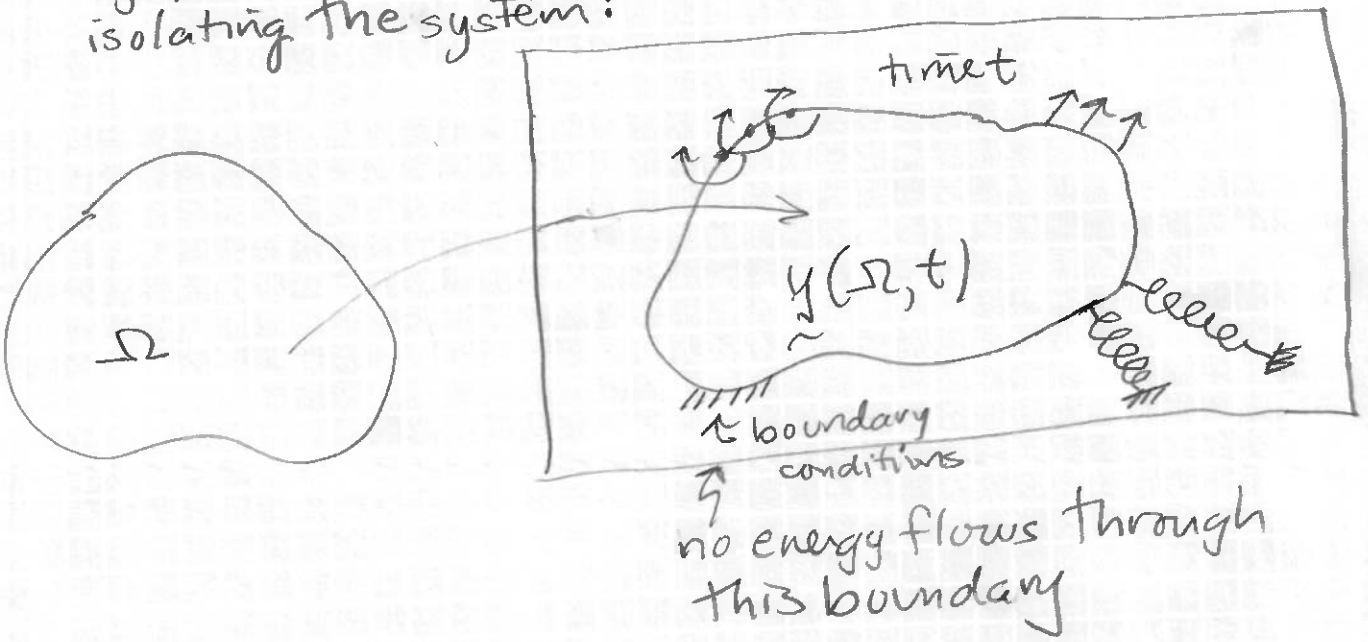
When thermal effects are not being considered,

$$\varphi(\underset{\sim}{F})$$

When the material is inhomogeneous,

$$\varphi(\underset{\sim}{F},\underset{\sim}{x}).$$

An argument based on the second law of thermodynamics (the Clausius-Duhem inequality) gives a formal justification of a minimization principle for elasticity (see J.L. Ericksen, IJSS (1966), p. 573), consistent also with the approach of Gibbs (Equilibrium of Heterogeneous Substances). This formal justification essentially identifies a Lyapunov functional. Gibbs idea is based on isolating the system:



The minimization principle is, formally,

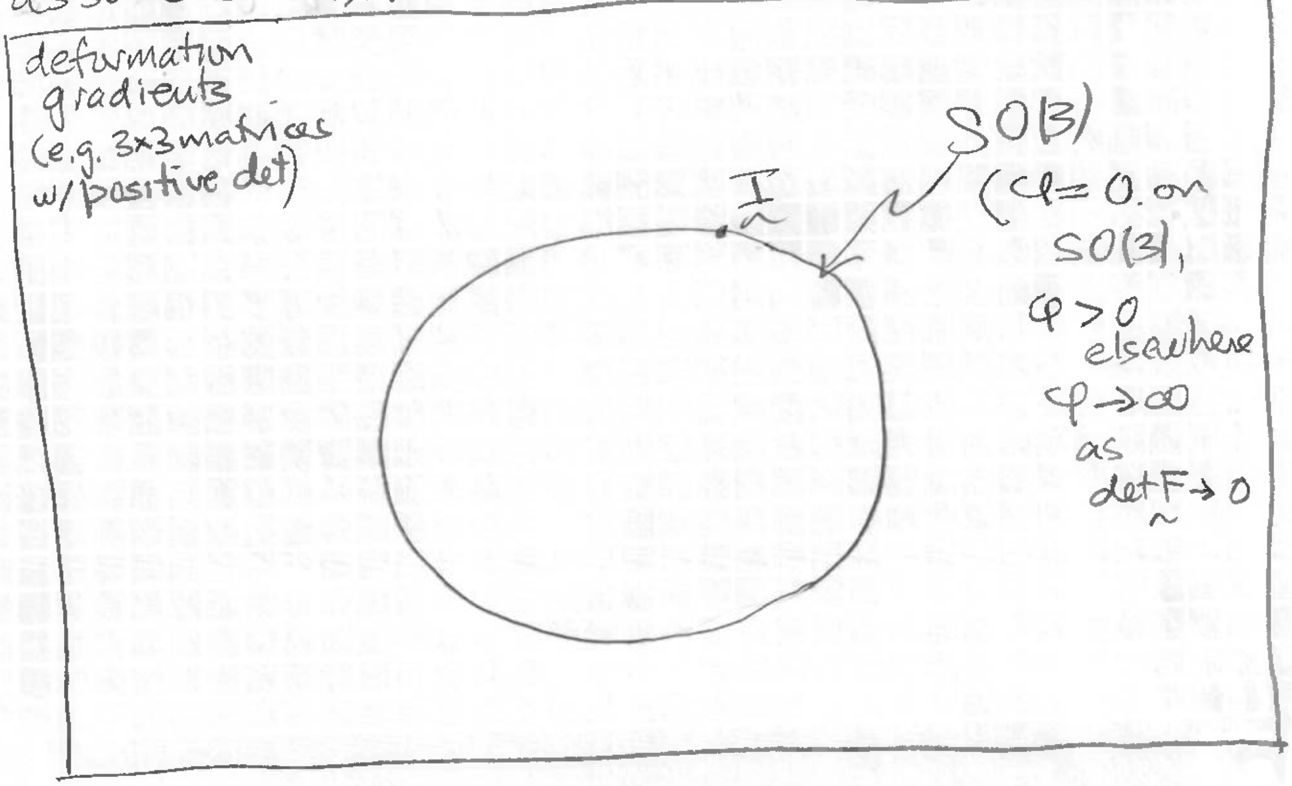
$$\min_{\substack{\underset{\sim}{y}(\underset{\sim}{x}) \,\in\, \text{boundary conditions} \\ \text{e.g. } \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{y}_0(\underset{\sim}{x}) \\ \underset{\sim}{x} \in \partial\Omega}} \left(\int_\Omega \varphi(\nabla \underset{\sim}{y}(\underset{\sim}{x}), \theta)\, d\underset{\sim}{x} + \underset{\substack{\uparrow \\ \text{energy of the loading device} \\ \text{(everything else in the box above)}}}{\mathcal{L}} \right)$$

(Note: this is very different from the formulation of linearized elasticity, where only displacement and traction at $\partial\Omega$ are needed.)

LECTURE 2

Topological elasticity (Liouville's theorem), compatibility, folding paper flat, balances of mass and momentum

At the end of Lecture 1 we introduced the free energy function. A picture I find very useful (though a bit misleading), is a picture of the domain of the free energy density $\varphi(\underset{\sim}{F})$, $\det \underset{\sim}{F} > 0$, introduced at the end of Lecture 1. The square below represents all tensors (from $\mathbb{R}^3$ to $\mathbb{R}^3$) w/ positive det i.e., deformation gradients. The circle represents $SO(3)$; $\underset{\sim}{I}$ is the identity. Typical free energy densities are, say, 0 on $SO(3)$ and grow rapidly as one moves away from $SO(3)$. Generally, we assumed $\varphi(\underset{\sim}{F}) \to \infty$ as $\det \underset{\sim}{F} \to 0$



One feature this picture does get right is that φ can never be a convex function. That is, consider an orthonormal basis $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ and two 180° rotations

$$-\underset{\sim}{I}+2\underset{\sim}{e}_1\otimes\underset{\sim}{e}_1 \,, \quad -\underset{\sim}{I}+2\underset{\sim}{e}_2\otimes\underset{\sim}{e}_2 \qquad \underset{\sim}{e}_1 \perp \underset{\sim}{e}_2$$

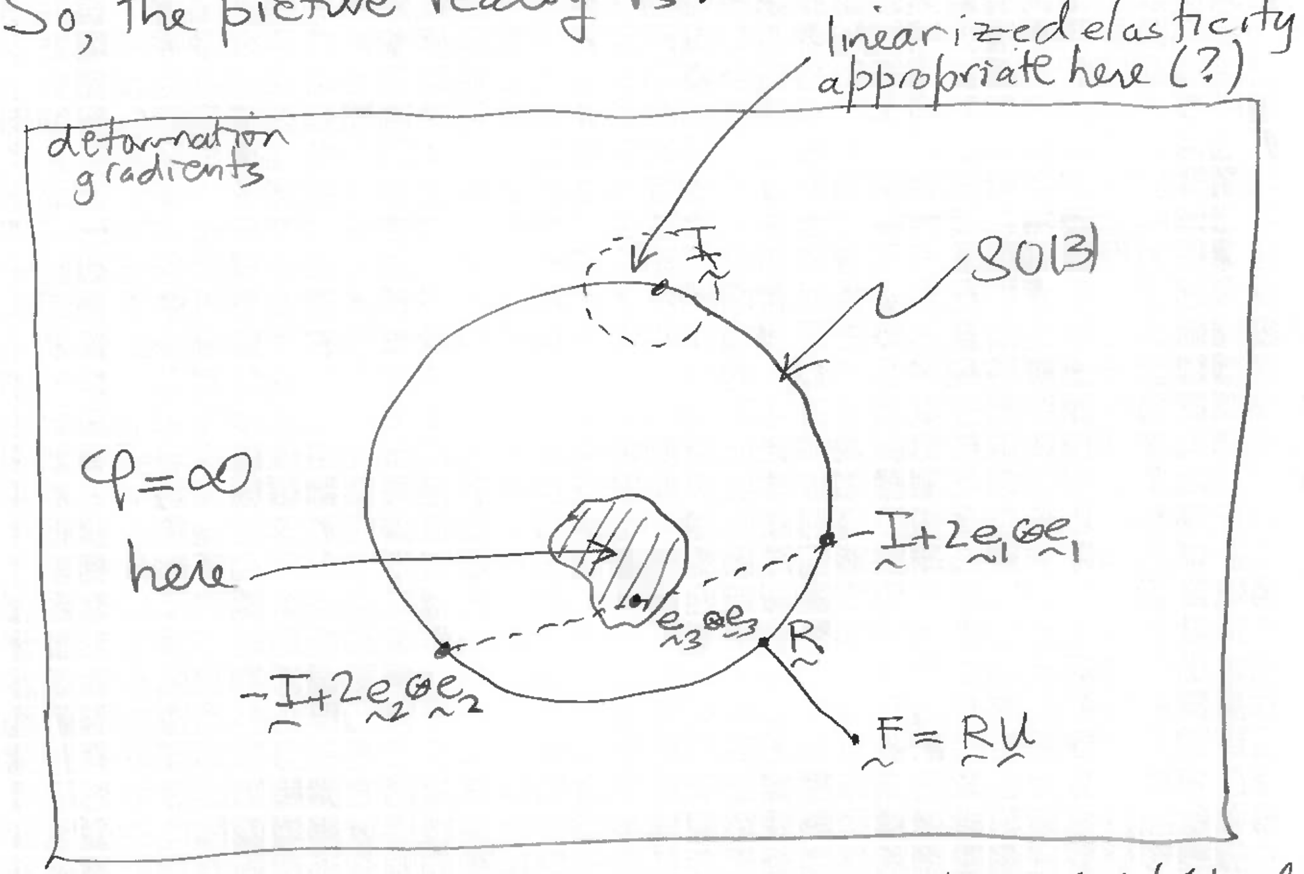
$$|\underset{\sim}{e}_1|=|\underset{\sim}{e}_2|=1$$

$\varphi(-\underset{\sim}{I}+2\underset{\sim}{e}_1\otimes\underset{\sim}{e}_1)=\varphi(-\underset{\sim}{I}+2\underset{\sim}{e}_2\otimes\underset{\sim}{e}_2)=0$, but

$$\varphi\left(\tfrac{1}{2}(-\underset{\sim}{I}+2\underset{\sim}{e}_1\otimes\underset{\sim}{e}_1)+\tfrac{1}{2}(-\underset{\sim}{I}+2\underset{\sim}{e}_2\otimes\underset{\sim}{e}_2)\right)=\varphi(-\underset{\sim}{I}+\underset{\sim}{e}_1\otimes\underset{\sim}{e}_1+\underset{\sim}{e}_2\otimes\underset{\sim}{e}_2)$$

$$=\varphi(\underset{\sim}{e}_3\otimes\underset{\sim}{e}_3)=\infty \; !$$

So, the picture really is:



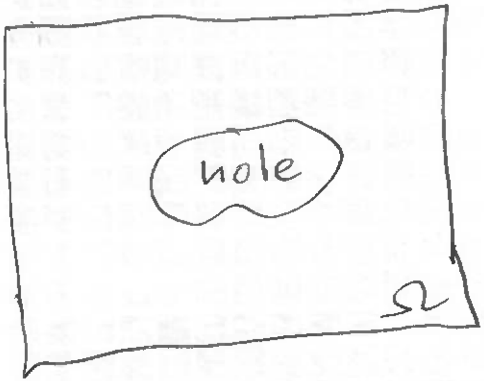
More details about this picture can be established. Linear elasticity is based on Taylor expansion about $\underset{\sim}{I}$. The closest rotation tensor to $\underset{\sim}{F}=\underset{\sim}{R}\underset{\sim}{U}$ is $\underset{\sim}{R}$ (prove this): $\min_{\underset{\sim}{Q}\in SO(3)} |\underset{\sim}{F}-\underset{\sim}{Q}| = |\underset{\sim}{R}\underset{\sim}{U}-\underset{\sim}{R}| = |\underset{\sim}{U}-\underset{\sim}{I}|$

Compatibility and Topological Elasticity

One of the most important topological concepts, with important implications for elasticity, is simple connectedness

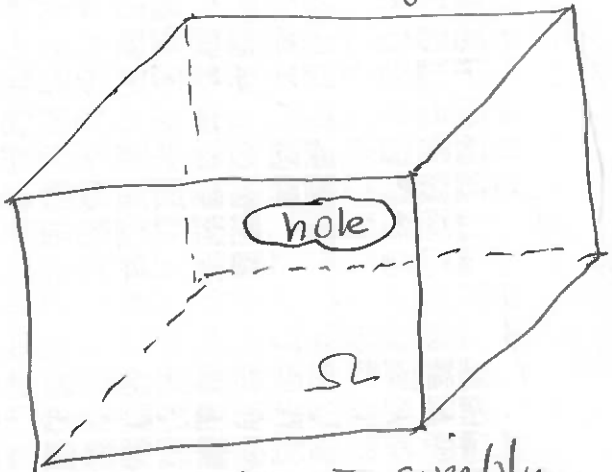
A simply connected domain is domain $\Omega \subset \mathbb{R}^n$ such that loops in Ω can be continuously contracted to a point.

2-D region



Not simply connected

3-D region



This region is simply connected. A donut is not simply connected.

- This is a topological concept because it is preserved by homeomorphisms (continuous invertible mappings whose inverse is continuous)
- In high dimensions (>3) there are many different kinds of holes, quantified by Betti numbers.

We will do the case of vector-valued mappings. The interesting case for elasticity is tensor valued mappings, i.e., the deformation gradient.

(Q: under what conditions on $\underset{\sim}{F}$ is there a differentiabl $\underset{\sim}{y}$ such that $\underset{\sim}{F} = \nabla \underset{\sim}{y}$?) This will follow immediatly from the vector-valued case:

Let $\underset{\sim}{u} : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ and assume Ω is open and simply connected. Assume $\underset{\sim}{u}$ is smooth. Then

$$\underset{\sim}{u} = \nabla \varphi \qquad (u_i = \varphi_{,i} \text{ in RCC})$$

if and only if

$$\nabla \underset{\sim}{u} = \nabla \underset{\sim}{u}^T$$

Proof. If $\underset{\sim}{u} = \nabla \varphi$ $(u_i = \varphi_{,i})$ then $u_{i,j} = \varphi_{,ij}$ $= \varphi_{,ji} = u_{j,i}$. Conversely, suppose $\nabla \underset{\sim}{u} = \nabla \underset{\sim}{u}^T$.

Define φ by integrating $\underset{\sim}{u}$ along line segments

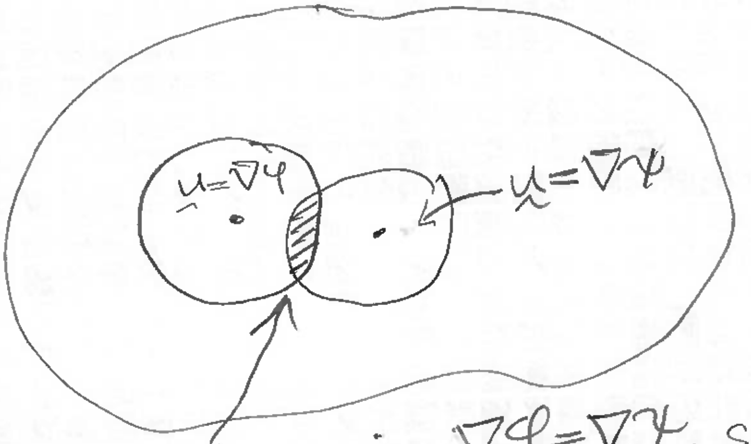
$$\varphi(\underset{\sim}{x}) = \int_0^1 \underset{\sim}{u}(\lambda \underset{\sim}{x} + (1-\lambda)\underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0)\, d\lambda$$

$$\nabla \varphi(\underset{\sim}{x}) = \int_0^1 \underset{\sim}{u}(\lambda \underset{\sim}{x} + (1-\lambda)\underset{\sim}{x}_0) + \lambda \nabla \underset{\sim}{u}^{\boxed{T}} (\underset{\sim}{x} - \underset{\sim}{x}_0)\, d\lambda$$

erase, by hypothesis

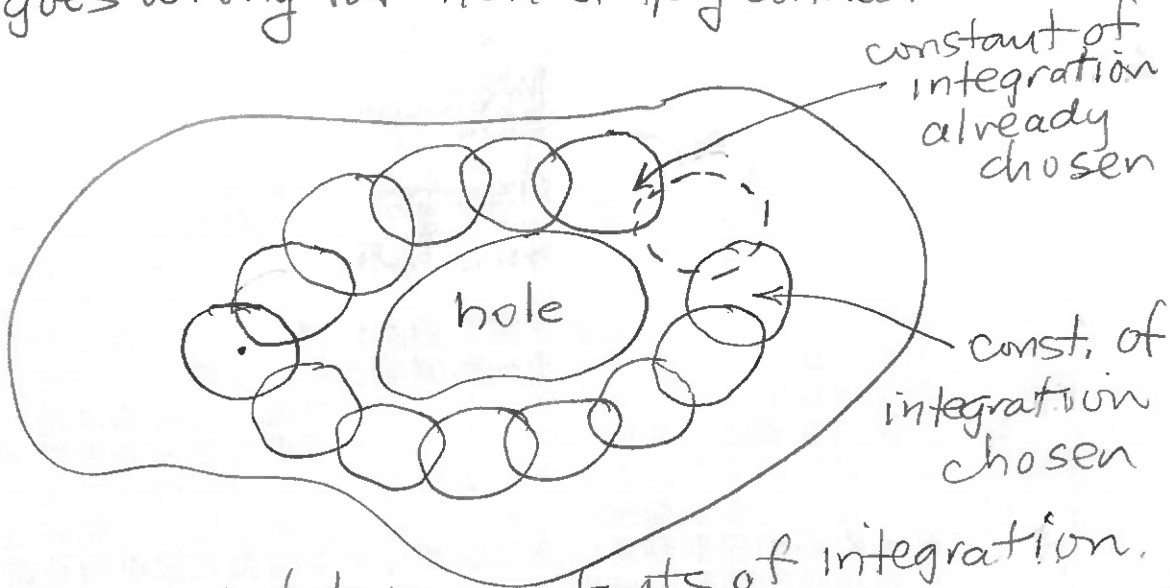
$$= \int_0^1 \frac{d}{d\lambda}\left(\lambda \underset{\sim}{u}(\lambda \underset{\sim}{x} + (1-\lambda)\underset{\sim}{x}_0\right) d\lambda = \underset{\sim}{u}(\underset{\sim}{x})$$

On a simply connected region, repeat on overlapping balls



On the overlap region $\nabla \varphi = \nabla \psi$, so $\psi = \varphi + \text{const}$. Simply choose const.$=0$. Repeat this process for additional balls. Then it is a matter of covering the region with balls.

What goes wrong for non-simply connected regions



Have to match two constants of integration for the dashed ball. Generically, this cannot be done.

The above also applies to tensors. In RCC

$$F_{ij} = y_{i,j} \Longleftrightarrow F_{ij,k} = F_{ik,j}.$$

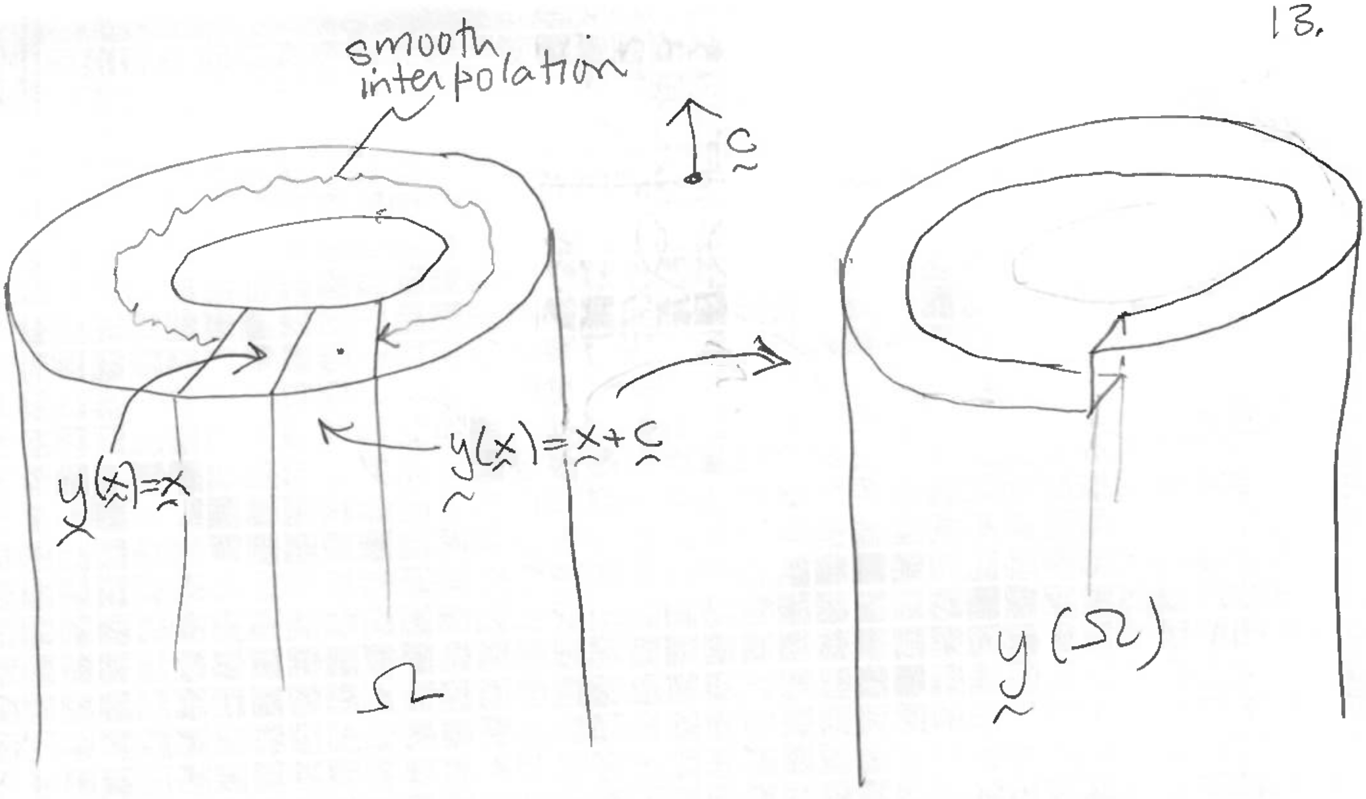
Fix i and this is the same as the above.

What physical phenomena are affected by having non-simply connected regions?

We will give two examples: 1) the Volterra dislocation, 2) folding paper flat.

The Volterra dislocation Ω is a hollow cylinder

Reference configuration Ω. Assign a deformation on this non-simply connected region as follows



- Notice that $\nabla \underset{\sim}{y}$ is completely smooth (could be C^∞)

- $\underset{\sim}{F}=\nabla \underset{\sim}{y}$ is smooth on Ω and satisfies

$$F_{ij,k}=\underset{\wedge}{F}_{ik,j}$$

(see p. 11-12)

- But, there is no continuous deformation $\underset{\sim}{y}: \Omega \to \mathbb{R}^3$ such that

$$\underset{\wedge}{F}=\nabla \underset{\sim}{y}$$

The structure of dislocations is fundamentally related to the lack of simple connectedness. To decide 'what is a dislocation' people typically interpolate lattice level motions by deformations, as above. The above is then the signature of a screw dislocation.

An example that illustrates the role of simple connectedness, and also the flexibility of the framework of elasticity theory to describe unusual phenomena that seem, at first, far from elasticity

Folding paper flat

Consider crumpling a piece of paper, and then pushing it down onto a flat surface. Neglecting the thickness of the paper, this is described by a deformation

$$\underset{\sim}{y} : \Omega \to \mathbb{R}^2, \quad \Omega \subset \mathbb{R}^2.$$

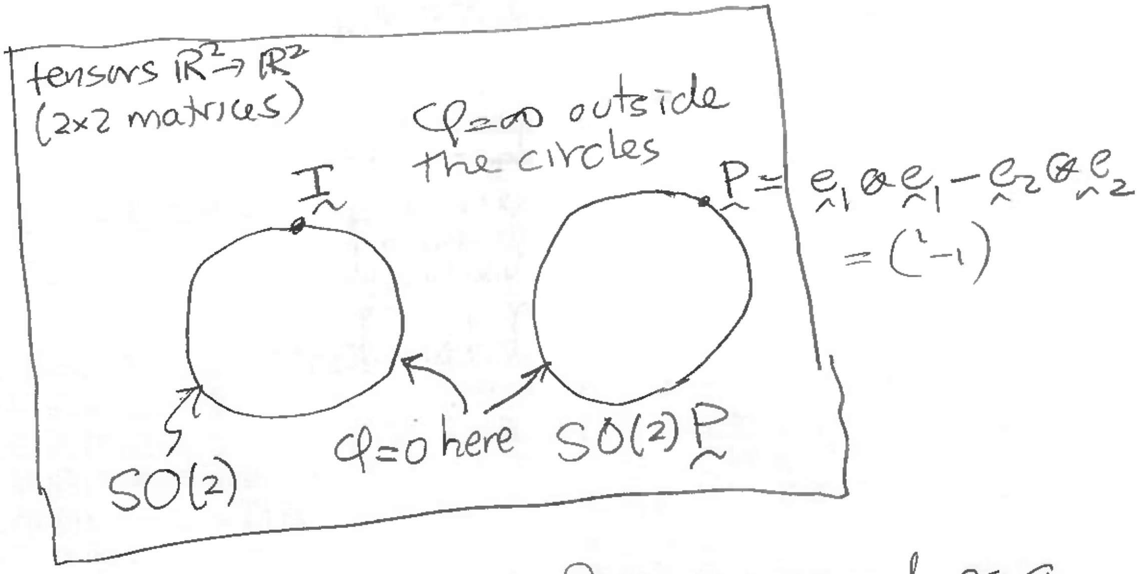
We will consider the paper to be unstretchable, be we have to give up the assumption $\det \nabla \underset{\sim}{y} > 0$, because some bits of paper will have turned over during folding. So, the correct assumption is

$$\underset{\sim}{F} = \nabla \underset{\sim}{y} \in O(2) = SO(2) \cup SO(2)\underset{\sim}{P}$$

where $\underset{\sim}{P}$ is any 2×2 tensor with determinant −1. Without loss of generality, we can take

$$\underset{\sim}{P} = \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 - \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 \qquad \underset{\sim}{e}_1, \underset{\sim}{e}_2 \text{ orthonormal}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{in this basis}$$

We can even think in terms of a free energy function that looks like this, when represented as on p. 9.:

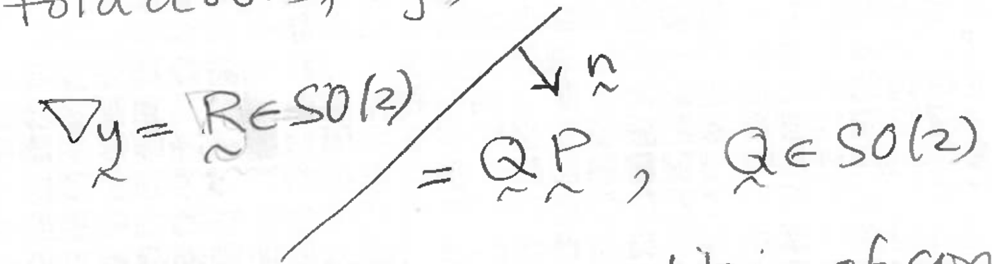


One can check that this free energy does a pretty good job of describing folding paper flat, no tearing allowed. Minimizers of this "free energy" satisfy

$\Omega \subset \mathbb{R}^2$

$\underset{\sim}{y} : \Omega \to \mathbb{R}^2$, $\nabla \underset{\sim}{y} \in SO(2) \cup SO(2)\underset{\sim}{P}$
continuous (no tearing) $= O(2)$

Liouville's theorem and the whole discussion above still holds for such y even though its gradient is (rather wildly) discontinuous, in some cases. At a crease in Ω where a fold occurs, e.g.,



we must satisfy the condition of compatibility

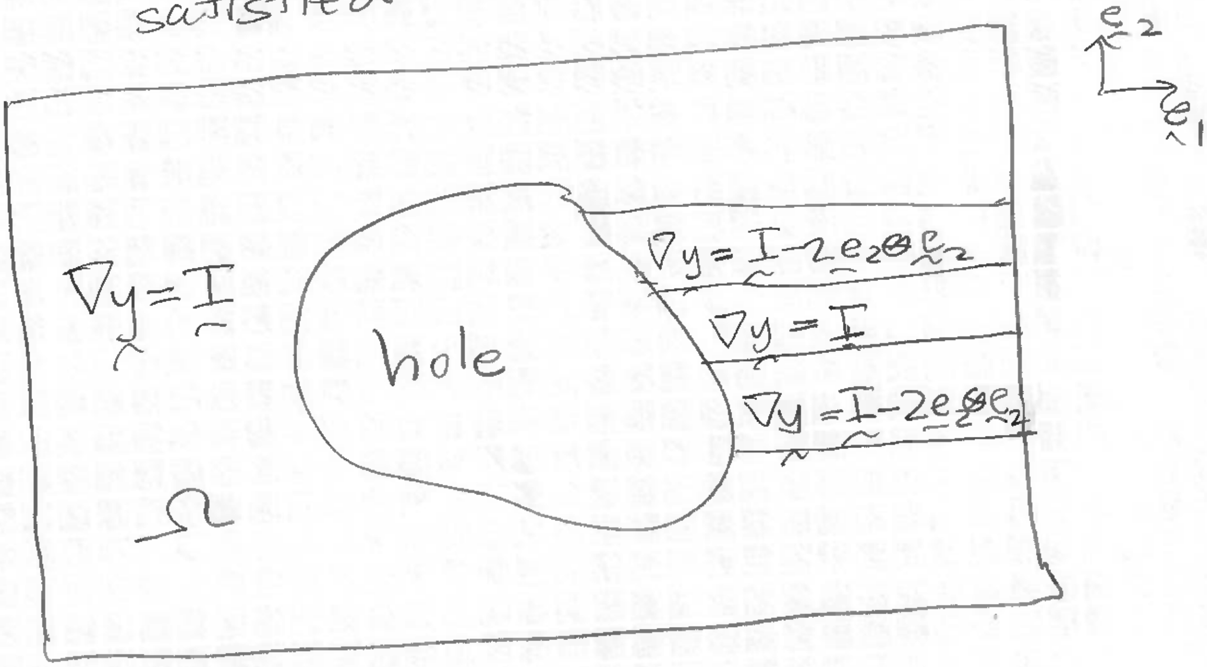
$$\underset{\sim}{Q}\underset{\sim}{P} - \underset{\sim}{R} = \underset{\sim}{a} \otimes \underset{\sim}{n}$$

$$(Q_{ij}P_{jk} - R_{ik} = a_i n_k)$$

If we satisfy this at all creases and otherwise assign $\nabla y \in O(2)$ this will be a possible folding plan. But there remain concerns:

1) This only assures that this single deformation is (theoretically) possible. To actually fold it, one has to deform up into 3D and back down to a plane, and this folding process may offer obstructions.

2) Before folding, the paper better be simply connected! For example, the following crease pattern is obviously not flat foldable, even though all rank-1 jump conditions are clearly satisfied:



Balances of Mass and Momentum

$\rho_0(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$, mass density in Ω

$\rho(\underset{\sim}{y}, t)$, $\underset{\sim}{y} \in y(\Omega, t)$, mass density in $y(\Omega, t)$

Mass is conserved: $\rho_0(\underset{\sim}{x}) = \rho(\underset{\sim}{y}(\underset{\sim}{x}, t), t) \det \nabla \underset{\sim}{y}(\underset{\sim}{x}, t)$

The derivative of the free energy density with respect to $\underset{\sim}{F}$ is the <u>Piola-Kirchhoff stress</u>

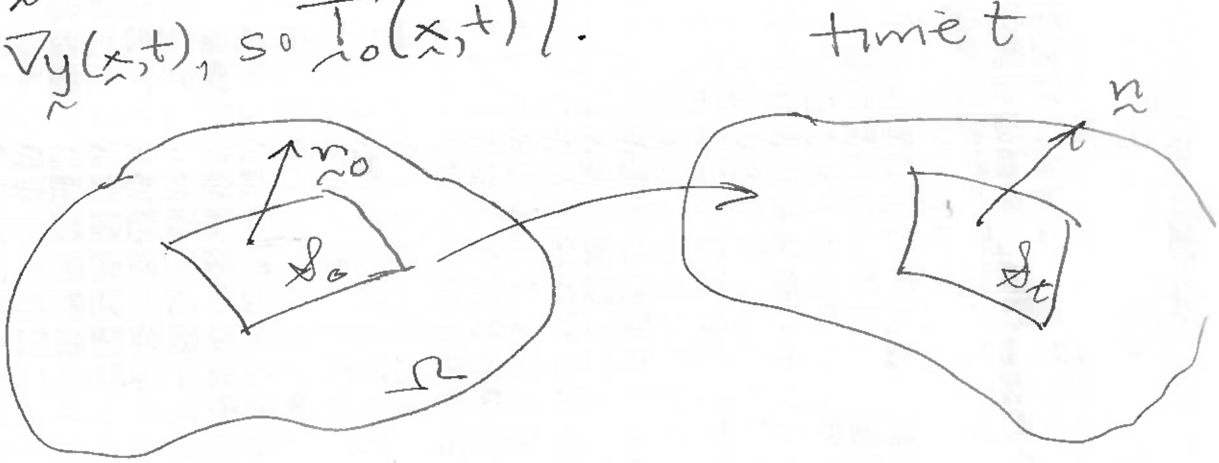
$$\underset{\sim}{T}_0 = \frac{\partial \varphi}{\partial \underset{\sim}{F}}$$

(This is consistent with the identification of φ as a Lyapunov function for the dynamical equations.)

Also, from the theory of stress (due to Cauchy), the integration of

$$\underset{\sim}{T}_0 \underset{\wedge}{n}_0$$

over a surface $S_0 \subset \Omega$ gives the force per unit area on $S_t = \underset{\sim}{y}(S_0, t)$ (In the time-dependent case $\underset{\sim}{F} = \nabla \underset{\sim}{y}(\underset{\wedge}{x}, t)$, so $\underset{\sim}{T}_0(\underset{\wedge}{x}, t)$).



$$\text{Force on } S_t = \int_{S_0} \underset{\sim}{T}_0 \underset{\sim}{n}_0 \, da_0$$

This way of calculating forces is consistent with the Lagrangian formulation of elasticity theory (independent variables $\underset{\wedge}{x} \in \Omega$ and t)

The Cauchy stress $\underset{\sim}{T}$ is defined by

$$\underset{\sim}{T} = \frac{1}{\det \underset{\sim}{F}} \underset{\sim}{T}_0 \underset{\sim}{F}^T$$

This formula is designed to guarantee that

$$\int_{\mathcal{S}_0} \underset{\sim}{T}_0 \underset{\wedge}{n}_0 \, da_0 = \int_{\mathcal{S}_t} \underset{\sim}{T} \underset{\wedge}{n} \, da.$$

(Easy way to see this: use a parametric representation, $\tilde{y}(\xi_1, \xi_2, t)$, of $\mathcal{S}_0$, and the corresponding parametric representation $\bar{\underset{\wedge}{y}}(\xi_1, \xi_2, t) = \underset{\sim}{y}(\tilde{y}(\xi_1, \xi_2, t), t)$ of $\mathcal{S}_t$).

Introducing a body force per unit mass $\underset{\sim}{b}$ and a mass density in the reference configuration $\rho_0(\underset{\wedge}{x}), \underset{\wedge}{x} \in \Omega$, we have the balance of linear momentum in the Lagrangian form used in elasticity:

$$\frac{d}{dt} \int_P \rho_0 \dot{\underset{\sim}{y}} \, d\underset{\wedge}{x} = \int_{\partial P} \underset{\sim}{T}_0 \underset{\wedge}{n}_0 \, da_0 + \overbrace{\int_P \rho_0 \underset{\sim}{b} \, d\underset{\wedge}{x}}^{\text{body force, e.g. gravity}}$$

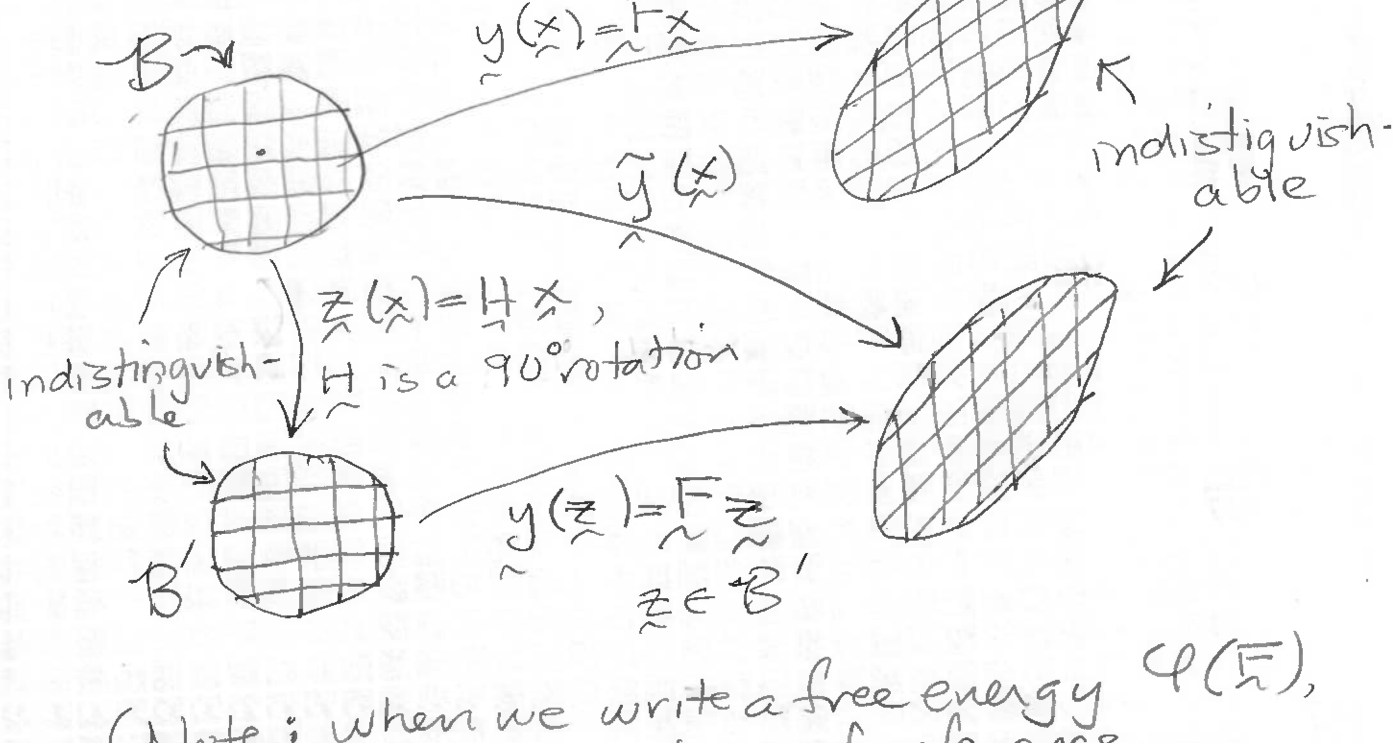
for all (measurable) $P \subseteq \Omega$. Since P is time-independent and ρ_0 is independent of t, we can localize: technically, we use the divergence theorem on the stress term, divide by vol.(P) and let $P \to \underset{\wedge}{x}$ (Lebesgue's theorem). We get the differential form

$$\rho_0 \ddot{\underset{\wedge}{y}} = \operatorname{div} \underset{\sim}{T}_0 + \rho_0 \underset{\sim}{b}.$$

Lecture 3

Material symmetry, linear vs. nonlinear elasticity, rank-1 convexity
(The vs. is intentional)

$\underset{\sim}{\varphi}$ may have symmetries beyond Galilean invariance (p. 6). These can be based on atomistic → continuum theory, in which case a key concept is the Cauchy-Born rule. Or, they can be found by purely macroscopic considerations, in which case "the free energy decides." The basic idea can be illustrated by a 2-D picture of a material with square symmetry at $\underset{\sim}{x} \in \Omega$:



(Note: when we write a free energy $\varphi(\underset{\sim}{F})$, it depends on the choice of reference configuration: so, we have to describe the deformation from $\mathcal{B}' \to \underset{\sim}{F}\mathcal{B}'$ in terms of a deformation from $\mathcal{B}$. Hence, $\underset{\sim}{\tilde{y}}(\underset{\sim}{x})$.

In fact,

$$\underset{\sim}{\tilde{y}}(\underset{\sim}{x}) = \underset{\sim}{y}(\underset{\sim}{z}(\underset{\sim}{x}))$$

$$\underset{\sim}{\nabla} \underset{\sim}{\tilde{y}}(\underset{\sim}{x}) = \underset{\sim}{\nabla} \underset{\sim}{y}(\underset{\sim}{z}(\underset{\sim}{x}))\nabla \underset{\sim}{z}(\underset{\sim}{x})$$

$$= \underset{\sim}{F}\,\underset{\sim}{H}$$

Thus, we say that H belongs to the symmetry group $\mathcal{G}$ of φ if

$$\varphi(\underset{\sim}{F}\,\underset{\sim}{H}) = \varphi(\underset{\sim}{F}) \text{ for all } \underset{\sim}{F},\ \det \underset{\sim}{F} > 0. \quad *$$

Clearly, $\mathcal{G}$ is a group; $\underset{\sim}{H}_1, \underset{\sim}{H}_2 \in \mathcal{G} \Rightarrow$

$$\left.\begin{aligned} &\varphi(\underset{\sim}{F}\,\underset{\sim}{H}_1 H_2) = \varphi(\underset{\sim}{F}\,\underset{\sim}{H}_1) = \varphi(\underset{\sim}{F}) \\ &\varphi(\underset{\sim}{F}) = \varphi(\underset{\sim}{F}\,\underset{\sim}{H}_1^{-1}\underset{\sim}{H}_1) = \varphi(\underset{\sim}{F}\,\underset{\sim}{H}_1^{-1}) \end{aligned}\right\} \text{for all } \underset{\sim}{F},\ \det \underset{\sim}{F} > 0$$

+ associative law.

It is generally assumed that $\mathcal{G} \subset \mathcal{U}$, the unimodular group: $\underset{\sim}{H} \in \mathcal{G} \Rightarrow \det \underset{\sim}{H} = 1$

If not, then since $\underset{\sim}{H}^n \in \mathcal{G}$ and $\det \underset{\sim}{H}^n = (\det \underset{\sim}{H})^n$ and

$$\left.\begin{aligned} (\det \underset{\sim}{H})^n &\to \infty \text{ if } \det \underset{\sim}{H} > 1 \\ &\to 0 \text{ if } \det \underset{\sim}{H} < 1 \end{aligned}\right\} \text{un-physical}$$

The largest possible group is then

$$\mathcal{U} = \{\underset{\sim}{H} : \det \underset{\sim}{H} = 1\}, \quad \text{unimodular group}$$

To understand what kind of material is described by the unimodular group, choose

$$\underset{\sim}{H} = (\det \underset{\sim}{F})^{1/3}\underset{\sim}{F}^{-1} \quad \text{in } * \text{ above}$$

Then, we get

$$\varphi(\underset{\sim}{F}) = \tilde{\varphi}(\det \underset{\sim}{F})$$

By the balance of mass (p. 16, bottom)

$$\varphi(\underset{\sim}{F}) = \bar{\varphi}(\rho/\rho_0),$$

so the Piola-Kirchhoff stress is

$$\underset{\sim}{T}_0 = \frac{\partial \varphi}{\partial \underset{\sim}{F}} = \bar{\varphi}'(\det \underset{\sim}{F})\, \underset{\sim}{F}^{-T} \quad \text{Piola-Kirchhoff stress}$$

$$\Rightarrow \underset{\sim}{T} = \frac{1}{\det \underset{\sim}{F}} \underset{\sim}{T}_0 \underset{\sim}{F}^T = \bar{\varphi}'(\rho/\rho_0)\, \underset{\sim}{I}, \quad \text{Cauchy stress}$$

an Euler fluid..

(Nonlinear elasticity) ∩ (Fluid mechanics)
= Euler fluid

...also equations of motion, etc.

Isotropic materials, $\mathcal{G} = SO(3)$

By Galilean invariance (p. 6, choosing $\underset{\sim}{Q} = \underset{\sim}{R}^T$)

$$\varphi(\underset{\sim}{F}) = \varphi(\underset{\sim}{U}) = \hat{\varphi}(\underset{\sim}{C})$$

$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U}$

$\underset{\sim}{C} = \underset{\sim}{F}^T$

(A positive-definite symmetric tensor has a a unique positive-definite square root).

$\underset{\sim}{F} \to \underset{\sim}{F}\underset{\sim}{H} \Rightarrow \underset{\sim}{C} \to \underset{\sim}{H}^T \underset{\sim}{C} \underset{\sim}{H}$. So, an isotropic material satisfies

$$\hat{\varphi}(\underset{\sim}{C}) = \hat{\varphi}(\underset{\sim}{R}\underset{\sim}{C}\underset{\sim}{R}^T)$$

for all $\underset{\sim}{C} = \underset{\sim}{C}^T > 0$ and all $\underset{\sim}{R} \in SO(3)$

By writing $\underset{\sim}{C}$ in spectral form, and choosing $\underset{\wedge}{R}$ (bottom p. 22) to map orthonormal eigenvectors of $\underset{\sim}{C}$ to a fixed orthonormal basis,

$$\varphi(\underset{\sim}{C}) = \hat{\varphi}(\lambda_1, \lambda_2, \lambda_3) \qquad \lambda_i \geq 0$$

The λ_i can also be chosen as the positive square roots of the eigenvalues of $\underset{\sim}{C}$; i.e., the principal stretches.

Aside from issues of smoothness, a function of $\lambda_1, \lambda_2, \lambda_3$ can be expressed as a function of I, II, III where

$$\begin{aligned} I &= \operatorname{tr} \underset{\sim}{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ II &= \tfrac{1}{2}\left((\operatorname{tr} \underset{\sim}{C})^2 - \operatorname{tr} \underset{\sim}{C}^2\right) = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \\ III &= \det \underset{\sim}{C} = \lambda_1^2 \lambda_2^2 \lambda_3^2 \end{aligned}$$

Using $\underset{\sim}{T}_o = \frac{\partial \varphi}{\partial \underset{\sim}{F}}$ and $\underset{\sim}{T} = \frac{1}{\det \underset{\sim}{F}} \underset{\sim}{T}_o \underset{\sim}{F}^T$, one gets that, for an isotropic material

$$\underset{\sim}{T} = f_1(I, II, III)\underset{\sim}{I} + f_2(I, II, III)\underset{\sim}{B} + f_3(I, II, III)\underset{\sim}{B}^2$$

(by changing the coefficients, one can replace $\underset{\sim}{B}^2$ by $\underset{\sim}{B}^{-1}$, according to the Cayley-Hamilton Theorem.)

Linearized elasticity

Linearized elasticity is based on a formal Taylor expansion about an equilibrium state, typically $\underset{\sim}{F}=\underset{\sim}{I}$, pictured schematically on p.9. Linearization in elasticity is done in an unusual way: instead of writing the equations of motion and linearizing them, it is done in two stages 1) linearize the formula for the Piola-Kirchhoff stress, and 2) then linearize the equations of motion.

The linearization goes like this: introduce the displacement

$$\underset{\sim}{u}(\underset{\sim}{x})=\underset{\sim}{y}(\underset{\sim}{x})-\underset{\sim}{x}$$

(or $\underset{\sim}{u}(\underset{\sim}{x},t)$) t is suppressed here, but could be included)

and the displacement gradient

$$\underset{\sim}{H}=\nabla\underset{\sim}{u}=\underset{\sim}{F}-\underset{\sim}{I}$$

To ease the calculation, we introduce a scalar small parameter ε and change $\underset{\sim}{H}\to\varepsilon\underset{\sim}{H}$, i.e.,

$$\underset{\sim}{F}_{\varepsilon}=\underset{\sim}{I}+\varepsilon\underset{\sim}{H}+\cdots$$

$$\underset{\sim}{C}_{\varepsilon}=\underset{\sim}{I}+\varepsilon(\underset{\sim}{H}+\underset{\sim}{H}^{T})+\cdots$$

$$\underset{\sim}{U}_{\varepsilon}=\underset{\sim}{I}+\varepsilon\underset{\sim}{E}+\cdots$$

$$\underset{\sim}{R}_{\varepsilon}=\underset{\sim}{I}+\varepsilon\underset{\sim}{W}+\cdots$$

$$\underset{\sim}{E}=\tfrac{1}{2}(\underset{\sim}{H}+\underset{\sim}{H}^{T})=\underset{\sim}{E}^{T}$$

$$\underset{\sim}{W}=\tfrac{1}{2}(\underset{\sim}{H}-\underset{\sim}{H}^{T})=-\underset{\sim}{W}^{T}$$

The first nonzero term for the free energy is at ε^2

$$\varphi(\underset{\sim}{F}_{\varepsilon})=\tfrac{1}{2}\varepsilon^{2}\underset{\sim}{E}\cdot\frac{\partial^{2}\varphi(\underset{\sim}{I})}{\partial\underset{\sim}{F}\,\partial\underset{\sim}{F}}\underset{\sim}{E}+\cdots$$

How does the invariance of the linear elasticity tensor work out?

$\overset{\underset{\uparrow}{Q \in SO(3)}}{}$

Frame indifference $\underset{\sim}{F} \sim \underset{\sim}{I}$ but $\underset{\sim}{Q}\underset{\sim}{F}$ is potentially far from $\underset{\sim}{I}$, so the only restriction consistent with this expansion is to truncate

$$\underset{\sim}{Q}_\varepsilon \doteq \underset{\sim}{I} + \varepsilon \underset{\sim}{W} \qquad \underset{\sim}{W}^T = -\underset{\sim}{W}$$

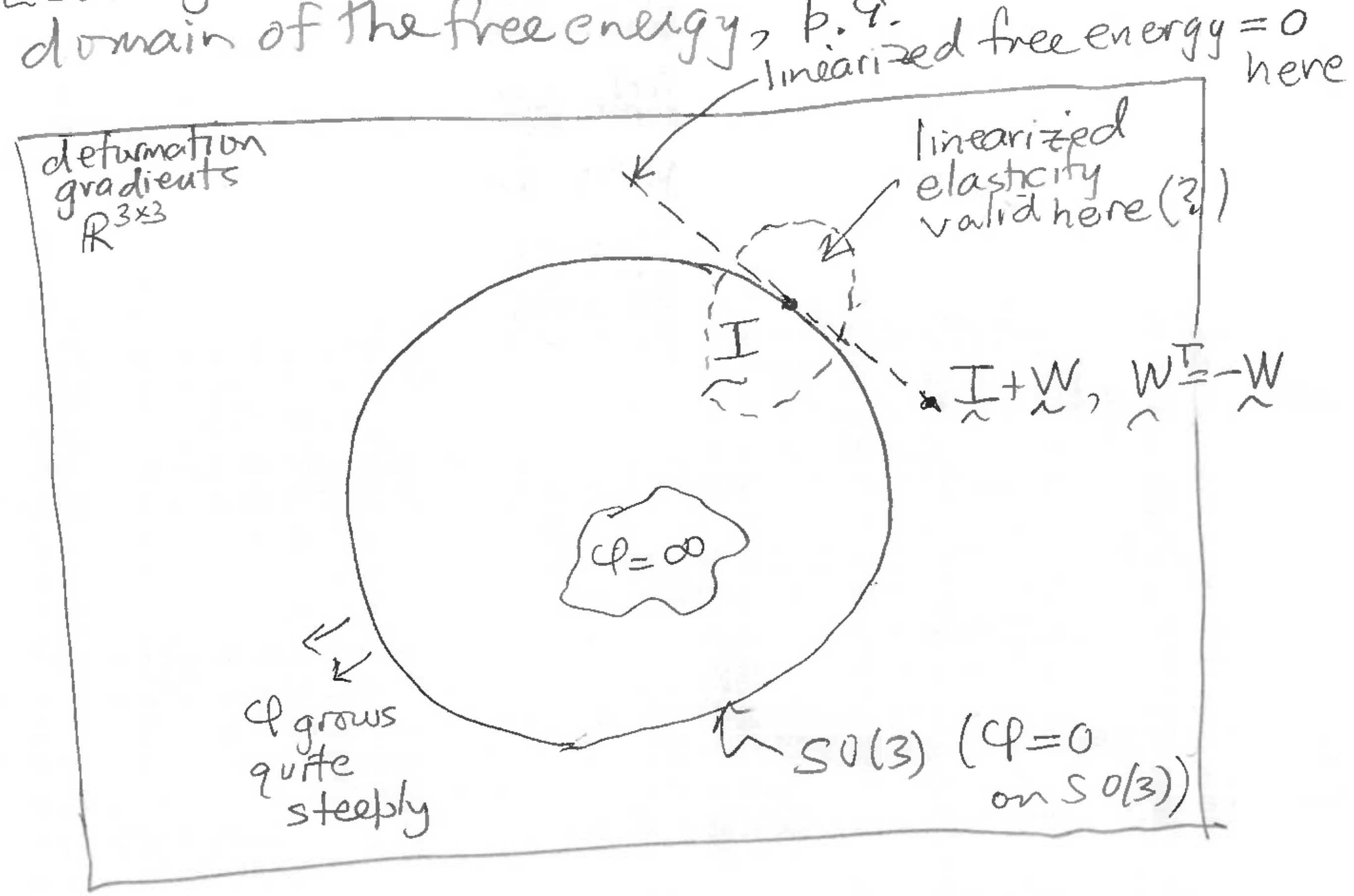
giving $$\underset{\sim}{W} \cdot \frac{\partial^2 \varphi}{\partial \underset{\sim}{F}\, \partial \underset{\sim}{F}} \underset{\sim}{W} = 0$$ "linearized Galilean invariance"

Material symmetry $\underset{\sim}{F} \sim \underset{\sim}{I}$, but $\underset{\sim}{F}\underset{\sim}{H}$ (see p. 20) is far from $\underset{\sim}{I}$! So, material symmetry seems to have no implications for linearized elasticity. But, if the symmetry group is a subgroup of SO(3), then we combine Galilean invariance and material symmetry to get

$$\underset{\underset{\text{Galilean invariance}}{\uparrow}}{\underset{\sim}{H}^T}\, \underset{\sim}{F}_\varepsilon\, \underset{\underset{\text{material symmetry}}{\uparrow}}{\underset{\sim}{H}} = \underset{\sim}{I} + \varepsilon \underset{\sim}{H}\, \underset{\sim}{E}\, \underset{\sim}{H} + \dots \sim \underset{\sim}{I} \qquad \underset{\sim}{H} \in SO(3)$$

Gives the standard restriction on linear elastic moduli

Let's go back to our picture of the domain of the free energy, p. 9.



In practical terms how far can we go away from $\underset{\wedge}{I}$ and trust results from linearized elasticity? A way to understand that is to solve the linear elasticity problem where the boundary displacement is given as an exact rigid rotation,

$$\underset{\sim}{u}(\underset{\wedge}{x}) = (\underset{\sim}{R} - \underset{\wedge}{I})\underset{\wedge}{x}, \quad \underset{\sim}{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\underset{\wedge}{x} \in \partial\Omega,$$

Then $\underset{\wedge}{u}(\underset{\wedge}{x})$, $\underset{\wedge}{x} \in \Omega$, satisfies equilibrium equations of linearized elasticity, and it is the unique solution under mild conditions with these boundary conditions

Of course, the stress is not zero, as would be predicted by the nonlinear theory. What is the discrepancy? We can understand that by using the moduli (for a typical steel),

$$E = 200\ \text{GPa}, \quad \nu = 0.3$$

To measure the error, we calculate the shearing stress intensity, given by

$$\sqrt{\frac{1}{2}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right.}$$

When this expression reaches the yield stress Y, the material is at yield, according to a typical model of yielding. For a typical high strength steel is approximately 400 MPa. At what angle θ is the shearing stress intensity 10% of the yield stress of steel? Of course, from a physical viewpoint, an exact rigid rotation should leave a stress-free reference configuration stress-free.

Answer: $\theta \cong 1.3^\circ$

Conclusion: one has to be extremely careful using linearized elasticity. By the way, many plate and shell theories look a lot like some kind of generalization of linear elasticity. For example, they may have energy densities that depend only on linear elastic moduli, e.g., Kirchhoff's plate theory. But the best of these (including any with the name Kirchhoff)' do not suffer from the striking defect mentioned on page 31: they are perfectly objective.