Lectures on Applied Topology

by Randall D. Kamien

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Based on notes by Glasha Osipycheva and Anish Pandya

The Lattice Laplacian & Hodge Decomposition.

1. Introduction. In this set of notes, we will explore the lattice Laplacian and the Hodge decomposition. These topics are an entry into the role of homology in physics. An application of homology theory is in the study of lattice models. We will begin by introducing the Euler characteristic of a manifold. The Euler characteristic is given by the alternating sum of the *Betti numbers* of a manifold. The generalization of the Laplacian in terms of the boundary and co-boundary operators will enable us to investigate the geometric content of the manifold in new (old) ways. This will be explored in the lecture on Witten's proof of the Morse inequalities. For the remainder of these notes, unless otherwise stated, assume that $\mathcal{M}$ is a closed, smooth manifold with boundary $\partial\mathcal{M}$. We will also be loose with definitions to hide our mathematical naïveté.

2. The Euler Characteristic. In a previous section, you have likely covered the Gauss-Bonnet theorem, which we state without proof here:

> **Theorem 2.1** (Gauss-Bonnet Theorem, 1827)
> Let $\mathcal{M}$ be a closed, smooth manifold with boundary $\partial\mathcal{M}$. Let K be the Gaussian curvature and let κ_G be the geodesic curvature on the boundary. Then,
>
> $$\int_{\mathcal{M}} K\,dA + \int_{\partial\mathcal{M}} \kappa_G\,ds = 2\pi \tag{1}$$

Consider a polygon on the surface of a domain $\Omega \subset \mathcal{M}$: The integral above does not equal 2π in this case since there are external angles (there is a jump in the tangent vector). If we consider tiling (or *triangulating*) the manifold with such polygons, the correct sum is:

$$2\pi = \int_{\mathcal{M}} K\,dA + \int_{\partial\mathcal{M}} \kappa_g\,ds + \sum_i \theta_i \tag{2}$$

where the external angle at vertex i of triangle j is $\theta_{ij} = \pi - \alpha_{ij}$ and α_{ij} is the internal angle. At each vertex of the triangulation, the internal angles must sum to 2π. When we sum over the edges, we get zero, since each edge is traversed in both its positive and negative orientation:

$$\sum_j \int_{\partial\mathcal{M}_j} \kappa_g\,ds = 0 \tag{3}$$

So, denoting the number of edges of tile j as E_j and summing over all the $\mathcal{M}_i$, we get:

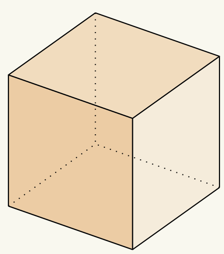
$$\sum_j 2\pi = \sum_j \int_{\mathcal{M}_j} K\, dA + \sum_j E_j \pi - \sum_i \sum_j \alpha_{ij} \tag{4}$$
$$2\pi F = \oiint_{\mathcal{M}} K\, dA + 2\pi E - 2\pi V \tag{5}$$

Since each edge appears in *two* tiles. We have:

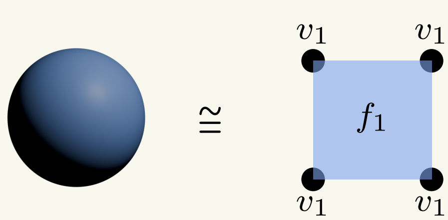
$$\frac{1}{2\pi} \oiint_{\mathcal{M}} \kappa\, dA = V - E + F \tag{6}$$

How is it that the integral over a closed manifold gives an *integer* multiple of 2π? TOPOLOGY! We call the right-hand side the *Euler characteristic* of a manifold, $\chi = V - E + F$. For the remainder of the notes, let V, E, F denote both the respective sets of vertices, edges, and faces, as well as the number of vertices, edges, and faces. The meaning will be clear from the context.

Example 2.2 (S^2)
Consider the sphere drawn below.



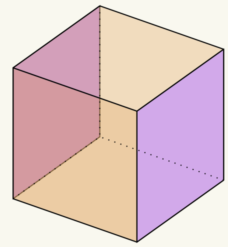
Count the number of faces, edges, and vertices respectively: 6, 12, 8 respectively. Verify if $\chi = 8 - 12 + 6$ equals 2. We can also represent S^2 with one vertex and one face as follows:



Okay, but how about something a little more interesting, like a torus:

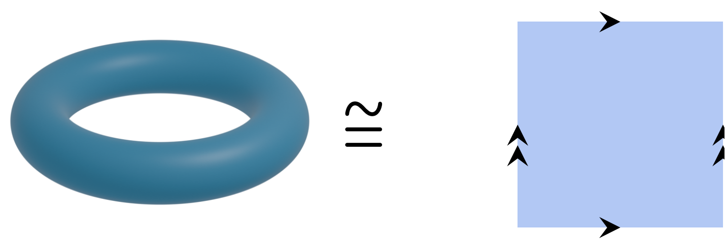
Example 2.3 ($S^1 \times S^1$)

How do you make a torus? In a torus, there is a hole, so we will have to make this hole by somehow puncturing a sphere and deforming it in the right shape. Start with a sphere and identify the two opposite faces:



To make this, we had to take two faces away from the sphere: $F = 6-2 = 4$. Then, $\chi = V - E + F = 8 - 12 + 4 = 0$. So, $\chi = 0$.

We can draw a torus in the following way:



Based on this iterative procedure, for every hole, we can glue two faces together. We can upgrade the notion of hole as *genus*.

Definition 2.4 (Genus of a manifold.)

The **genus** of a manifold is the number of "holes."

Consider drawing a genus 2 manifold; it is not two copies of the above — why? This suggests the following formula for the Euler characteristic:

Definition 2.5 (Euler characteristic.)

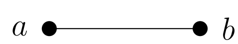
The **Euler characteristic** χ of a manifold $\mathcal{M}$ of genus g is given by:

$$\begin{aligned} \chi(g) &\coloneqq 2 - 2g && (7) \\ &= \frac{1}{2\pi} \oiint K dA && (8) \\ &= V - E + F && (9) \end{aligned}$$

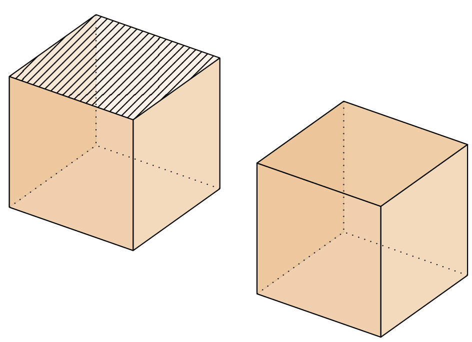
First, we did not rely on the microscopic structure of the polygon embedded into the manifold: this is not clear upon first glance. Additionally, all we assumed is that $\mathcal{M}$ is a closed, smooth manifold. Those assumptions showed up in the existence of some integrals. Some manifolds admit an approximation via *cell complexes*, which are collections of vertices, edges, faces (and so on...) which are constructed in a particular way.

If I add an edge to the graph, I will also add two extra vertices and one extra face: $\Delta F = 1, \Delta V = 2, \Delta E = 1$. There is a pattern with the Euler characteristic of lines, edges, and spheres: First, just consider a single vertex:

Then, we have $V = 1, E = 0, F = 0$. So, $V - E + F = 1$. We would agree there's one object here. Okay, but what if we have two vertices and one edge:



Then, $V - E + F = 2 - 1 + 0 = 1$. There is still one *object.* Now what about a cube with the top face removed? If I put the cap on, does it change things?



For the first cube, $V - E + F = 8 - 12 + 5 = 1$. For the cube with a top, I will get 2 for the same sum. If I subtract the number counting *3*-cells (or closed voids), call it c, I will still count 1 object. This brings us to the more general concept of *cellular homology.* The void that is filled when the cap is put on the cube is a *3-cell.* Another formation is *simplicial homology. Simplexes* are formed from functions of the standard simplex. Apparently, for finite dimensions, the homological properties of simplicial complexes and cellular complexes are the same, so we won't have to worry too much about the distinction at the moment.

For the remainder of the notes, we will assume that the sets V, E, F are finite[*]. We will often refer to them as C^0, C^1, C^2: these are the *additive groups* formed by vertices, edges, and faces (respectively). These groups are Abelian groups;

[*] Although, this is not strictly necessary.

they contain oriented combinations of vertices, edges, and faces (respectively) with integer coefficients (what group axiom would break if they were not oriented?). For example, a graph with two vertices, a, b can have elements of C^0 of the form $\alpha a + \beta b$, where $\alpha, \beta \in \mathbb{Z}$ and α, β represent some physically relevant quantity on the vertices. More generally, we could also have α, β in $\mathbb{R}$ (or $\mathbb{C}$) in which case we can interpret these sets as vector spaces – linear combinations of vectors where the basis vectors are abstract objects representing the vertices, edges, and faces. They are *not* vectors in the embedding space. We can define C^k be defined in an analogous manner.

Definition 2.6 (Cell complexes & Orientation.)
The manifolds we will consider are **oriented.**

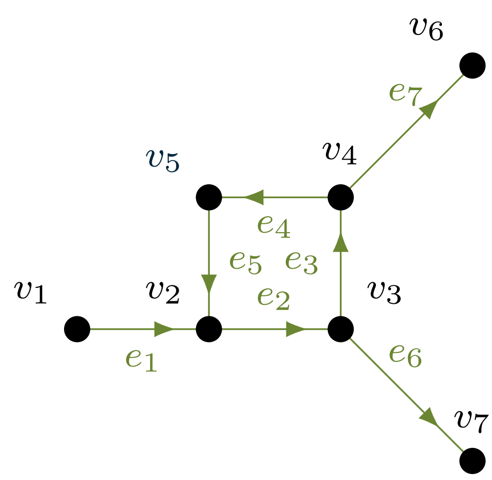
1. The **0-cells** are discrete points (vertices). The Abelian group of formal combinations of 0-cells is denoted C^0.

2. The **1-cells** can be defined inductively as oriented edges. The Abelian group of formal combinations of 1−cells is denoted C^1.

3. The **2-cells** are oriented faces (and so on.. for n-cells).The Abelian group of formal combinations of 2−cells is denoted C^2.

This generalizes to k−cells and C^k.

3. The Boundary Operator. To analyze k−cells, we want to understand group homomorphisms between $C^0, C^1, \ldots, C^k$. This is a natural idea to consider in physics; for example: what is the relationship between node-potentials and currents of an electrical network? To start, consider the following graph:



We can algebraically characterize the properties of this graph in terms of the V, E, F. For now, let's consider the formal space generated by linear combinations of the vertices, edges, and faces, and store their coefficients in a column vector as follows:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_V \\ \beta_1 \\ \vdots \\ \beta_E \\ \gamma_1 \\ \vdots \\ \gamma_F \end{pmatrix}$$

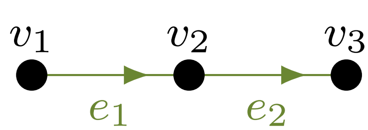
where, an element in the space is an arbitrary linear combination: $\Delta = \alpha_1 v_1 + \cdots \alpha_V v_V + \beta_1 e_1 + \cdots + \beta_E e_E + \gamma_1 f_1 + \gamma_F f_F$, with coefficients in some set of numbers, like $\mathbb{Z}$ or $\mathbb{R}$ *. We can also refer to the individual vector of vertices, $\vec{v} = (v_1, \cdots, v_V)^T$, edges, $\vec{e} = (e_1, \cdots, e_E)^T$, and faces, $\vec{f} = (f_1, \cdots, f_F)^T$.

* For example, if this graph were an electrical circuit, a quantity defined on edges might be the current between two nodes.

Intuitively, what should something called the *boundary operator* do? If I give the boundary operator an edge, it should give me back an oriented linear combination of the *boundary* vertices. If I input a face, it should give me some linear combination of edges. If I give it a vertex, I expect it to give me zero, since vertices have no boundaries. Let's do an example for a graph before getting into the details.

Consider the vector space† formed by taking the coefficients of the formal combinations in $\mathbb{Z}_2$. Suppose $\Delta = (0, \ldots, 1, 0, \ldots)^T$. What would we expect $\partial\Delta$ to be? We expect $\partial e_2 = v_3 - v_2$. Further, $\partial^2 e_2 = 0$. If f_1 denotes the face formed by the closed square, we might expect $\partial f_1 = e_2 + e_3 + e_4 + e_5$ (because we are using $\mathbb{Z}_2$ we don't need to mind the orientation of the face). If we take ∂v_1 we want 0 since vertices have no boundary. In this way, the boundary operator is a homomorphism, $\partial : C^k \to C^{k-1}$ Let's do another example, this time where the coefficients are no longer in $\mathbb{Z}_2$ but rather $\mathbb{Z}$. Consider the following graph:

† Really what one means here is *module*, but this detail is about definition, not function.



We can recap the boundary operator actions simply:

$$
\begin{aligned}
\partial e_1 &= v_2 - v_1 \\
\partial e_2 &= v_3 - v_2
\end{aligned}
$$

Now, take $\Delta = 3e_1 + 4e_2$. What is the action of the boundary operator on this new object? It's given by:

$$
\begin{aligned}
\partial\Delta &= 3\ \partial e_1 + 4\ \partial e_2 \\
&= 3(v_2 - v_1) + 4(v_3 - v_2) \\
&= -v_2 - 3v_1 + 4v_3
\end{aligned}
$$

This object does not have an interpretation based on the above diagram (other than the property of being the boundary of Δ). Even so, we expect *linearity* of the boundary operator, and so we expect its action to be accordingly. Let's write down a definition:

Definition 3.1 (The Boundary Operator.)
Let $\Delta = \alpha^i f_i \in C^k$. The **boundary operator** is a homomorphism of C^k into C^{k-1}, satisfying:

$$\partial\Delta = \alpha^i(\partial f_i)$$

where ∂f_i is the boundary-operator restricted to the the i^{th} generator of C^k. In general, ∂f_i will give an alternating sum of elements in C^{k-1}.

The image of ∂_{k+1} is called the space of *boundaries* and the kernel of ∂_k is the space of *cycles*; cycles are sums of elements of C^k with *no boundaries*. The quotient group $H_k \coloneqq \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$ is the k^{th} **homology group**. We will not delve further into homology in these notes, but note that H_k is the set of k-cells that have no boundary but are not the boundary of something else. It is like saying that two gauge fields are equivalent if they differ by a gauge transformation (more later). For a fixed labeling of the cell-complex, we can write a matrix representation of the boundary operator for the concatenated space of V, E, F as:

$$\begin{pmatrix} \mathbf{0} & \boldsymbol{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{D}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\boldsymbol{D_1}$ is a $V \times E$ block matrix and $\boldsymbol{D_2}$ similarly is an $E \times F$ block matrix. $(\boldsymbol{D_1})_{ij}$ is ± 1 if vertex i is on the boundary of edge j, where the sign indicates orientation. Similarly, $(\boldsymbol{D_2})_{ij}$ is ± 1 if edge i is on the boundary of face j. These matrices are also the incidence matrices from graph theory. Check the encoding of the following statements in the block description of the matrix:

1. Faces don't have vertices as their boundaries.

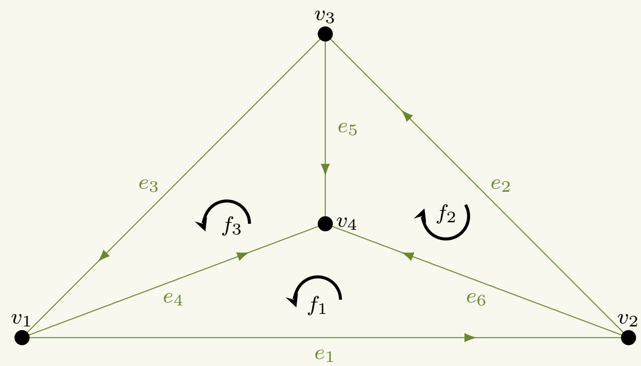
2. The boundaries of edges are not faces.

3. The boundary of a vertex is zero.

The matrices $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are called incidence matrices in algebraic graph theory. By construction, $\partial^2 = 0$ on any tuple of the form $(v_1, \cdots, v_V, e_1, \cdots, e_E, f_1, \cdots f_F)^T$. If we don't want to consider all of the V, E, F together, we can use the notation $\partial_k : C^k \to C^{k-1}$. So, ∂_1 would be the boundary operator on edges, and ∂_2 would be the boundary operator on faces and so on. This construction makes the linear algebra of the usual cases of the boundary operator clear. Additionally, the matrix representation makes the *dual* operator clear: $\partial^\dagger$. This is called the *co-boundary operator*. The co-boundary operator is a homomorphism: $\partial^\dagger_{k-1} : C^{k-1} \to C^k$. We write $\partial^\dagger$ to be the co-boundary operator defined on the (V, E, F) space, W. Clearly, by extension, we could add 3-cells, 4-cells, *etc.* and build larger and larger operators out of these block operators ∂_k.

Example 3.2

What is $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ for the following example?



Then, we can write $\boldsymbol{D_1}$ as:

$$\boldsymbol{D_1} = \begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Similarly, we can write $\boldsymbol{D_2}$ as:

$$\boldsymbol{D_2} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Notice that the fourth face f_4 is the remainder of the plane.

Recall the following theorem from algebra:

Theorem 3.3 (Rank-Nullity Theorem.)

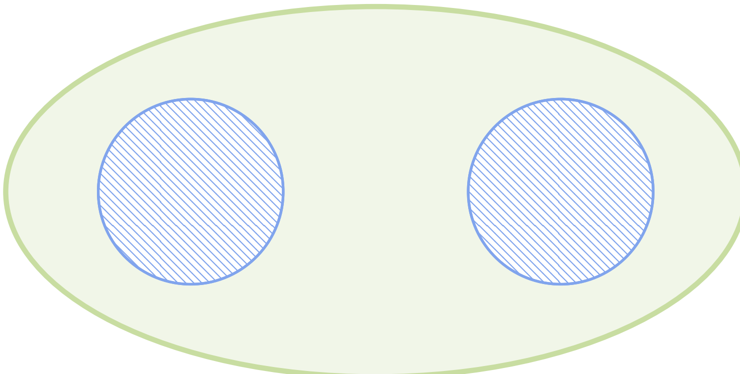
Let $T : H \to G$ be a linear operator between vector spaces, where H is finite-dimensional. Then,

$$\dim H = \dim \ker(T) + \dim \operatorname{im}(\mathrm{T}) \tag{10}$$

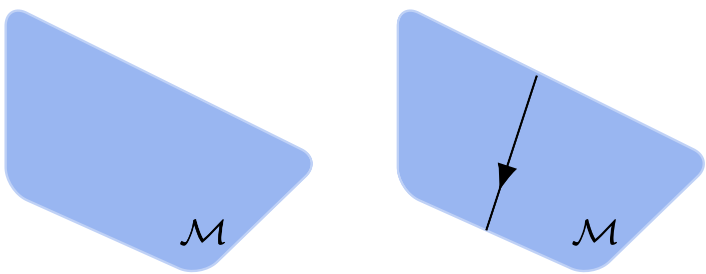
For $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$:

$$\dim \operatorname{im}(\boldsymbol{D_1}) + \dim \ker(\boldsymbol{D_1}) = E$$
$$\dim \operatorname{im}(\boldsymbol{D_2}) + \dim \ker(\boldsymbol{D_2}) = F$$

4. Betti Numbers From the previous section, we know the boundary of a boundary is zero ($\partial_{k-1}\partial_k\Delta = 0$). Does that mean $\partial\Delta = 0$ implies that Δ is the boundary of something? Consider a pair of pants:



The pants have two holes (blue, striped). The boundary of these holes will be zero, but they are not boundaries of another element. Suppose we have a triangulation of a manifold (shown below).



If we draw an extra edge, the image of $\boldsymbol{D_2}$ would be changed by a single face and the kernel of $\boldsymbol{D_1}$ is changed by 1 loop. Similar to what we saw in the discussion of the Euler characteristic ($V - E + F - C + \cdots$), we can consider:

$$b_1 = \dim\ker(\boldsymbol{D_1}) - \dim\operatorname{im}(\boldsymbol{D_2})$$

Let's invent $\boldsymbol{D_0}$ a $0 \times V$ matrix acting on the vertices (and sending them to 0) to write:

$$b_0 = \dim\ker(\boldsymbol{D_0}) - \dim\operatorname{im}(\boldsymbol{D_1})$$

If we consider the combined system:

$$\begin{cases} b_0 & = \dim\ker(\boldsymbol{D_0}) - \dim\operatorname{im}(\boldsymbol{D_1}) \\ b_1 & = \dim\ker(\boldsymbol{D_1}) - \dim\operatorname{im}(\boldsymbol{D_2}) \\ b_2 & = \dim\ker(\boldsymbol{D_2}) - \dim\operatorname{im}(\boldsymbol{D_3}) \end{cases}$$

But we know that $\dim\ker(\boldsymbol{D_0}) = V$ and $\dim\operatorname{im}(\boldsymbol{D_3}) = 0$ (assume there is nothing for $\boldsymbol{D_3}$ to multiply). Then, if we consider the following combination of the Betti numbers, $b_0 - b_1 + b_2$, we see:

$$\begin{aligned} b_0 - b_1 + b_2 &= V - (\dim\operatorname{im}(\boldsymbol{D_1}) + \dim\ker(\boldsymbol{D_1})) + (\dim\operatorname{im}(\boldsymbol{D_2}) + \dim\ker(\boldsymbol{D_2})) \\ &= V - E + F \end{aligned}$$

Definition 4.1 (Betti Numbers)

The **Betti numbers of a manifold** $\mathcal{M}$, b_0, b_1, b_2, are defined in terms of the boundary operator ∂ restricted to the relevant subspace. Specifically,

$$\begin{aligned} b_0 &:= \dim\ker(\boldsymbol{D_0}) - \dim\operatorname{im}(\boldsymbol{D_1}) \\ b_1 &:= \dim\ker(\boldsymbol{D_1}) - \dim\operatorname{im}(\boldsymbol{D_2}) \\ b_2 &:= \dim\ker(\boldsymbol{D_2}) - \dim\operatorname{im}(\boldsymbol{D_3}) \end{aligned}$$

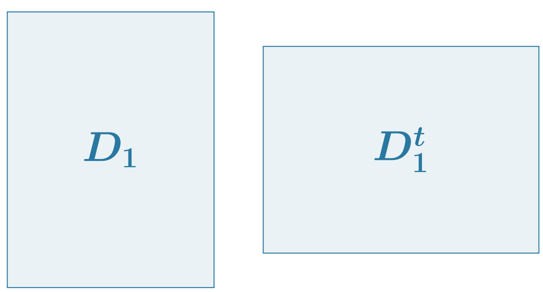
Example 4.2 (S^2)

From the representation of S^2 as a cell complex, we can see that: $b_0 = 1, b_1 = 0, b_2 = 1$, so $\chi = 2$.

Example 4.3

For a torus, $b_1 = 2, b_2 = 1, b_0 = 1$, so $\chi = 1 - 2 + 1 = 0$.

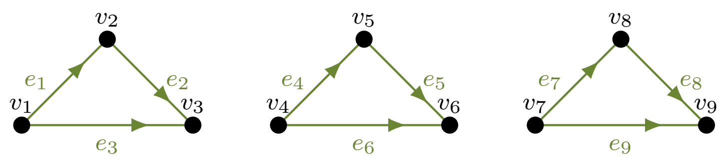
But what do the Betti numbers tell us about a manifold? Let's first consider b_0:



From (linear) algebra, we know that $\dim\operatorname{im}(\boldsymbol{D_1^T}) = \dim\operatorname{im}(\boldsymbol{D_1})$ (row rank = column rank). Then, b_0 satisfies:

$$\begin{aligned} b_0 &= V - \dim\operatorname{im}(\boldsymbol{D_1}) \\ &= V - \dim\operatorname{im}(\boldsymbol{D_1^T}) \\ &= \dim\ker(\boldsymbol{D_1^T}) \end{aligned}$$

Recall that $\boldsymbol{D_1^T}$ is the matrix representation of a dual boundary operator $\partial^\dagger : V \to E$, satisfying the matrix equation $D_1^T \Delta = 0$. What does it mean for this equation to have multiple non-trivial solutions? It means that the triangulation of the manifold has multiple connected components.
Check that $b_0 = 3$ for the following graph:



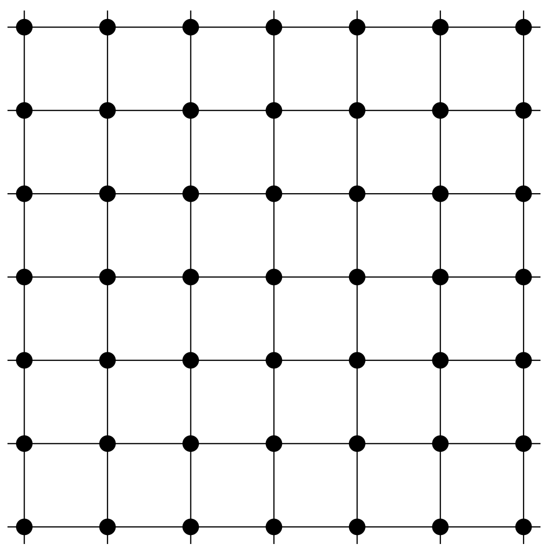
The Betti numbers are the (minimal) number of generators of the homology groups previously mentioned. That is, H_k is generated by b_k elements.

> **Theorem 4.4** (Poincaré Duality, 1893)
> If $\mathcal{M}$ is a closed, orientable $n-$ dimensional manifold, then $b_i = b_{n-i}$.

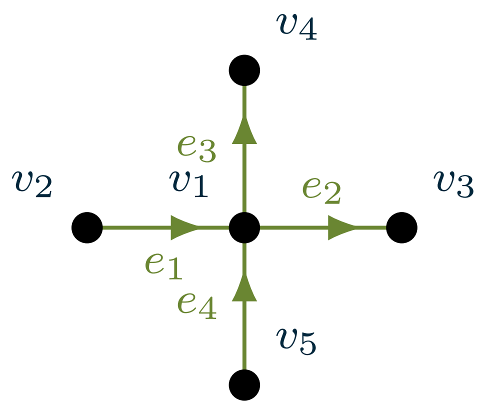
Prove this by pure thought.

5. The Lattice Laplacian. In this section, we will explore the Laplacian, which can be constructed out of the boundary operator.

The Lattice Laplacian can be constructed in multiple ways : in these notes, we take the topological perspective and consider the Laplacian as an operator constructed from the boundary and co-boundary operators. If one takes a graph-theoretic or combinatorial perspective, there are other objects which act in similar ways that are also called *the* Laplacian, but we will defer their discussion to other sources. With the new operators we've built, we'll consider a relevant setting for physics: the square lattice.



Fix an orientation on the lattice. First, let's consider the relationship between edges and vertices. Locally, around a vertex v_1, the edges and vertices look like the graph below:



Then, let $\Delta = \alpha^i v_i$, for $\alpha^i \in \mathbb{R}$. Clearly, the boundary of any vertex is zero, so $\partial\Delta = 0$. What about the action of $\partial^\dagger$?
We can write $\boldsymbol{D_1}$ and $\boldsymbol{D_1^\dagger}$ as:

$$\boldsymbol{D_1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \boldsymbol{D_1^T} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

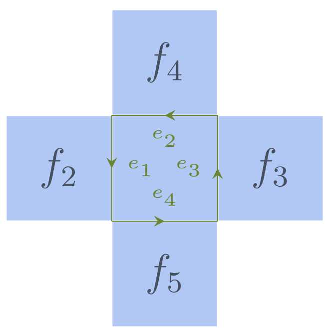
Then, if we act with $\boldsymbol{D_1^\dagger}$:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \alpha^4 \\ \alpha^5 \end{pmatrix} = \begin{pmatrix} \alpha^1 - \alpha^2 \\ \alpha^3 - \alpha^1 \\ \alpha^4 - \alpha^1 \\ \alpha^1 - \alpha^5 \end{pmatrix}$$

Let $\boldsymbol{e_1} = (1, 0, \cdots)^T$, the unit tuple with a 1 in coefficient to v_1 and zeros elsewhere. Then, the quadratic form $\boldsymbol{e_1^T}\partial\partial^\dagger\boldsymbol{\Delta}$ has a familiar form:

$$
\begin{aligned}
\boldsymbol{e_1^T}\partial\partial^\dagger\boldsymbol{\Delta} &= \begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & -1 & 1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\alpha^1-\alpha^2\\ \alpha^3-\alpha^1\\ \alpha^4-\alpha^1\\ \alpha^1-\alpha^5\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}4\alpha^1-\alpha^2-\alpha^3-\alpha^4-\alpha^5\\ -(\alpha^1-\alpha^2)\\ \alpha^3-\alpha^1\\ \alpha^4-\alpha^1\\ -(\alpha^1-\alpha^5)\end{pmatrix}\\
&= -\left[(\alpha_2-2\alpha_1+\alpha_3)+(\alpha_4-2\alpha_1+\alpha_5)\right]
\end{aligned}
$$

Okay, so what is this object we have constructed? Notice it is simply the finite difference Laplacian! Now, just as we considered the vertex-edge relationships, what if we consider the face-edge relationships. We can draw an analogous diagram for a square lattice as:



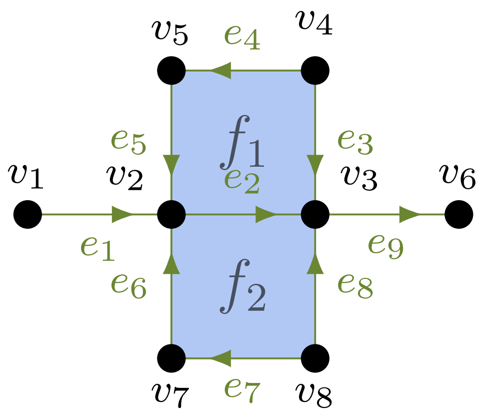
The missing face in the center is f_1. Much as we did with the vertices, let $\Delta = \gamma^i f_i$. Since there are no higher cells, $\partial^\dagger\Delta = 0$. What is the boundary of Δ? We can do this algebraically too, without matrix multiplication:

$$
\begin{aligned}
\partial\Delta &= \partial\left(\gamma^1 f_1+\gamma^2 f_2+\gamma^3 f_3+\gamma^4 f_4+\gamma^5 f_5\right)\\
&= \gamma^1\,(\partial f_1)+\gamma^2(\partial f_2)+\gamma^3(\partial f_3)+\gamma^4(\partial f_4)\\
&= \gamma^1(e_1+e_2+e_3+e_4)+\gamma^2(-e_1+\mathrm{S}_1)+\gamma^3\,(-e_2+S_2)+\gamma^4(-e_3+S_3)+\gamma^5(-e_4+S_4)\\
&= (\gamma^1-\gamma^2)e_1+(\gamma^1-\gamma^3)e_2+(\gamma^1-\gamma^4)e_3+(\gamma^1-\gamma^5)e_4+S
\end{aligned}
$$

where the S_i are edges that are *not* on in the boundary of f_1. Let $\boldsymbol{f_1}$ be the face vector with a coefficient of 1 in front of f_1 and zero elsewhere. We can make a quadratic form like before. In this case, the only entry that matters is the first, after applying the co-boundary operator to $\partial\Delta$.

$$\begin{aligned}\boldsymbol{f}_{\mathbf{1}}^T\partial^\dagger\partial\Delta &= (\gamma^1-\gamma^3)+(\gamma^1-\gamma^3)+(\gamma^1-\gamma^4)+(\gamma^1-\gamma^5)\\ &= -(\gamma^2-2\gamma^1+\gamma^3)-(\gamma^4-2\gamma^1-\gamma^5)\end{aligned}$$

and so, $\partial^\dagger\partial$ is a Laplacian on the faces. The operator $\partial\partial^\dagger$ acts on vertices yielding a finite-difference Laplacian on elements of C_0. On the other hand, $\partial^\dagger\partial$ acts on elements of C_2 in a similar way— what about edges? Consider the following figure:



Let Δ be of the form $\beta^i e_i$. Then:

$$\begin{aligned}\partial\Delta &= \beta^i(\partial e_i)\\ &= -\beta^1 v_1+(\beta^6-\beta^2+\beta^1+\beta^5)v_2+(\beta^8-\beta^9+\beta^3)v_3\\ &-(\beta^3+\beta^4)v_4+(\beta^4-\beta^5)v_5+\beta^9 v_6+(\beta^7-\beta^6)v_7-(\beta^7+\beta^8)v_8\end{aligned}$$

We can form both of the quadratic forms like last time:

$$\begin{aligned}\boldsymbol{e}_{\mathbf{2}}^\dagger\partial^\dagger\partial\Delta &= -(\beta^9-2\beta^2+\beta^1)+S\\ \boldsymbol{e}_{\mathbf{2}}^\dagger\partial\partial^\dagger\Delta &= -(\beta^4+2\beta_2+\beta^7)-S\end{aligned}$$

where now S includes contributions from other edges but, fortunately, these contributions cancel when adding the two expressions (check for yourself!). Thus, we see that we can recover something looking like a finite-difference Laplacian by adding the two components $\partial^\dagger\partial$ and $\partial\partial^\dagger$. Notice that the quadratic forms above resemble the inner product induced by a bilinear form. Eventually, we will see that the utility of this formalism is the ability to forget some of the individual properties of differential objects like div, grad, and curl, and gain a unified treatment of these operators. The sum of the two terms, $\partial^\dagger\partial$ and $\partial\partial^\dagger$, acts on (V, E, F) tuples with the previously mentioned properties (check this is true). So, we will call this new operator *the* Lattice Laplacian.

Definition 5.1 (Lattice Laplacian)
The **Lattice Laplacian** is given by:

$$-\nabla^2 := \partial^\dagger\partial + \partial\partial^\dagger \tag{11}$$

Let us recapitulate the matrix corresponding to the Laplacian:

$$[\partial] = \begin{pmatrix} 0 & \boldsymbol{D_1} & 0 \\ 0 & 0 & \boldsymbol{D_2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\partial^\dagger] = \begin{pmatrix} 0 & 0 & 0 \\ \boldsymbol{D_1^\dagger} & 0 & 0 \\ 0 & \boldsymbol{D_2^\dagger} & 0 \end{pmatrix}$$

Then,

$$[\partial\partial^\dagger] = \begin{pmatrix} \boldsymbol{D_1 D_1^\dagger} & 0 & 0 \\ 0 & \boldsymbol{D_2 D_2^\dagger} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and:

$$[\partial^\dagger\partial] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boldsymbol{D_1^\dagger D_1} & 0 \\ 0 & 0 & \boldsymbol{D_2^\dagger D_2} \end{pmatrix}$$

Then, the matrix of the Laplacian assumes the form:

$$[-\nabla^2] = \begin{pmatrix} \boldsymbol{D_1 D_1^\dagger} & 0 & 0 \\ 0 & \boldsymbol{D_1^\dagger D_1} + \boldsymbol{D_2 D_2^\dagger} & 0 \\ 0 & 0 & \boldsymbol{D_2^\dagger D_2} \end{pmatrix}$$

Check that the action of every block of this matrix makes sense on a vector of the form $(v_1, \cdots, v_V, e_1, \cdots, e_E, f_1, \cdots, f_F)^T \in W$.

Example 5.2

Continuing from example 3.2, we can calculate the Laplacian by constructing the matrices $D_1^\dagger D_1$ and $D_2 D_2^\dagger$:

$$D_1^\dagger D_1 = \begin{pmatrix} 2 & -1 & -1 & 1 & 0 & -1 \\ -1 & 2 & -1 & 0 & -1 & 1 \\ -1 & -1 & 2 & -1 & 1 & 0 \\ 1 & 0 & -1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$$D_2 D_2^\dagger = \begin{pmatrix} 2 & 1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & -1 \\ 1 & 1 & 2 & 1 & -1 & 0 \\ -1 & 0 & 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -1 & 2 & -1 \\ 1 & -1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

Check to see that $[-\nabla^2] = 4\mathbb{I}$. The remaining components of the Laplacian on W are given by:

$$D_1 D_1^\dagger = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

$$D_2^\dagger D_2 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

The first matrix is sometimes called the *vertex* Laplacian, and the second is sometimes called the *face* Laplacian.

6. Hodge Decomposition. Since, $\partial^2 = (\partial^\dagger)^2 = 0$, we can also write $-\nabla^2 = (\partial + \partial^\dagger)^2$. Now, recall some facts from linear algebra.

Proposition 6.1

Let $T : W \to W$, where T is a linear operator defined on a finite-dimensional vector space. Then, W admits the decomposition:

$$W \cong \ker(T) \oplus \mathrm{im}(T^\dagger)$$

Proof. Recall that for any (topologically closed) vector subspace $U \subset W$, we admit the direct sum decomposition of W as $W \cong U \oplus U^{\perp}$, where $U^{\perp}$ is the *orthogonal complement* of $U^{\perp}$. That is, U is the set of all vectors that have zero component in U:

$$U^{\perp} := \{t \in W \,|\, t \cdot u = 0, \quad \forall u \in U\}$$

First, suppose $U = \ker(T)$. Then, U is closed (T is linear, W is finite dimensional). Fix an arbitrary element of $\ker(T)$, ℓ. If $w \in \mathrm{im}(T^{\dagger})$, then there exists $u \in W$ such that $T^{\dagger}u = w$. The inner product between these elements becomes:

$$\begin{aligned} \ell^T w &= \ell^T T^{\dagger} u \\ &= 0 \end{aligned}$$

where we have used the definition of adjoint of T and the fact that ℓ is in the kernel. So, we have shown that $\ker(T) \perp \mathrm{im}(T^{\dagger})$, and we admit the direct sum decomposition $W \cong \ker(T) \oplus \mathrm{im}(T^{\dagger})$. □

A corollary of this proposition is:

Corollary 6.2
If $T : W \to W$ is a linear operator defined on a finite-dimensional vector space W, then W admits the direct sum decomposition:

$$W \cong \ker(T + T^{\dagger})^2 \oplus \mathrm{im}(T + T^{\dagger})^2$$

Note that $T + T^{\dagger}$ is self-adjoint. We will use that in the following.
Next, we move to prove the Hodge Decomposition theorem for the case of finite-dimensional W as described above.

Theorem 6.3 (Hodge Decomposition, 1930s)
Let ∇^2 be the Laplacian operator, $-\nabla^2 = (\partial + \partial^{\dagger})^2$. Then,

$$W \cong \ker(-\nabla^2) \oplus \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^{\dagger})$$

Proof. From the previous proposition and since $-\nabla^2$ is self-adjoint, we know:

$$W \cong \ker(-\nabla^2) \oplus \mathrm{im}(-\nabla^2)$$

To continue, we must demonstrate: $\mathrm{im}(-\nabla^2) = \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^{\dagger})$. W can also be decomposed as a direct sum $W \cong \ker(\partial + \partial^{\dagger}) \oplus \mathrm{im}(\partial + \partial^{\dagger})$. We proceed in two lemmas: first to prove $\mathrm{im}(\partial + \partial^{\dagger}) = \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^{\dagger})$ and then to show $\ker(\partial + \partial^{\dagger}) = \ker(\partial + \partial^{\dagger})^2$.

Lemma 6.4

$$\mathrm{im}(\partial + \partial^\dagger) = \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^\dagger)$$

Suppose that $u \in \mathrm{im}(\partial)$ and $v \in \mathrm{im}(\partial^\dagger)$. Then, there exists s, t respectively such that $\partial s = u$ and $\partial^\dagger t = v$. Then, the inner product of u and v is:

$$\begin{aligned} u \cdot v &= s^T \partial^\dagger \partial^\dagger t \\ &= 0 \end{aligned}$$

so, $\mathrm{im}(\partial)$ and $\mathrm{im}(\partial^\dagger)$ are orthogonal subspaces. Since they are perpendicular, if $u \in \mathrm{im}(\partial + \partial^\dagger)$ then it must have components in the union of the two images, so $\mathrm{im}(\partial + \partial^\dagger) \subseteq \mathrm{im}\partial \cup \mathrm{im}\partial^\dagger$.
Now consider the complement $S^\perp$ in W of $S = \mathrm{im}\partial \cup \mathrm{im}\partial^\dagger$. We know that $S^\perp = \ker\partial \cap \ker\partial^\dagger$. Consider $t \in S^\perp$. We have

$$0 = (\partial t + \partial^\dagger t)^T u = t^T (\partial + \partial^\dagger) u$$

so $S^\perp \perp \mathrm{im}(\partial + \partial^\dagger)$ and $\mathrm{im}\partial \cup \mathrm{im}\partial^\dagger \subseteq \mathrm{im}(\partial + \partial^\dagger)$, proving the Lemma.

Lemma 6.5
To complete the proof, we can write:

$$\ker(\partial + \partial^\dagger) = \ker(\partial + \partial^\dagger)^2$$

Suppose $u \in \ker(\partial + \partial^\dagger)$, then $u \in \ker(\partial + \partial^\dagger)^2$. Alternatively, suppose $u \in \ker((\partial + \partial^\dagger)^2)$. If $(\partial + \partial^\dagger)^2 u = 0$, then $|(\partial + \partial^\dagger)u| = 0$, so $u \in \ker((\partial + \partial^\dagger))$. □

Thus, every vector $v \in V$ can be written as the sum:

$$v = \partial\alpha + \partial^\dagger\beta + \gamma$$

where γ satisfies $\nabla^2\gamma = 0$. Such functions γ are called *harmonic*. In three-dimensions this is called the *Helmholtz-Hodge decomposition*.