1.1

stationary state: $0 = \nabla^2 c$

spherical coordinates: $0 = \partial_r (r^2 \partial_r c)$

$$\Rightarrow \quad \partial_r c = \frac{a}{r^2}$$

$$\Rightarrow \quad c = b - \frac{a}{r}$$

BCs: $c(r \to \infty) = c_\infty \longrightarrow b = c_\infty$

$c(r=R) = c^{eq}_{out} \Longrightarrow c^{eq}_{out} = c_\infty - \frac{a}{R}$

$$\Rightarrow a = R\left(c_\infty - c^{eq}_{out}\right)$$

$$\Rightarrow c(r) = c_\infty + \left(c^{eq}_{out} - c_\infty\right)\frac{R}{r}$$

$$j = -D c'(r) = D\left(c^{eq}_{out} - c_\infty\right)\frac{R}{r^2}$$

$$J_{out} = 4\pi R^2 j(R) = 4\pi R D\left(c^{eq}_{out} - c_\infty\right)$$

6.2

stationary state:

$$0 = \frac{D}{r^2}\,\partial_r\left(r^2 c'(r)\right) + k\left(c_0 - c(r)\right) \qquad \Big|\cdot \frac{r^2}{D}$$

$$0 = \partial_r\left(r^2 c'(r)\right) + \frac{r^2}{\lambda^2}\left(c_0 - c(r)\right) \qquad \lambda^2 = \frac{D}{k}$$

$$f(r) = c_0 - c(r) \qquad c'(r) = -f'(r)$$

$$2r f'(r) + r^2 f''(r) = \frac{r^2}{\lambda^2} f(r)$$

$$\partial_r\left(r^2 f'(r)\right) = \frac{r^2}{\lambda^2} f(r)$$

$$f(r) = \frac{g(r)}{r} \implies f'(r) = g(r) + r g'(r)$$

$$\partial_r\left(r g'(r) - g(r)\right) = \frac{r}{\lambda^2} g(r)$$

$$\cancel{g'(r)} + r g''(r) - \cancel{g'(r)} = \frac{r}{\lambda^2} g(r)$$

$$\implies g(r) = a e^{r/\lambda} + b e^{-r/\lambda}$$

BCs: $c(r\to\infty)$ is finite $\Rightarrow a = 0$

$$c(r=R) = c_{out}^{eq} = c_0 + b\,\frac{e^{-R/\lambda}}{R}$$

$$\Rightarrow b = R\left(c_{out}^{eq} - c_0\right) e^{R/\lambda}$$

$$\Rightarrow c(r) = c_0 + \left(c_{out}^{eq} - c_0\right)\frac{R}{r}\, e^{\frac{R-r}{\lambda}}$$

$$\Rightarrow c(r\to\infty) = c_0 \quad \checkmark$$

$$j = -D\,\partial_r c = D\left(c_{out}^{eq} - c_0\right)\frac{R(r+\lambda)}{r^2\lambda}\, e^{\frac{R-r}{\lambda}}$$

$$\Rightarrow J = 4\pi R^2 j(R) = 4\pi R D\left(c_{out}^{eq} - c_0\right)\left(1 + \frac{R}{\lambda}\right)$$

$\rightarrow$ Reactions negligible if $R \ll \lambda$

1.3

Stationary state: $0 = \Gamma_{in} \frac{\cancel{4\pi}}{3} R^3 - \cancel{4\pi} R D (c_{out}^{eq} - c_\infty)$

$$\Rightarrow 0 = \frac{\Gamma_{in}}{3} R^3 + D(c_\infty - c_{out}^{(0)}) R - \frac{D c_{out}^{(0)}}{\ell}$$

$$0 = \underbrace{\frac{\Gamma_{in}}{3}}_{a} R^3 + \underbrace{D \frac{\Gamma_{out}}{h_{out}}}_{b} R - \underbrace{\frac{D c_{out}^{(0)}}{\ell}}_{c}$$

Limit $R \ll \ell$: $a \simeq 0$

$$\Rightarrow R_* = \frac{c}{b} = \ell \frac{c_{out}^{(0)} h_{out}}{\Gamma_{out}}$$

Limit $R \gg \ell$: $c \approx 0$

$$\Rightarrow R_* \approx \sqrt{-\frac{3D \Gamma_{out}}{h_{out} \Gamma_{in}}}$$

Bonus : $R = R_* + \varepsilon$

$$0 = a \cancel{R_*^3} + 3a R_*^2 \varepsilon + b(\cancel{R_*} + \varepsilon) - c$$

since $a R_*^3 + b R_* = 0 \Rightarrow 3a R_*^2 = -3b$

$$\Rightarrow 0 = -2b\varepsilon - c \Rightarrow \varepsilon = -\frac{c}{2b}$$

$$\Rightarrow R_* \approx \left(-\frac{3D \Gamma_{out}}{h_{out} \Gamma_{out}} \right)^{1/2} - \frac{\ell h_{out} c_{out}^{(0)}}{2 \Gamma_{out}}$$

1.4 $\partial_t R = \underbrace{\frac{1}{\Delta c}\left[\Gamma_{in}\frac{4\pi}{3}R^3 - 4\pi R D\left(\frac{\ell}{R} - \frac{\Gamma_{out}}{h_{out}}\right)\right]}_{= f(r)}$

Stability analysis: $R = R_* + \varepsilon$

$\Rightarrow \underbrace{\partial_t R_*}_{=0} + \partial_t \varepsilon \approx \underbrace{f(R_*)}_{=0} + \underbrace{f'(R_*)}_{\text{growth rate } \omega}\varepsilon$

$\Rightarrow$ Check sign of $f'(R)$

$f'(R) \propto 3aR^2 + b$

Limit $R \ll \ell$: $a \approx 0 \Rightarrow$ sign given by b

$b = D\frac{\Gamma_{out}}{h_{out}} < 0$ if $\Gamma_{out} < 0$

$\Rightarrow$ small radius stable for internally-maintained droplets

Limit $R \gg \ell$: $3aR_*^2 = -3b$

$\Rightarrow f'(R) \propto -2b < 0$ if $\Gamma_{out} > 0$

$\Rightarrow$ large radius stable for externally-maintained droplets