

Classification by thermodynamic variable.

Pure LC fluids: *thermotropic*. Meaning *temperature* is the important thermodynamic variable governing phase behavior.

LCDs are mixtures of liquid crystals, but have no solvent.

LC solutions (polymer LC, colloidal LC, amphiphile LC): *lyotropic*. Meaning *concentration* is the important thermodynamic variable governing phase behavior.

Appearance of nematic liquid crystal.



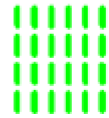
Cholesterol melts from a solid to a turbid liquid at 145 °C.

Turbid liquid turns to a clear liquid at 178 °C.

Turbid liquid is cholesteric (chiral nematic) liquid crystal.

Hybrid molecular form and molecular frustration

octane



crystal



-91 °C

isotropic liquid

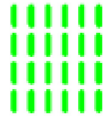
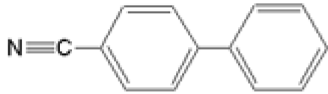


+98 °C

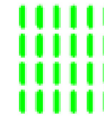


gas

4-cyanobiphenyl

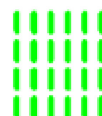


crystal



88 °C

8CB: octane + 4-cyanobiphenyl



crystal Solid



21.5 °C



Smectic A



33.5 °C



Nematic



40.5 °C

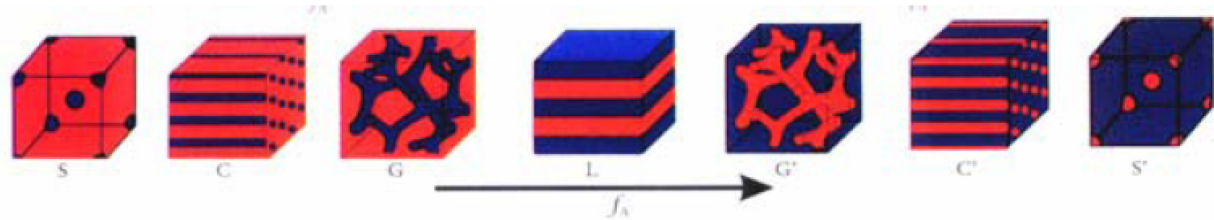
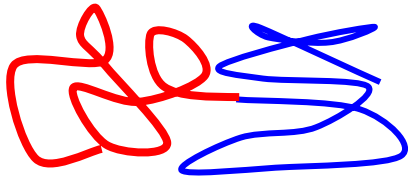


Isotropic

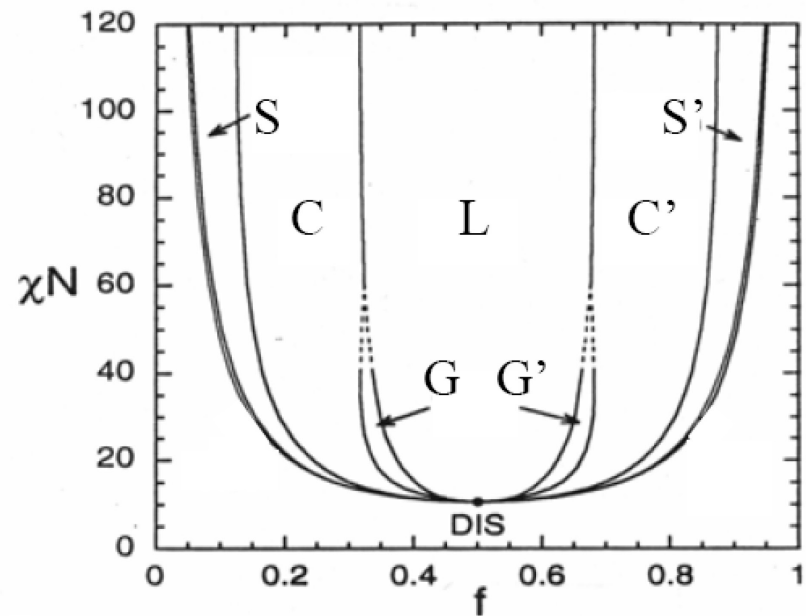
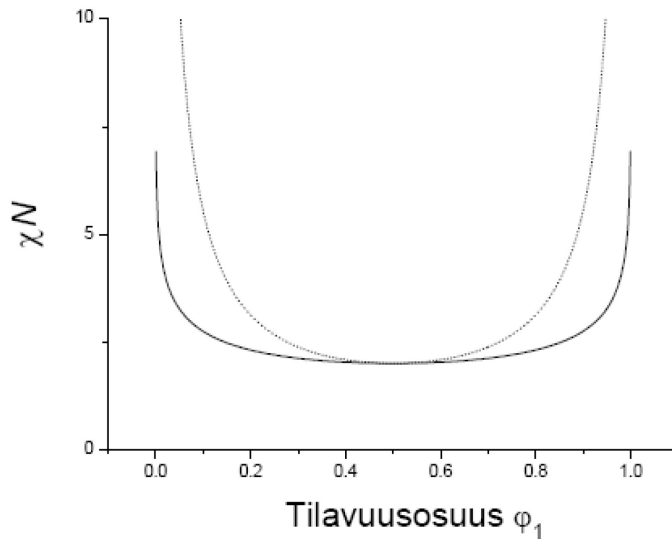
Low molecular weight LCs are *thermotropics*. Meaning no solvent, only liquid crystal.
Commercial LCs are mixtures.

Mean-Field Diblock Phase Diagram

(Matsen & Bates 95, Fredrickson & Bates)



Vertaa: Flory-Huggins theory for polymer blends (symmetric case $N_1 = N_2 = N$)



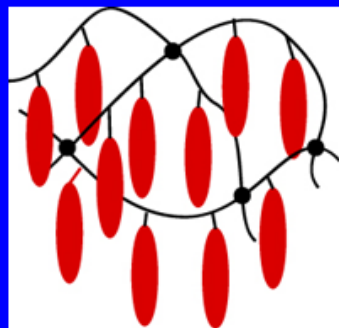
huom! lohkopolymeereillä $N = N_1 + N_2$

Most block co-polymers are melts
but solvent can be added without bulk phase separation

Examples of LC Gels

lyotropic

1. Liquid Crystal Elastomers - Weakly crosslinked liquid crystal polymers



Nematic



Smectic-C

2. Tanaka gels with hard-rod dispersion

3. Anisotropic membranes

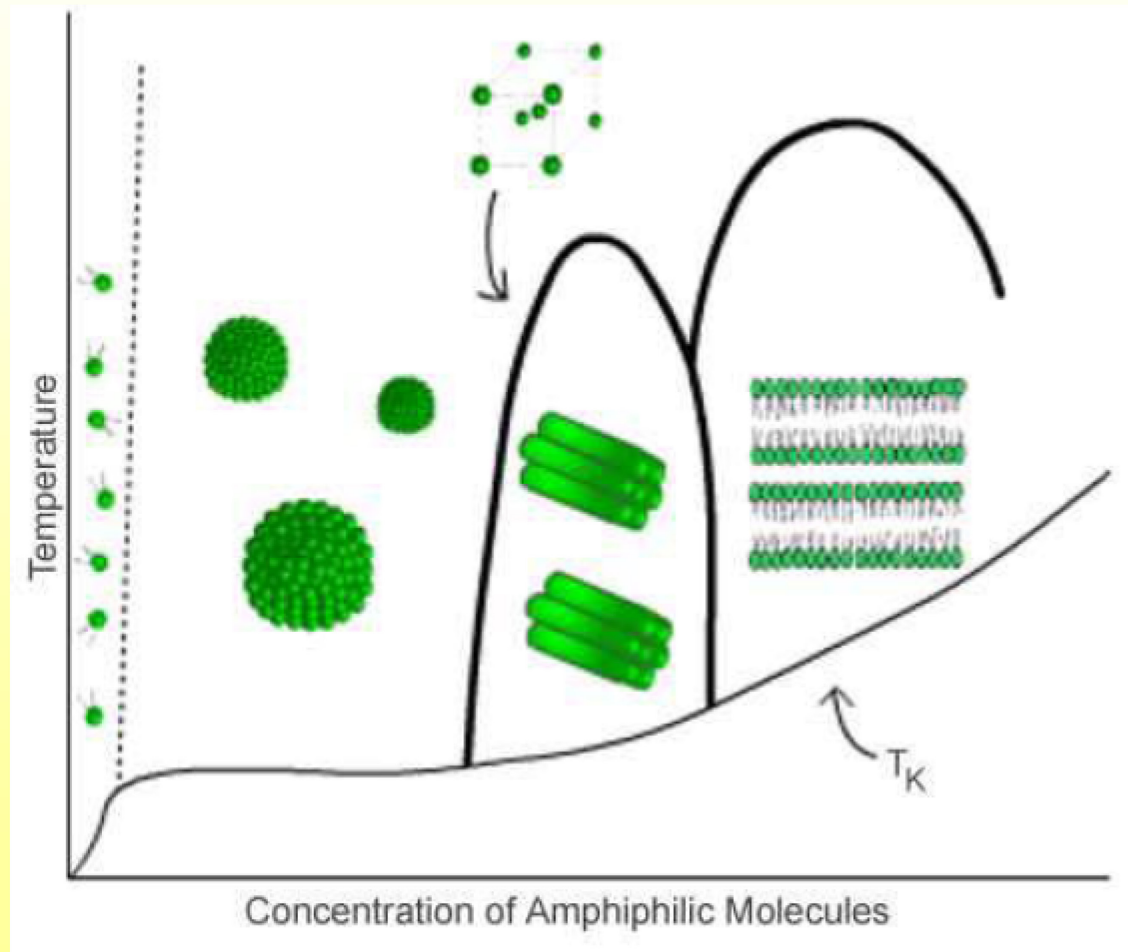
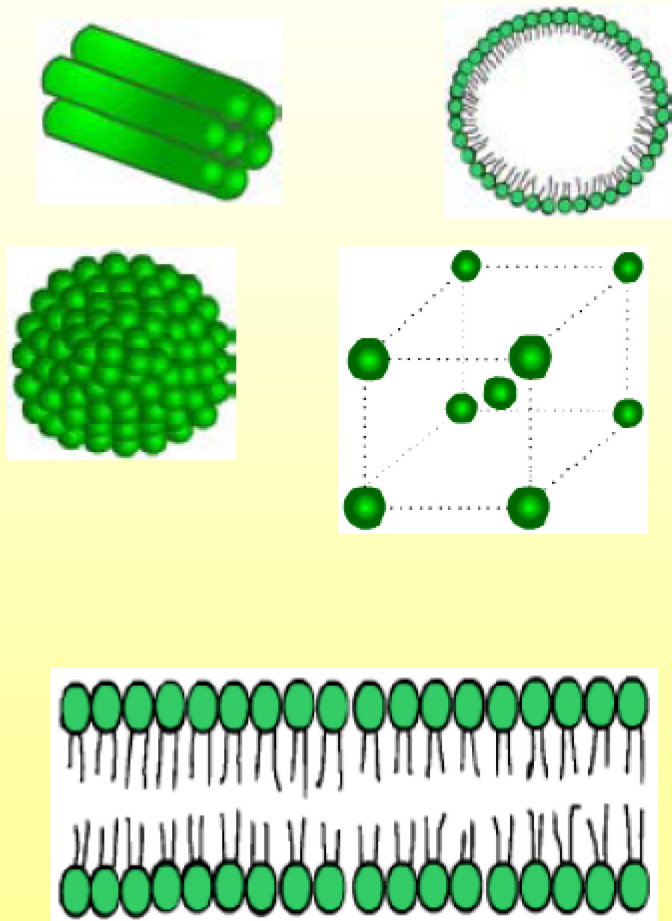


4. Glasses with orientational order



Lyotropic Liquid Crystals

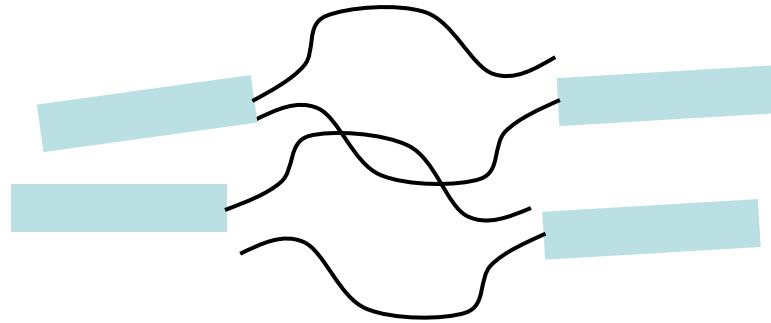
Lyotropic liquid crystals are multi component systems formed in mixtures of amphiphilic molecules and a polar solvent. Amphiphilic molecules are consisted of a hydrophilic polar head attached to a hydrophobic hydrocarbon tail containing one or two alkyl chains.



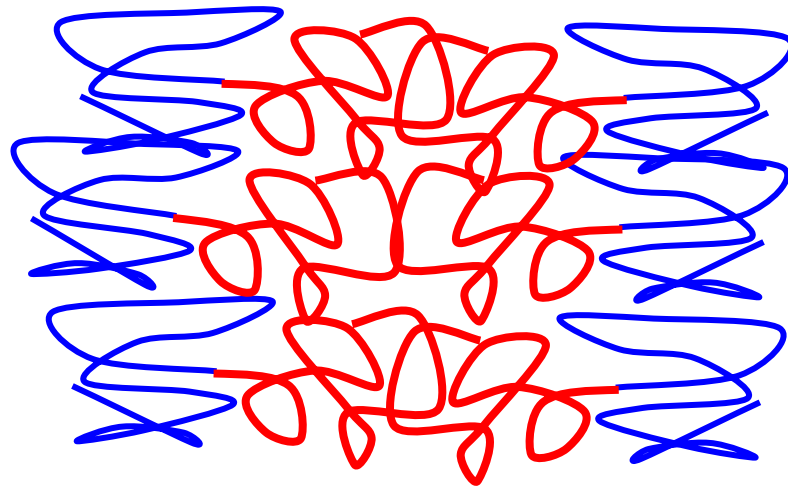
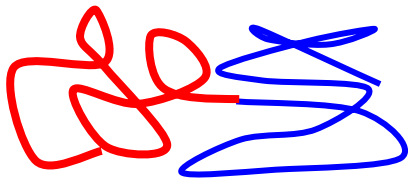
Big point.

Mesophase occurrence often caused by same reason: molecular frustration

Thermotropic LC



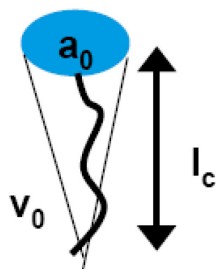
Block co-polymer



Mesophase occurrence often caused by same reason: molecular frustration

Lyotropic liquid crystals

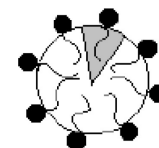
$$k = \frac{v_0}{a_0 l_c}$$



$$k < 1/3$$



Cone

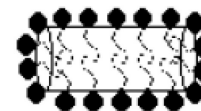


Spheres

$$1/3 < k < 1/2$$



Truncated
Cone

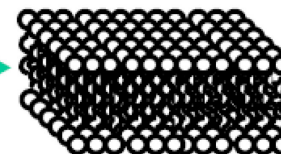


Rods

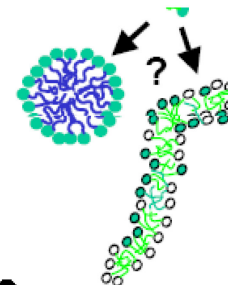
$$k > 1/2$$



Cylinder



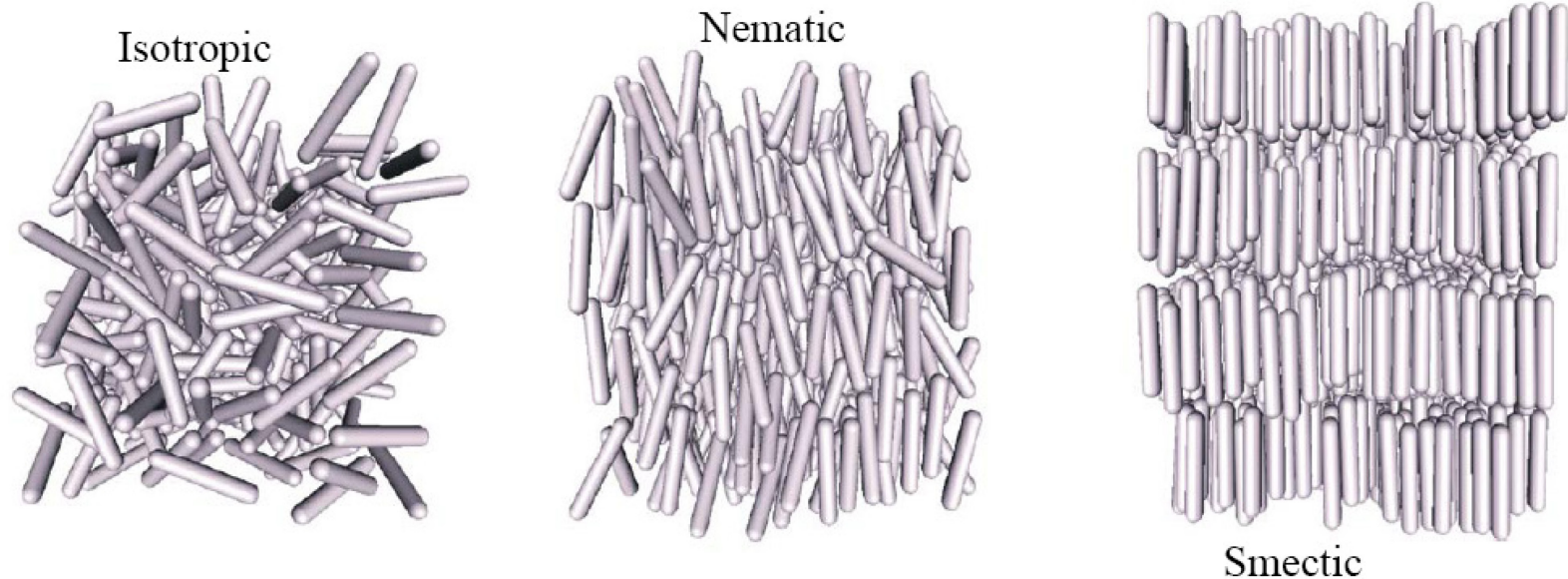
Bilayers



Mesophase occurrence often caused by same reason: molecular frustration



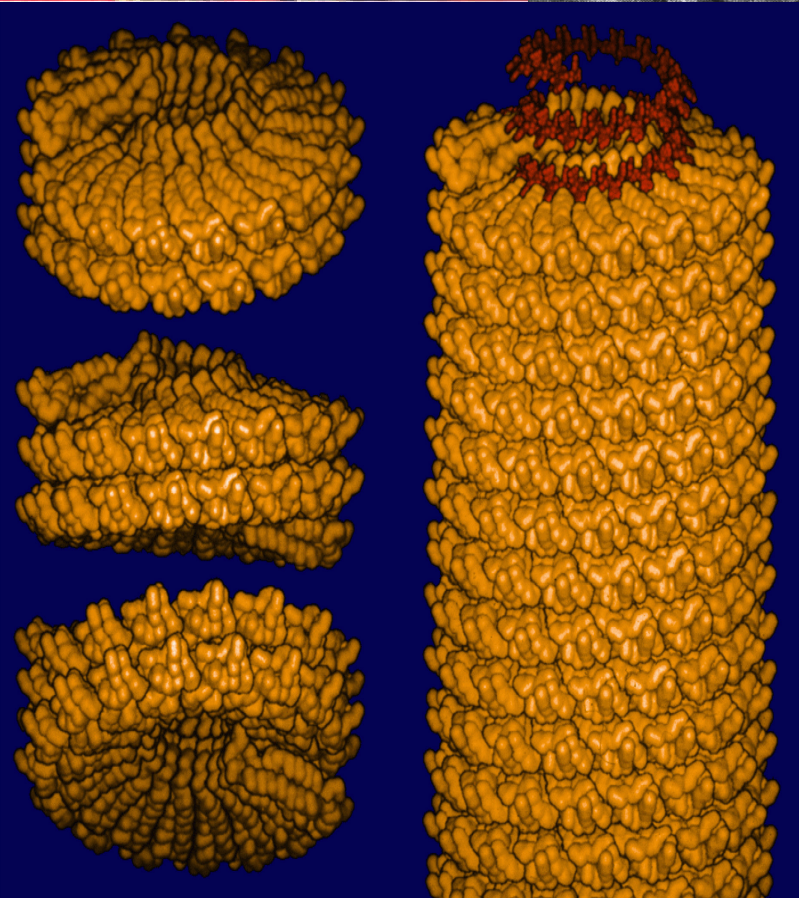
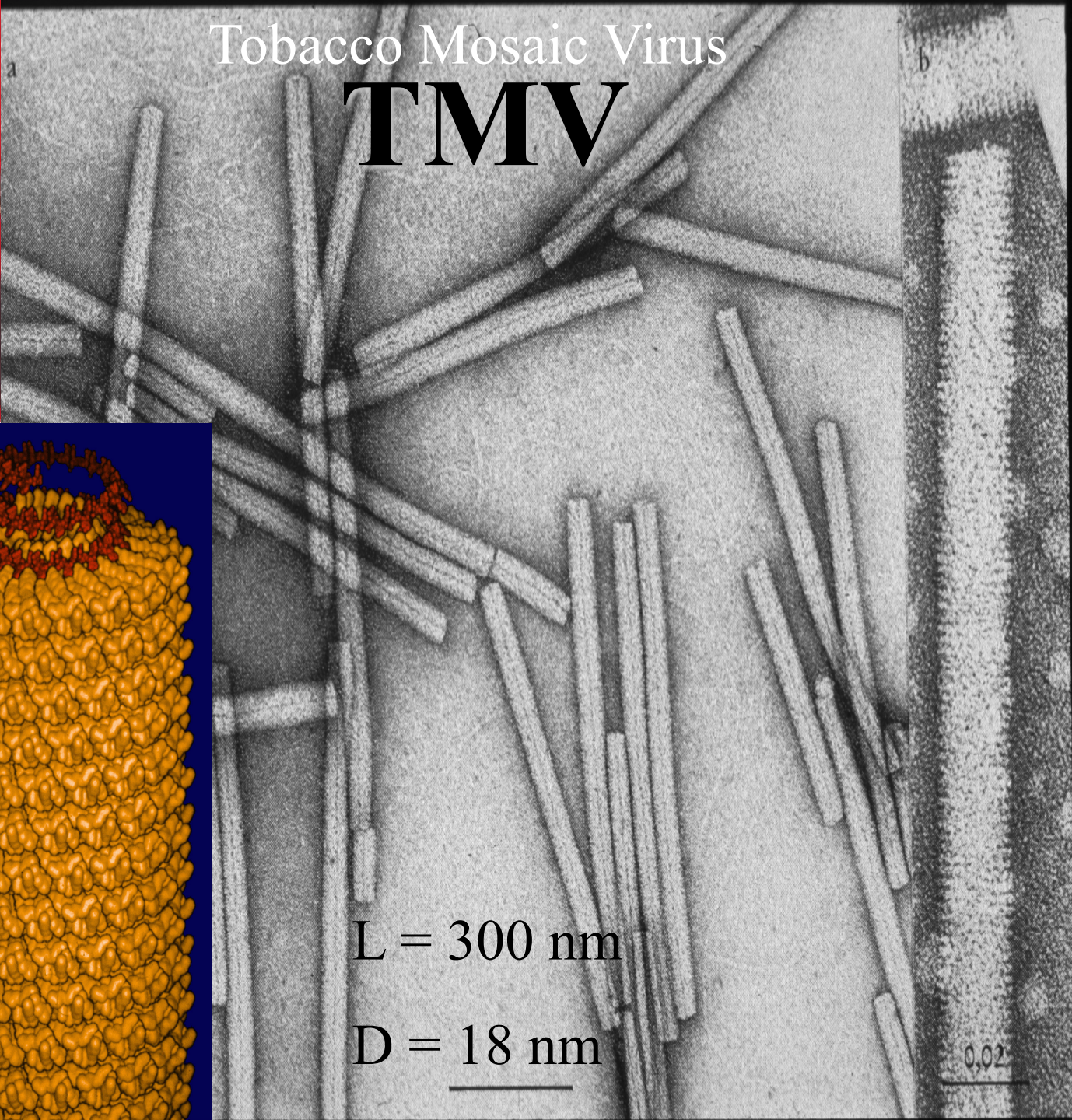
Hard rods. Perfect lyotropic. Onsager model.





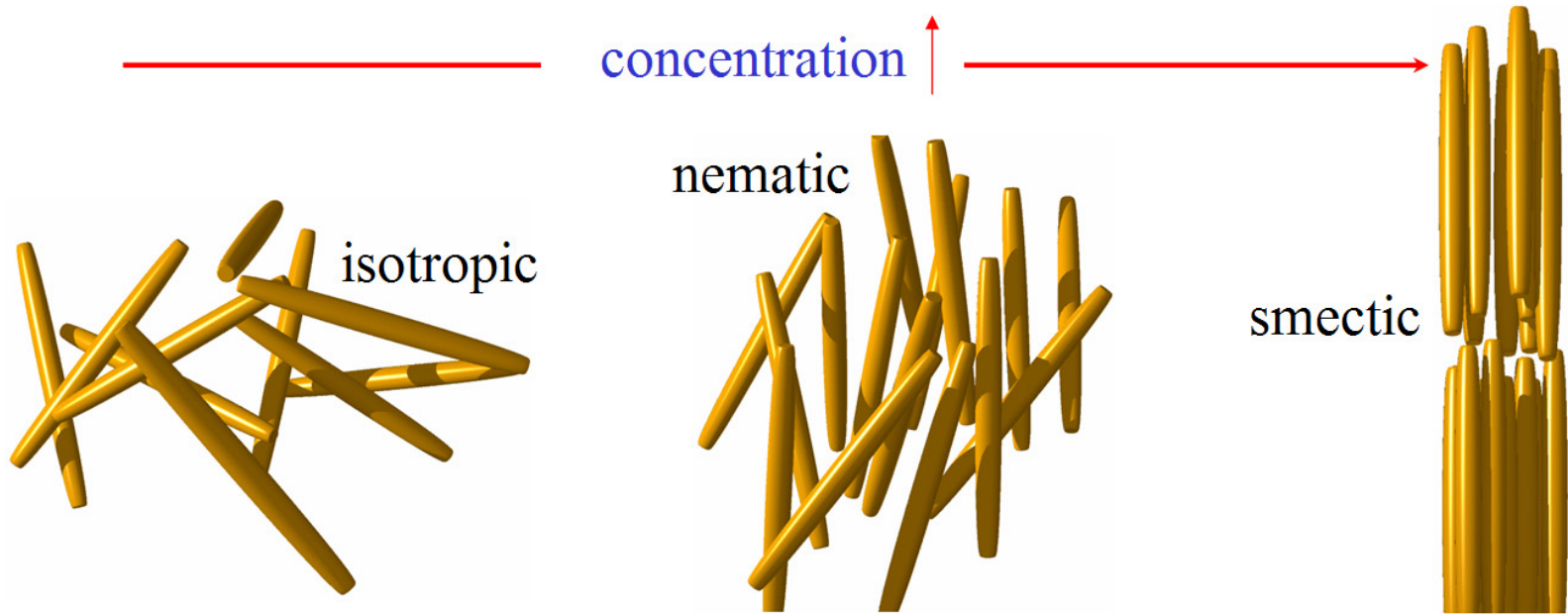
Tobacco Mosaic Virus

TMV



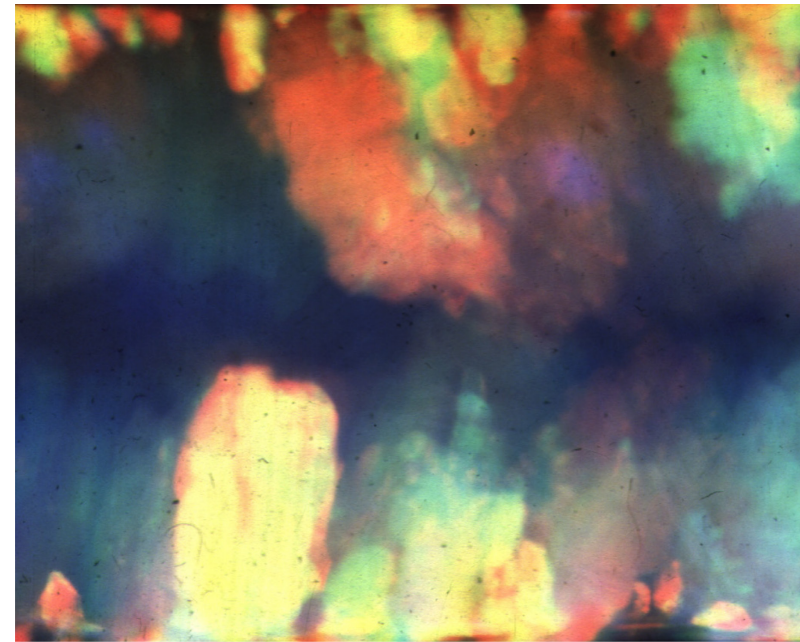
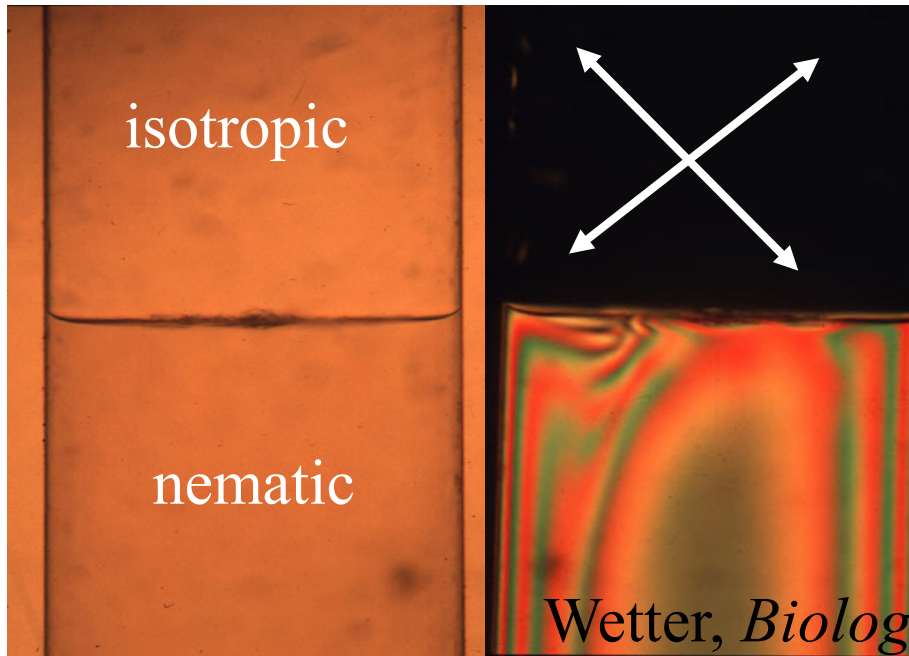
$L = 300 \text{ nm}$

$D = 18 \text{ nm}$



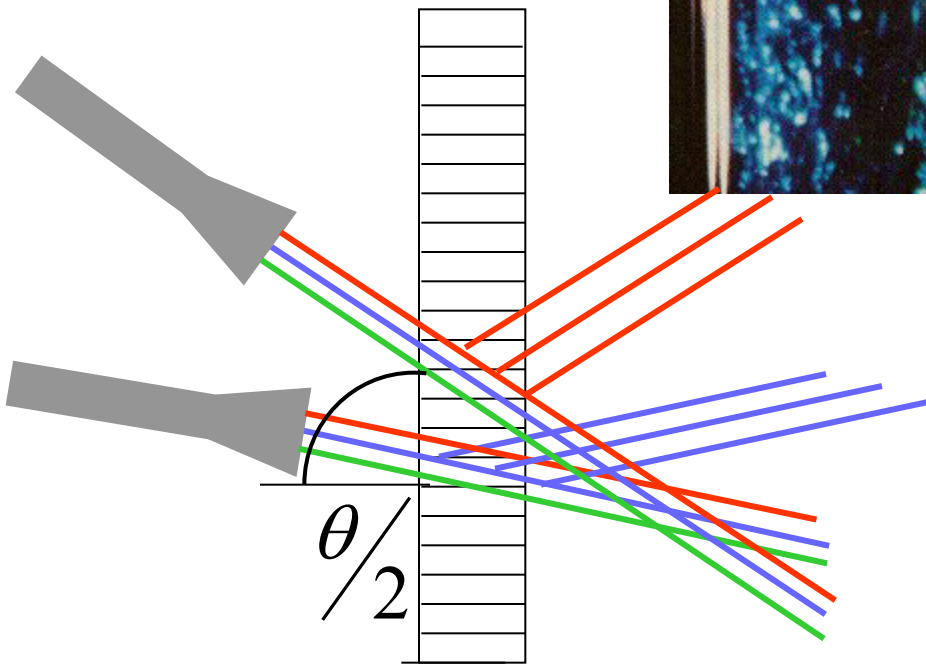
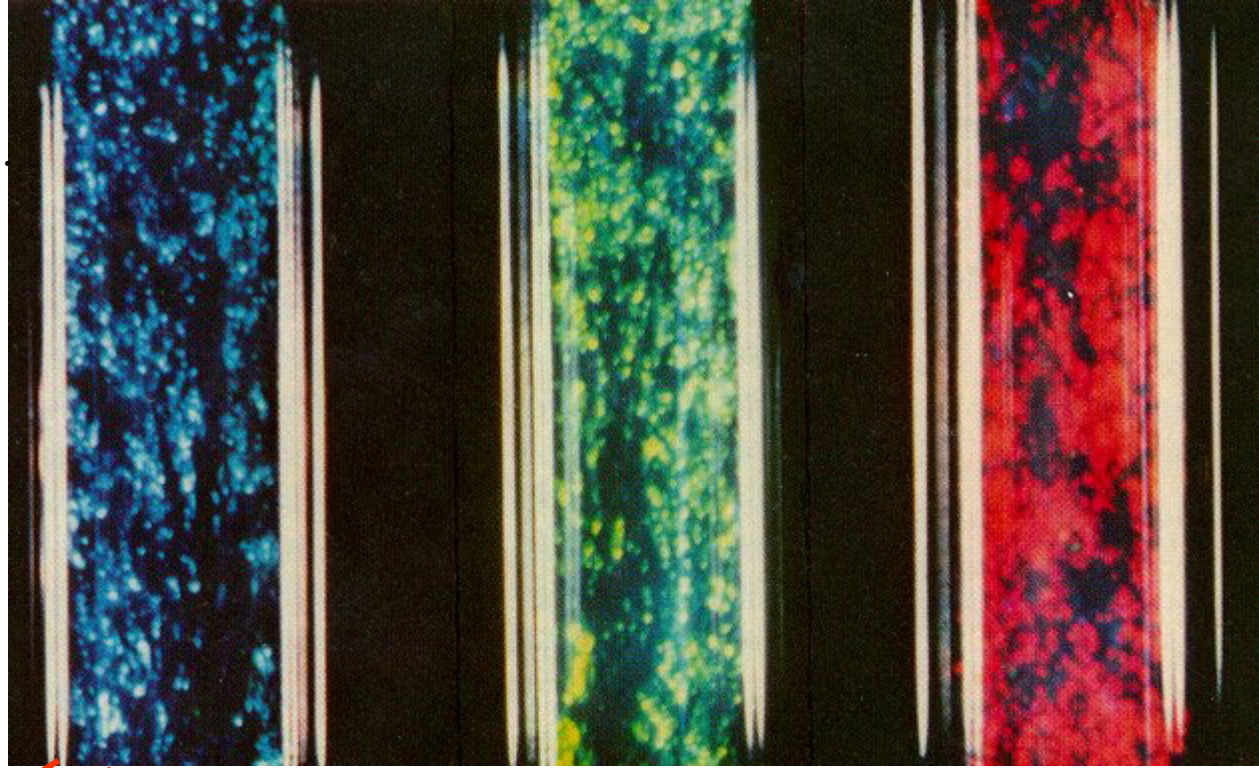
white light

crossed polarizers



Wetter, *Biologie in unserer Zeit* 3, 81-89 (1985)

C. Wetter
Biologie in unserer Zeit **3**, (1985).

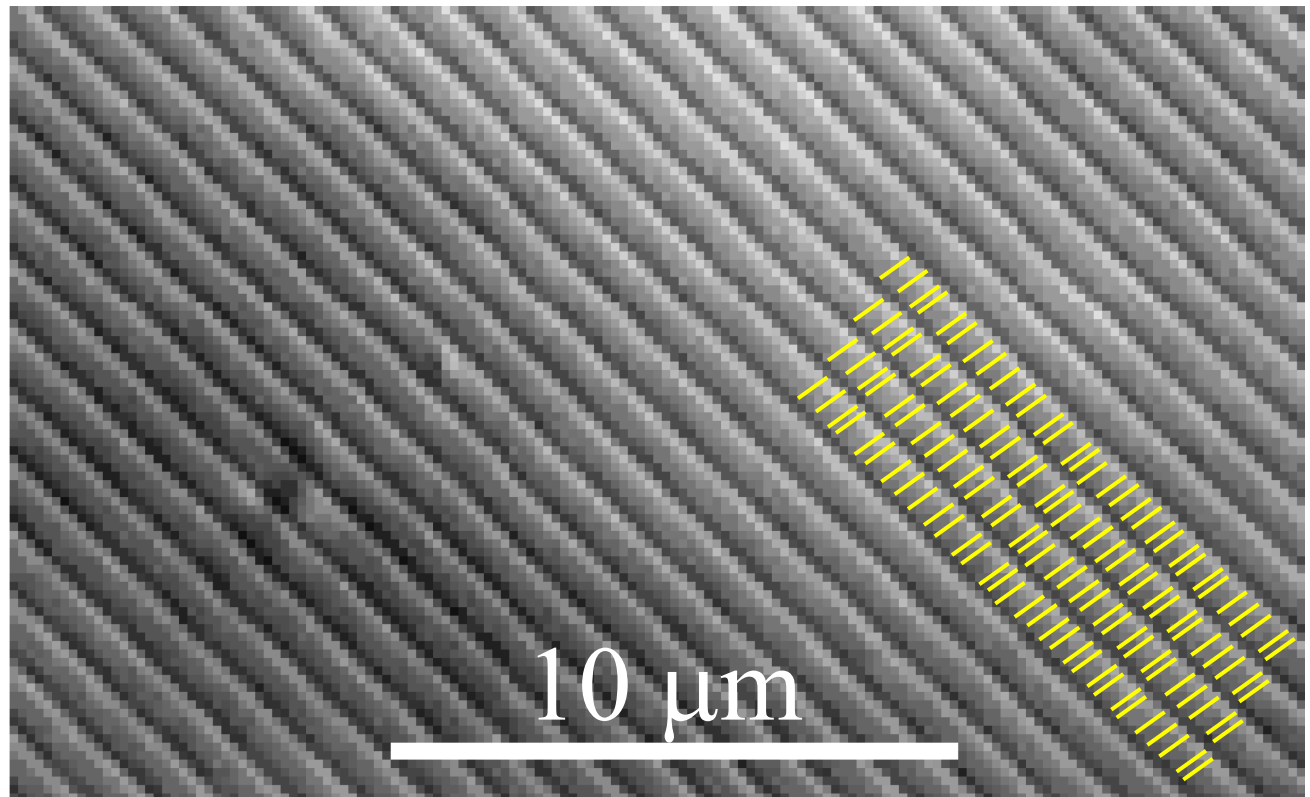
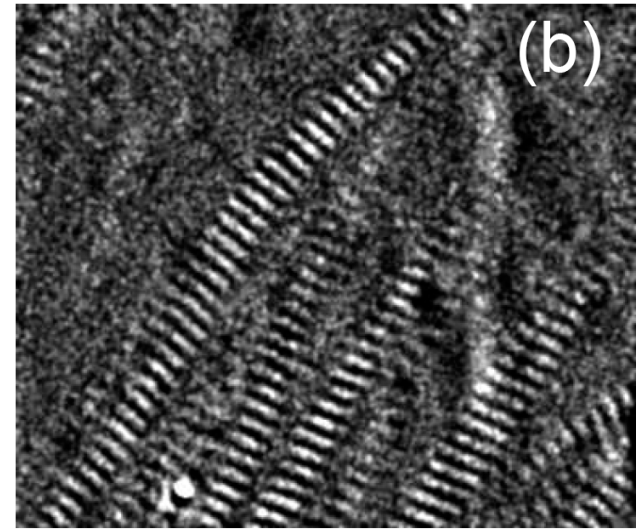
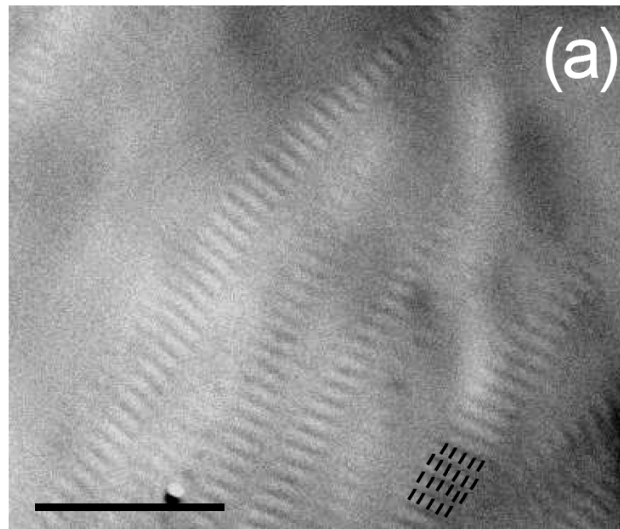


Bragg diffraction with
white light

$$2d \sin \theta = \lambda$$

Smectic Phase
of *fd* virus

co-existence of
nematic and smectic



Chaikin exaggerates when he says
real space beats Fourier space.

Real space (microscopy) is unparalleled for study of local structure:
defects, microstructure, single molecule dynamics,
high frequency spatial modulations

Reciprocal, or Fourier space is best for 3D averaged structure:
order parameters, symmetry, phase transitions, hydrodynamic modes,
high frequency spatial (and temporal) modulations

Soft matter is best studied experimentally
using

microscopy
PLUS
scattering

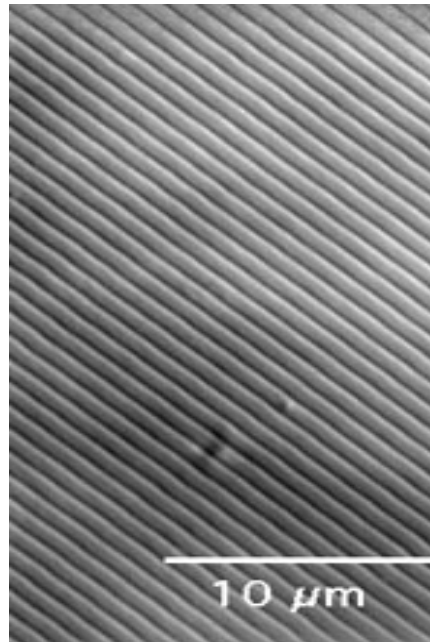
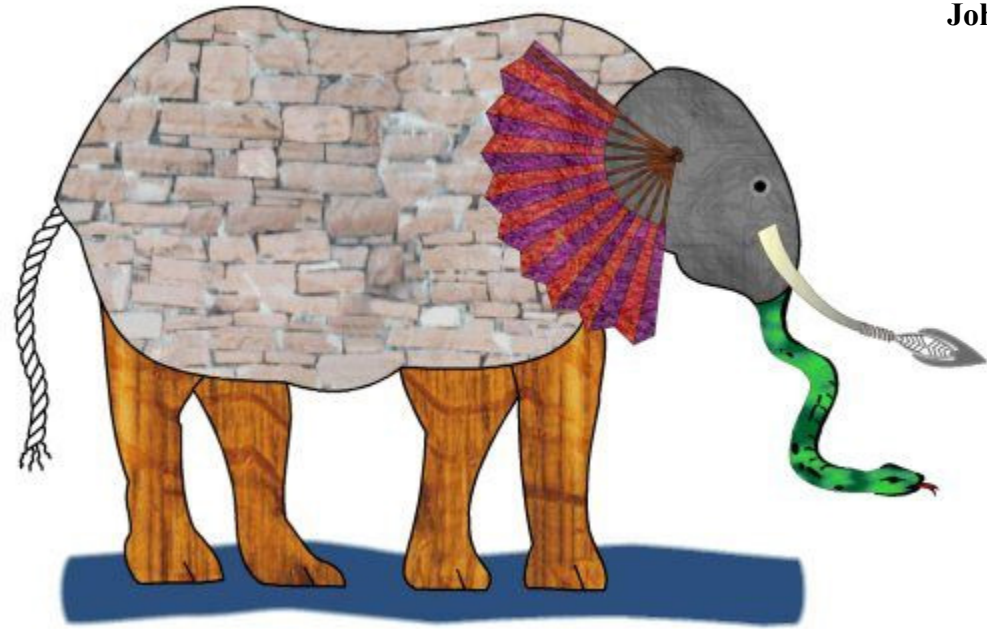
John Godfrey Saxe's (1816-1887) version of the famous Indian legend,

It was six men of Indostan,
To learning much inclined,
Who went to see the Elephant
(Though all of them were blind),
That each by observation
Might satisfy his mind.

The *First* approach'd the Elephant,
And happening to fall
Against his broad and sturdy side,
At once began to bawl:
"God bless me! but the Elephant
Is very like a wall!"

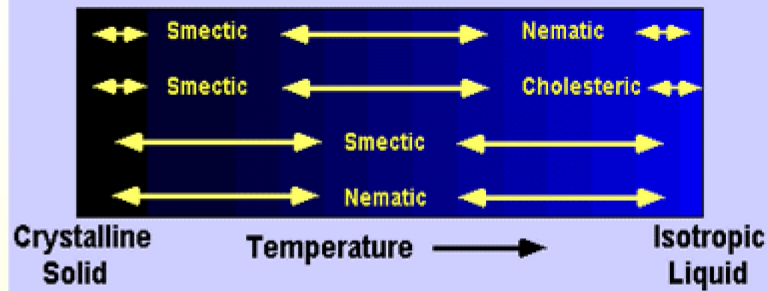
The *Second*, feeling of the tusk,
Cried, -"Ho! what have we here
So very round and smooth and sharp?
To me 'tis mighty clear,
This wonder of an Elephant
Is very like a spear!"

The *Third* approach'd the animal,
And happening to take
The squirming trunk within his hands,
Thus boldly up and spake:
"I see," -quoth he- "the Elephant
Is very like a snake!"



Phase transitions

Spectrum of Phases Exhibited by Liquid Crystals



Higher symmetry: decreasing order

I - isotropic liquid

N - nematic

N^A - chiral nematic (formally Ch for cholesteric)

A F

C G

C* H — smectics

B

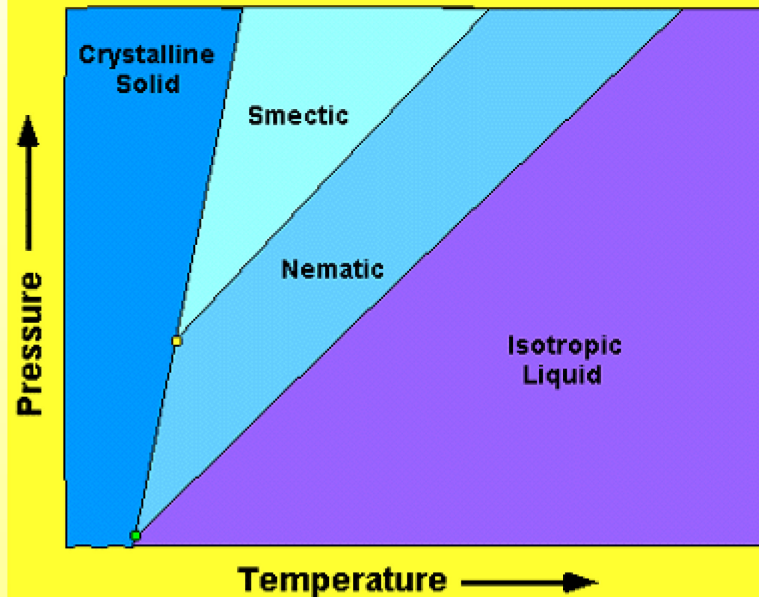
E

K - crystals

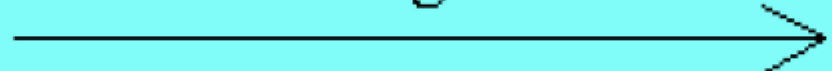
K₁ - highest temperature crystals

K₂ - second highest temperature crystals

Intermediate Phase Region for a Liquid Crystal

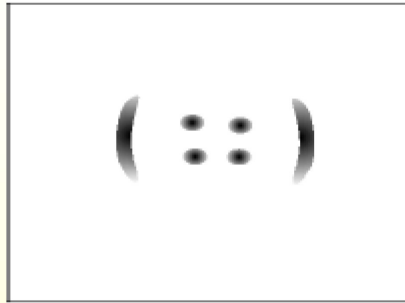


K-H-S_K-E-G-J-L-F-S_I-B-C-A-N-I
decreasing order

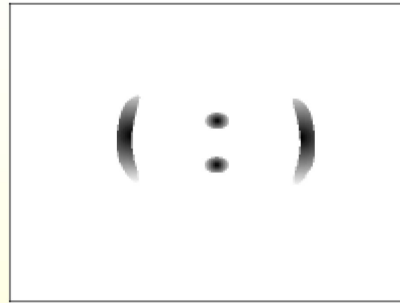


Characterization, SAXS

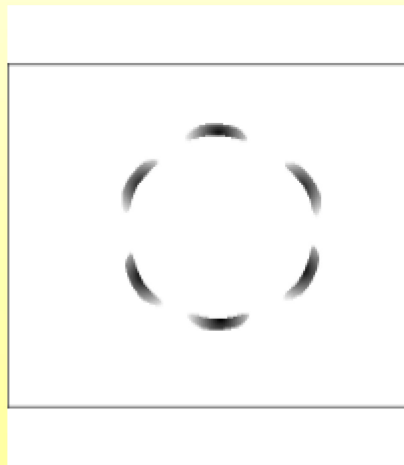
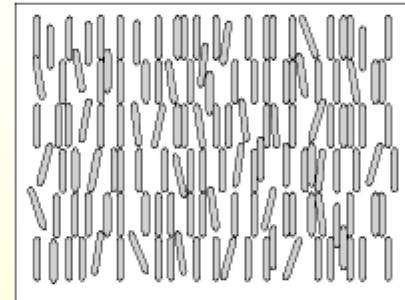
Small angle X-ray scattering



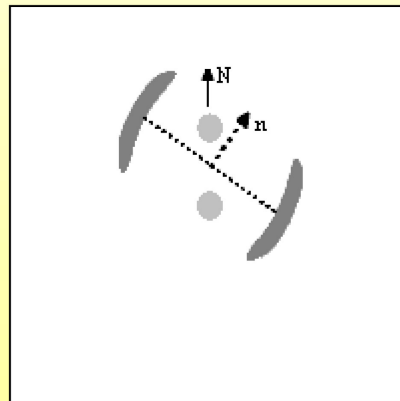
nematic



Smectic A



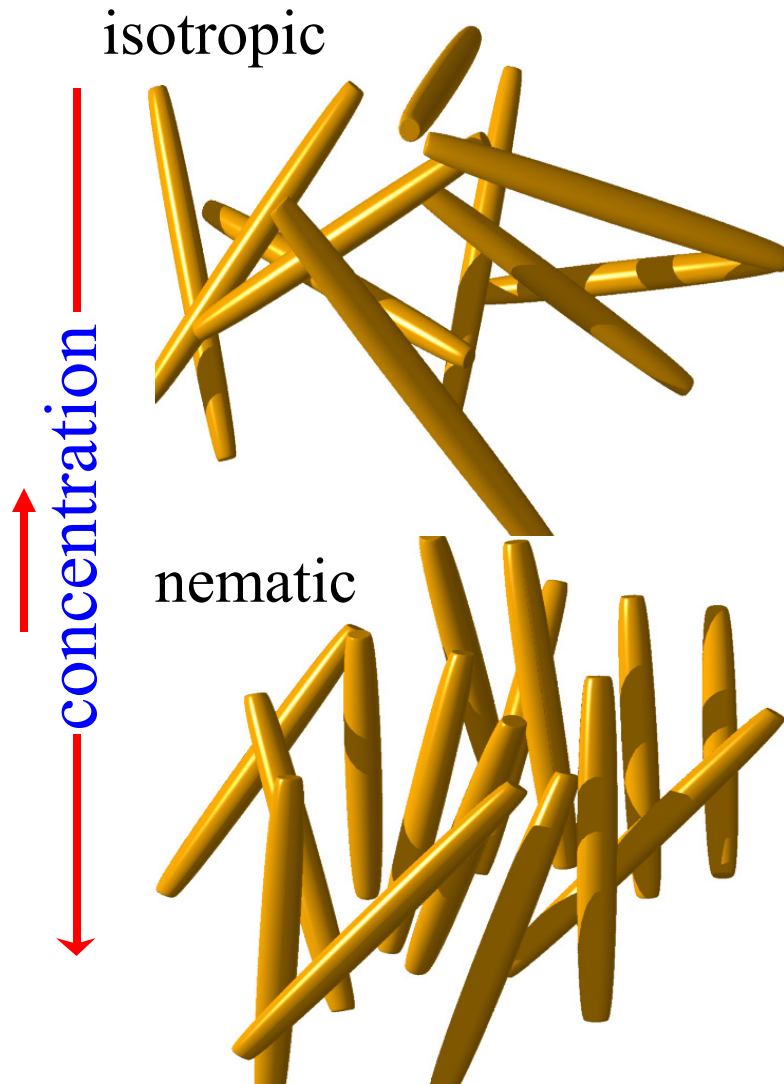
hexatic



Smectic C

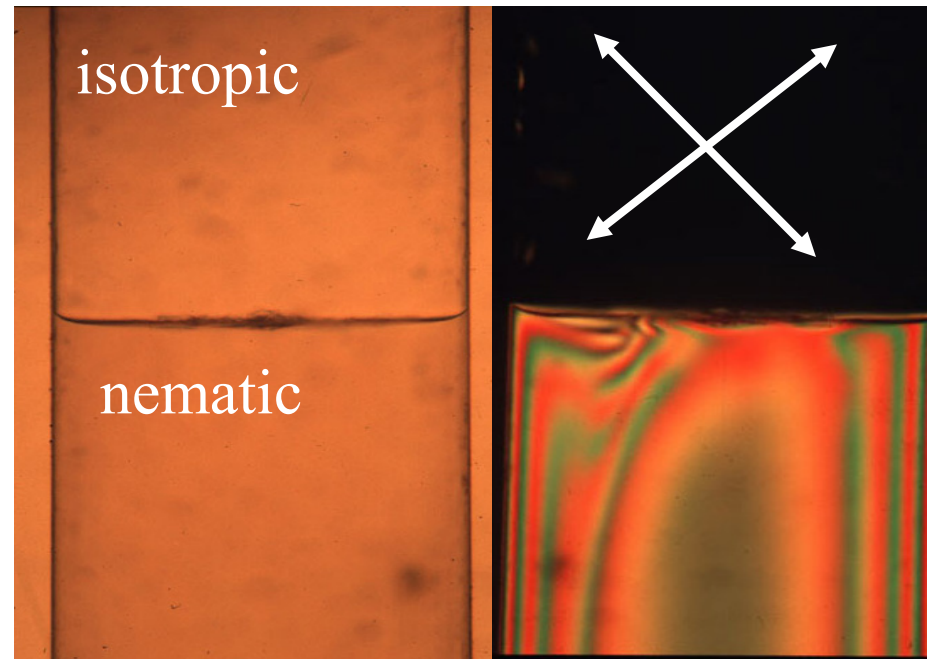


isotropic - nematic two phase coexistence

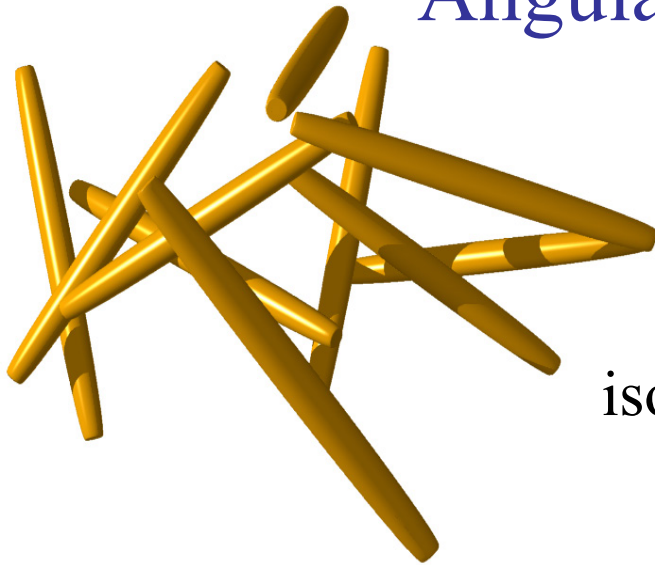


white light

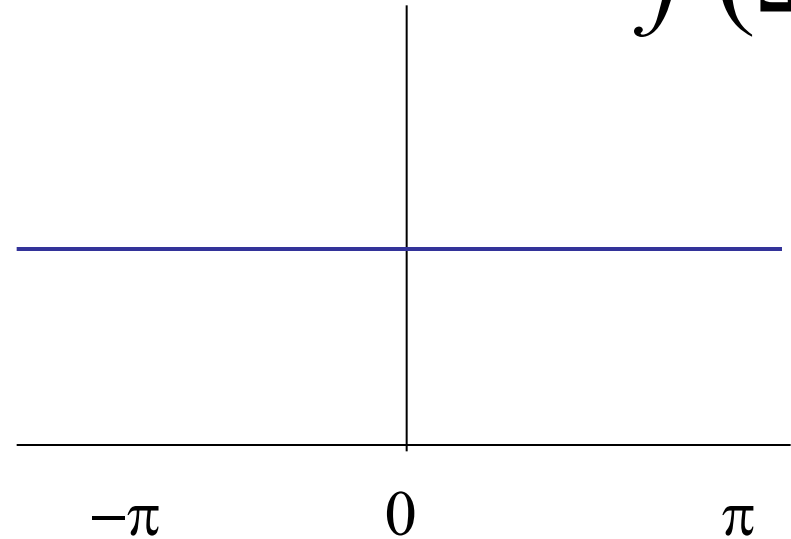
crossed polarizers



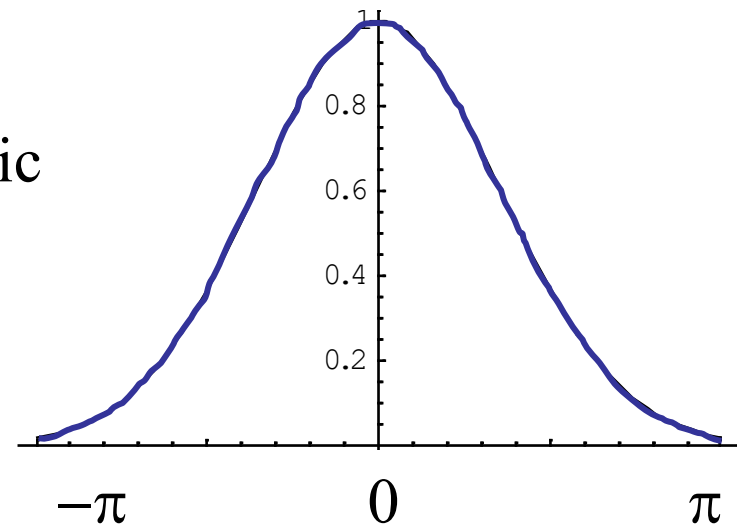
Angular distribution function $f(\Omega)$



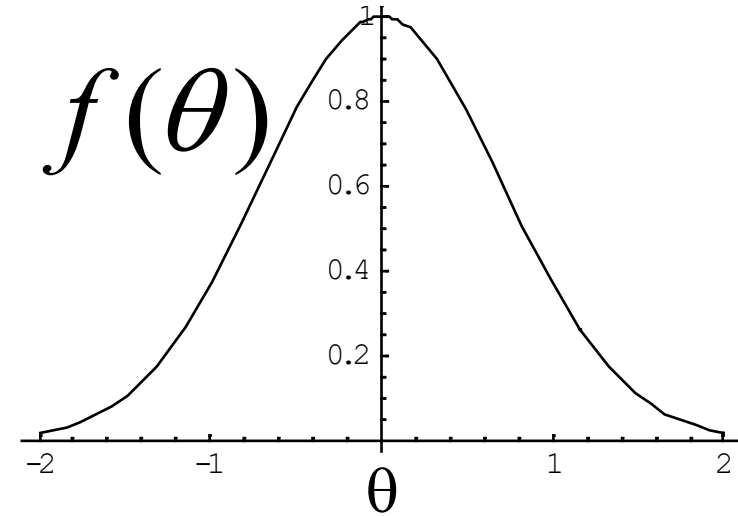
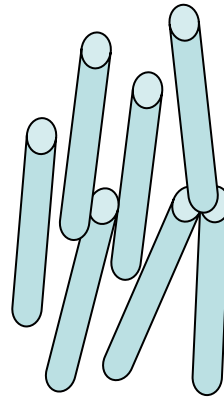
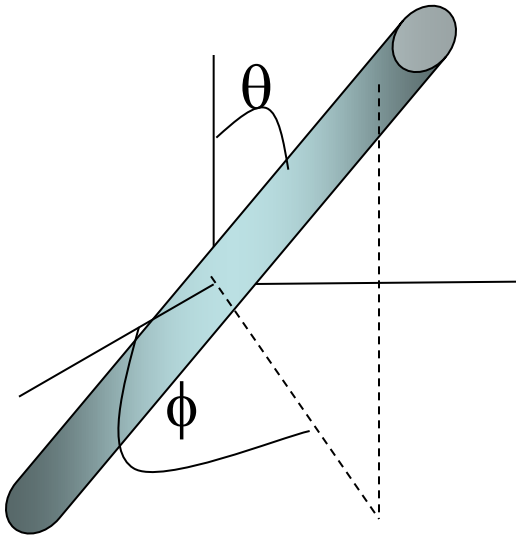
isotropic



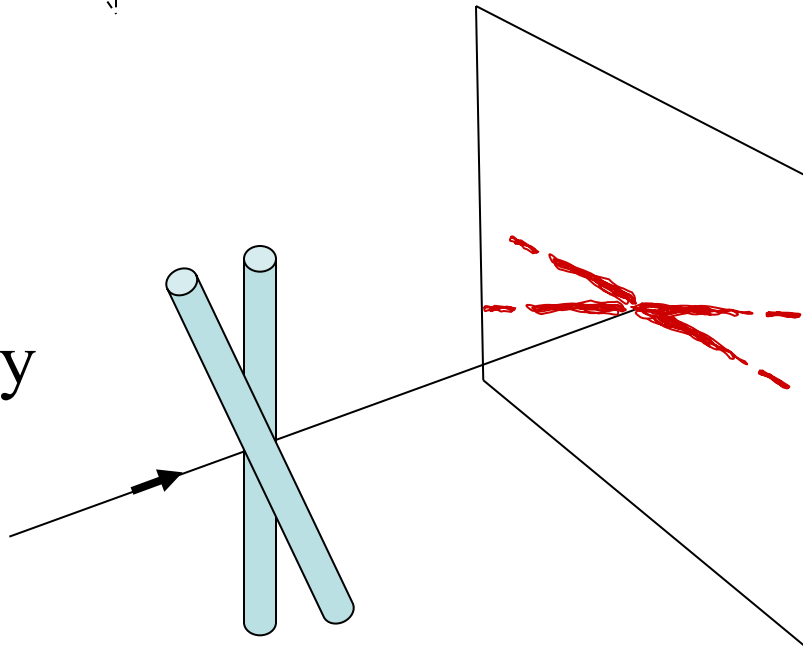
nematic



Measuring nematic order of colloidal rods by x-ray diffraction



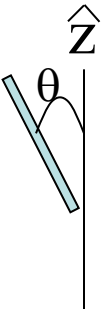
x-ray

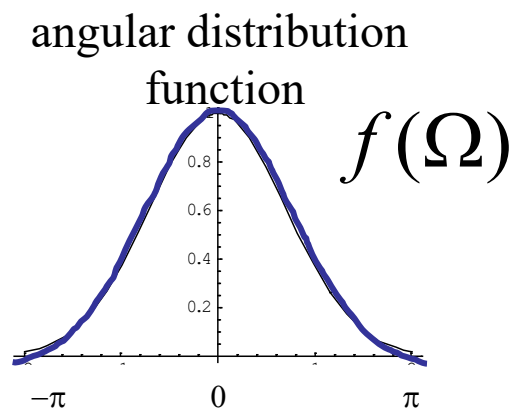
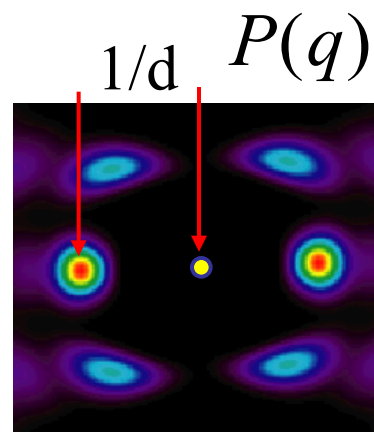
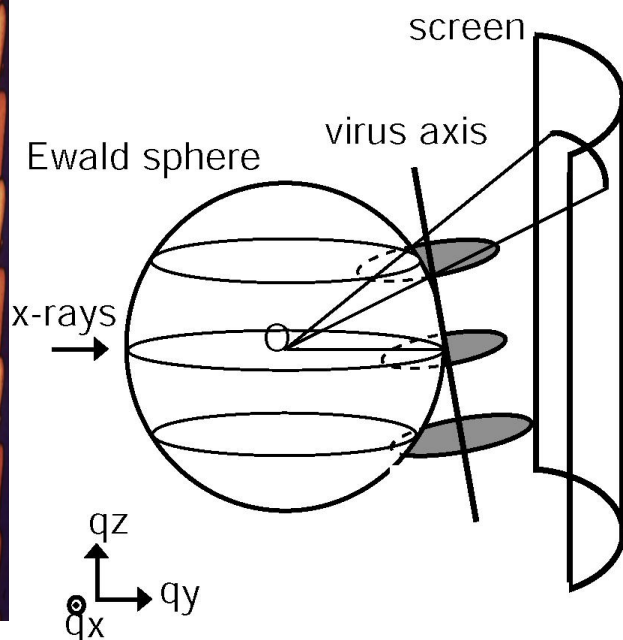
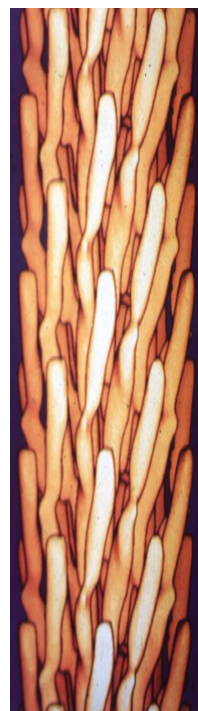
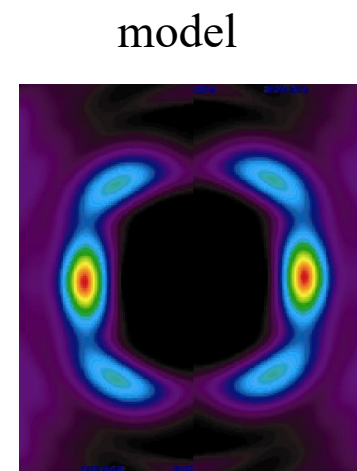
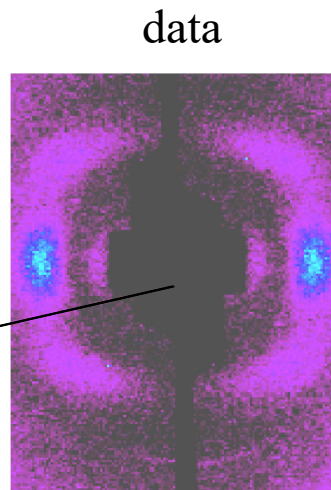
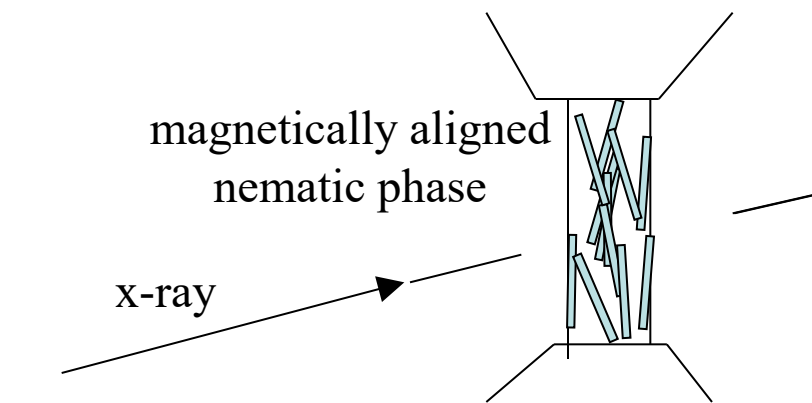


$$I(\mathbf{q}) = \int f(\Omega) P(\mathbf{q}, \Omega) d\Omega$$

angular distribution, $f(\theta)$

form factor, $P(\mathbf{q}, \Omega)$





$$I(q) = \int f(\Omega) P(q) d\Omega$$

Clay liquid crystals

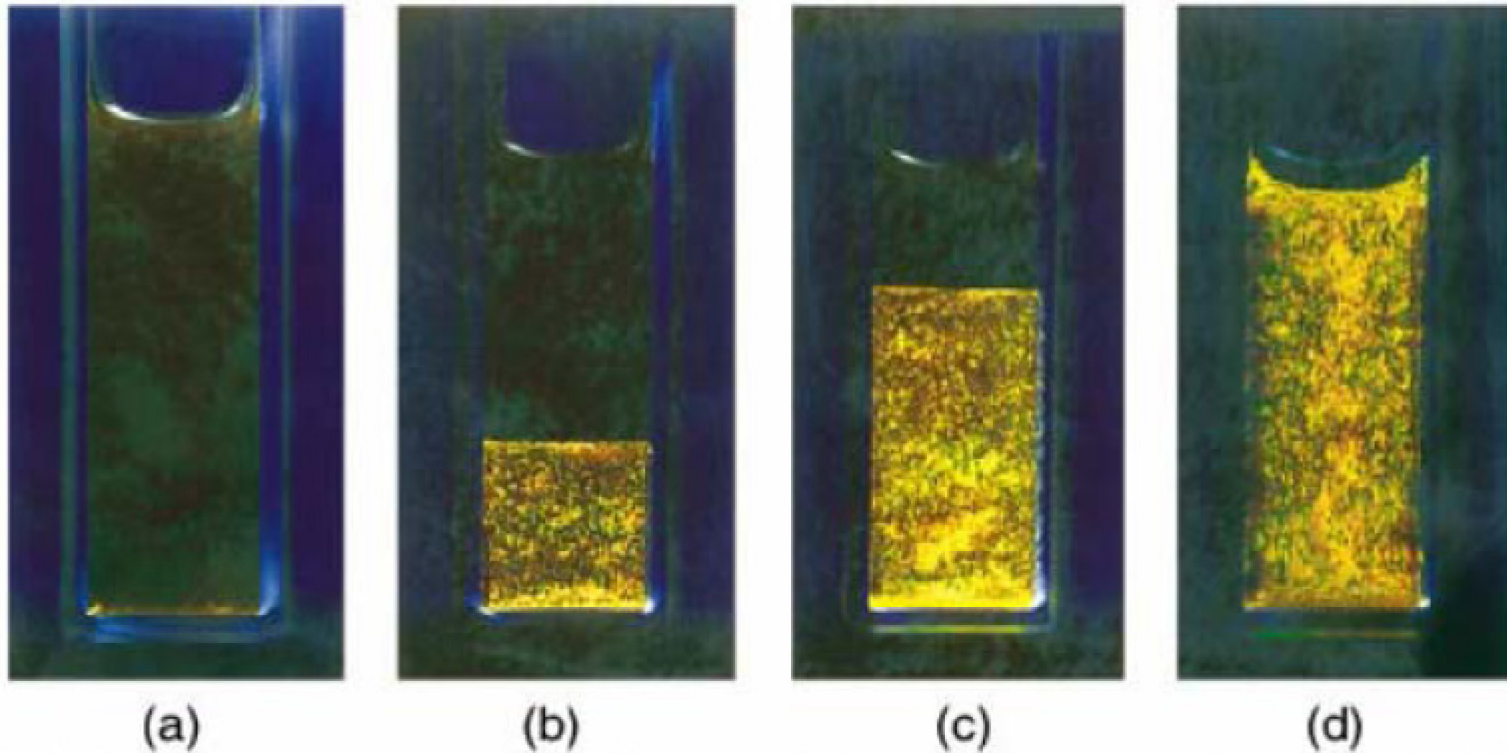
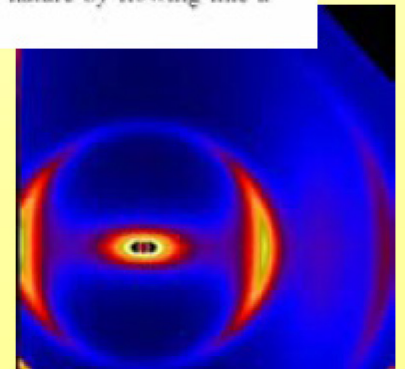


Figure 3. Isotropic–nematic phase separation as observed between crossed polarizers, at volume fractions ranging from (a) 15.9, (b) 16.5, (c) 16.9, to (d) 17.4% (v/v). The isotropic upper phase shows streaming birefringence when the tube is slightly shaken, indicating the presence of free alignable platelets. The lower phase is permanently birefringent due to nematic ordering and shows its liquid crystalline nature by flowing like a moderately viscous liquid when the tube is tilted.

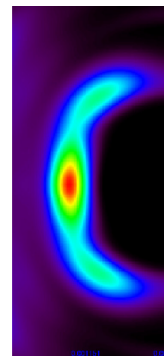
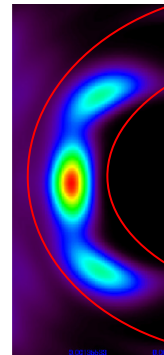
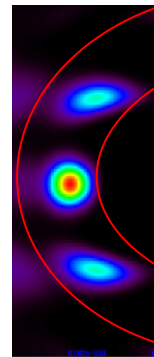
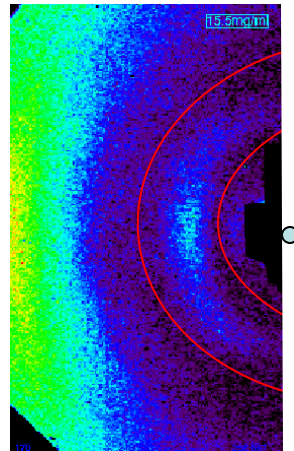
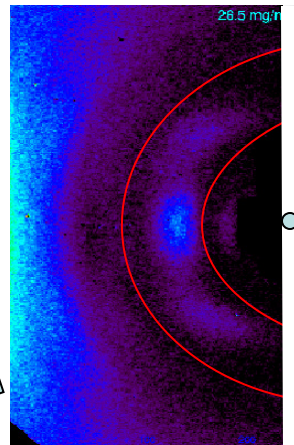
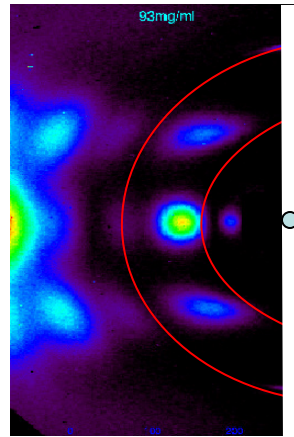
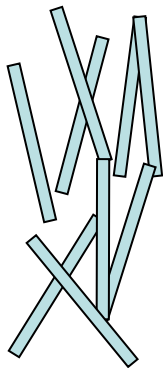
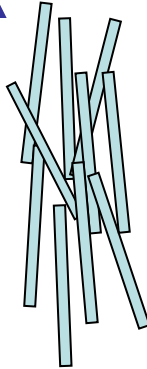


data

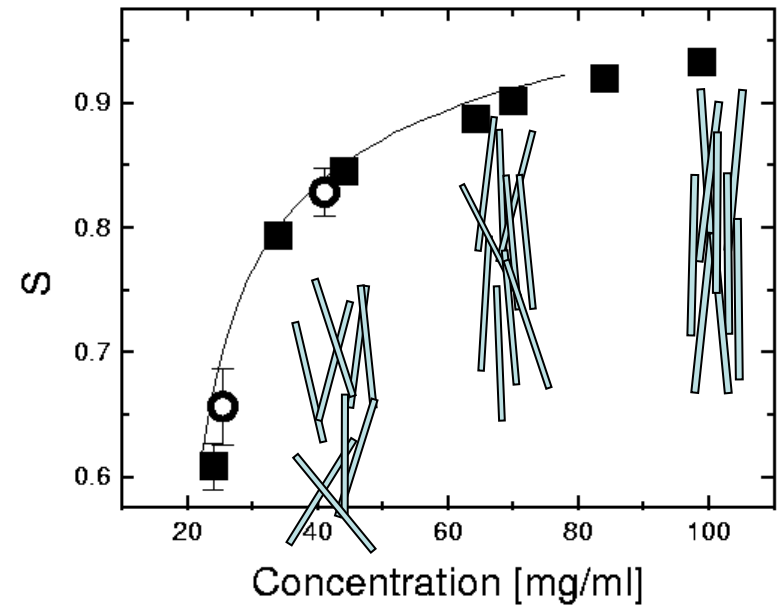
model



c ↑



$I(q) = \int f(\Omega) P(q) d\Omega$
Angular distribution

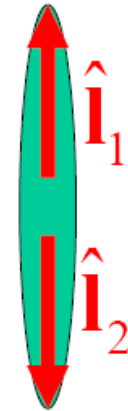


$$S = \int P_2(\cos(\theta)) f(\theta) d\cos(\theta)$$

intraparticle scattering

Nematic liquid crystals

- orientational order parameter?
- two equivalent directions:



$$l_x = \sin \theta \cos \phi$$

$$l_y = \sin \theta \sin \phi$$

$$l_z = \cos \theta$$

- expressions which treat these the same:

~~$$\hat{\mathbf{l}}_1 + \hat{\mathbf{l}}_2 = 0!$$~~

~~$$\frac{1}{2}(\hat{\mathbf{l}}_1 \hat{\mathbf{l}}_2 + \hat{\mathbf{l}}_2 \hat{\mathbf{l}}_1) = -\hat{\mathbf{l}}_1 \hat{\mathbf{l}}_1 = -\hat{\mathbf{l}}_2 \hat{\mathbf{l}}_2 = -\hat{\mathbf{l}}\hat{\mathbf{l}}$$~~

~~$$\hat{\mathbf{l}}_1 \hat{\mathbf{l}}_1 \hat{\mathbf{l}}_1 + \hat{\mathbf{l}}_2 \hat{\mathbf{l}}_2 \hat{\mathbf{l}}_2$$~~ not necessary

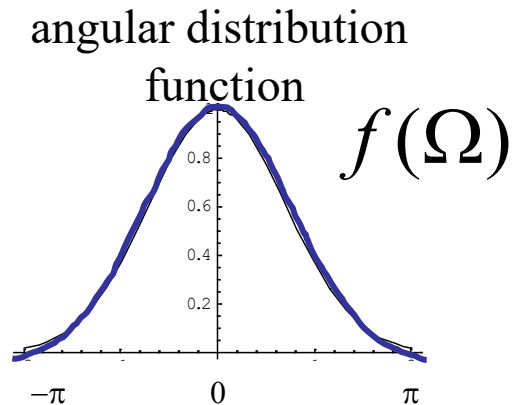
use $\hat{\mathbf{l}}\hat{\mathbf{l}}$

Nematic Liquid Crystals

- orientational variable:
 - second rank tensor

$$\sigma = \frac{1}{2}(3\hat{\mathbf{l}}\hat{\mathbf{l}} - \mathbf{I})$$

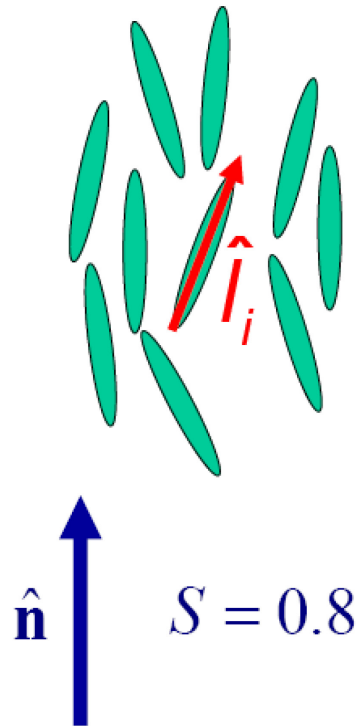
$$\sigma = \begin{bmatrix} \frac{1}{2}(3l_x^2 - 1) & \frac{3}{2}l_xl_y & \frac{3}{2}l_xl_z \\ \frac{3}{2}l_y l_x & \frac{1}{2}(3l_y^2 - 1) & \frac{3}{2}l_y l_z \\ \frac{3}{2}l_z l_x & \frac{3}{2}l_x l_y & \frac{1}{2}(3l_z^2 - 1) \end{bmatrix}$$



- orientational order parameter:
 - traceless symmetric

$$Q = \langle \frac{1}{2}(3\hat{\mathbf{l}}\hat{\mathbf{l}} - \mathbf{I}) \rangle$$

Nematic liquid crystals



- order parameter:
 - symmetric traceless tensor

$$\mathbf{Q} = \langle \frac{1}{2} (3\hat{\mathbf{l}}\hat{\mathbf{l}} - \mathbf{I}) \rangle$$

- eigenvectors:
 - direction of alignment:
nematic director $\hat{n}$

$$\mathbf{Q} = S \frac{1}{2} (3\hat{n}\hat{n} - \mathbf{I})$$

- eigenvalues:
 - degree of alignment:
order parameter S

$$S = \langle \frac{1}{2} (3(\hat{\mathbf{l}} \cdot \hat{n})^2 - 1) \rangle$$

Boulder 2002

1

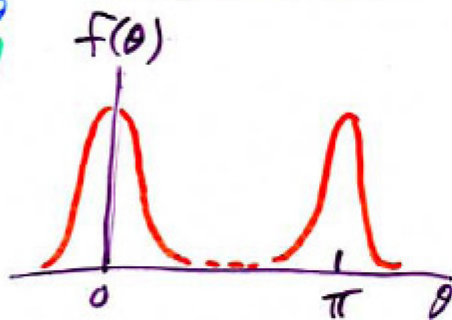
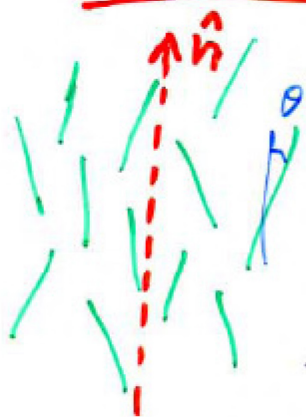
Liquid Crystals:

Basics

Elasticity, static equilibrium, textures

Phases

Nematic: parallel ordering of rodlike (or platelike) molecules.



Order parameter: $\underline{Q}$ is a second rank tensor

$$|\underline{Q}| = S = \left\langle \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right\rangle$$

$\hat{n}$ = "director" ($\hat{n} \leftrightarrow -\hat{n}$)

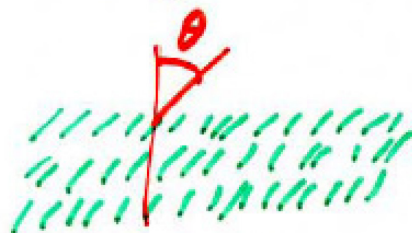
Nematic ground state: $\hat{n}$ uniform in space.

Low energy deformations: S constant, $\hat{n}(\mathbf{r})$ varying.

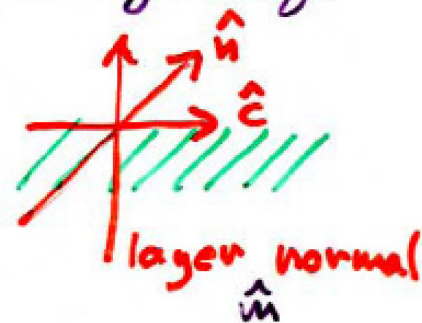
Smectic: 1D periodic, 2D fluid, layers of molecules.



Sm A



Sm C



SmC ground state: $\hat{c}$ uniform in space; layers flat.
 Low energy deformations: We will consider only flat layers, and $c(\vec{r})$ varying. "Almost" a 2D nematic, but $\hat{c} \leftrightarrow -\hat{c}$, for a thin film of SmC.

Nematic Elasticity (Curvature elasticity)

Expand $\hat{n}(\mathbf{r})$ in a Taylor series, with a local coordinate system, $\hat{n}_0 = \hat{z}$. Thus $\hat{n} = \hat{n}_0 + \delta n_x \hat{x} + \delta n_y \hat{y}$, ($\delta n_z = 0$ since $\hat{n}$ is a unit vector.)

$$\begin{aligned}\hat{n}(\vec{r}) = \hat{n}_0 + x \left(\frac{\partial n_x}{\partial x} \hat{x} + \frac{\partial n_y}{\partial x} \hat{y} \right) + y \left(\frac{\partial n_x}{\partial y} \hat{x} + \frac{\partial n_y}{\partial y} \hat{y} \right) \\ + z \left(\frac{\partial n_x}{\partial z} \hat{x} + \frac{\partial n_y}{\partial z} \hat{y} \right)\end{aligned}$$

Construct free energy from strains: following symmetry

- 1) ∞ rotation about $\hat{n}_0$
- 2) reflections $\perp$ and $\parallel$ to $\hat{n}_0$, center of inversion ($\hat{n} \leftrightarrow -\hat{n}$)

And, for bulk energy, eliminate terms of the form $\nabla \cdot \vec{u}$, where $\vec{u}$ is a vector field, since these can be integrated to surface terms.

This leaves only 3 terms in the free energy density:

$$f = \frac{1}{2} K_{11} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right)^2 + \frac{1}{2} K_{22} \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right)^2 + \frac{1}{2} K_{33} \left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_y}{\partial z^2} \right)$$

$$f = \frac{1}{2} K_{11} (\nabla \cdot \hat{n})^2 + \frac{1}{2} K_{22} (\hat{n} \cdot \nabla \times \hat{n})^2 + \frac{1}{2} K_{33} (\hat{n} \times \nabla \times \hat{n})^2$$

$$(\vec{S})^2 = |\hat{n} (\nabla \cdot \hat{n})|^2$$

Splay 

torsion "t"

twist 

$$(\hat{n} \times \nabla \times \hat{n})^2 = |\vec{B}|^2$$

Bend 

Notes on elastic moduli K_{ii} , K_s , K_B :

Nematic: Picture extreme curvature:

$$\frac{\partial n_x}{\partial x} \sim \frac{1}{l} \text{ with } l = \text{molecular length. This}$$

"disorders" the nematic $\rightarrow$ isotropic. $\Delta f \sim \Delta H_{NI} \sim 1 \text{ kC/mol}$

$$\frac{1}{2} K_{ii} \left(\frac{1}{l} \right)^2 = \frac{\text{molecular interaction energy}}{l^3}$$

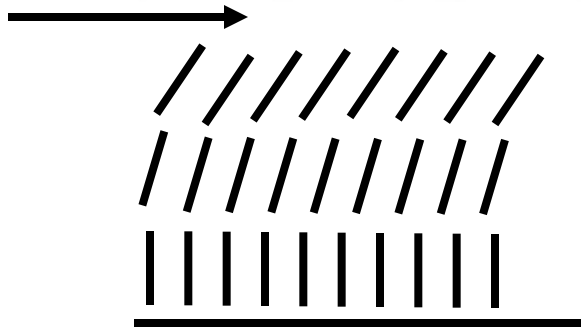
$$[K_{ii}] \sim \frac{[\text{energy}]}{[l]} \sim 10^{-12} \text{ J/m}.$$

Elastic constants K have units of force.

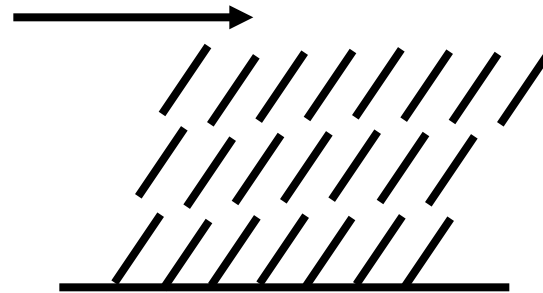
Boundary Conditions

Strong Anchoring: Molecules interact at a boundary so that their orientation at the boundary is fixed.

Weak Anchoring: Elastic torques in the interior are strong enough to compete with anchoring.



Strong anchoring



Weak anchoring

Boundary Value Problems : nematic

For some volume of nematic, with specified boundary conditions the free energy in the volume is:

$$F = \int dx dy dz f(\hat{n}(\vec{r})).$$

In general, find $\hat{n}(\vec{r})$ that minimizes F . For any particular problem, choose coordinate system, make assumptions (guesses!) about the form of solution, generate f , and use Calculus of Variations to minimize F .

Example: Planar structures: $\hat{n} \parallel (x, y)$ plane

$$n_x = \cos \theta, \quad n_y = \sin \theta, \quad n_z = 0 \quad (\text{clearly } |\mathbf{n}| = 1)$$

$$\nabla \cdot \hat{n} = -\sin \theta \frac{\partial \theta}{\partial x} + \cos \theta \frac{\partial \theta}{\partial y}$$

$$\hat{n} \cdot \nabla \times \hat{n} = -\frac{\partial \theta}{\partial z}$$

$$\mathbf{n} \times \nabla \times \mathbf{n} = \hat{x} \left(\sin \theta \cos \theta \frac{\partial \theta}{\partial x} + \sin^2 \theta \frac{\partial \theta}{\partial y} \right) + \hat{y} \left(-\cos^2 \theta \frac{\partial \theta}{\partial x} - \sin \theta \cos \theta \frac{\partial \theta}{\partial y} \right)$$

Assume $K_{11} = K_{22} = K$ then

$$F = \int dx dy dz \left[\frac{1}{2} K \left(\left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 \right) + \frac{1}{2} K_{22} \left(\frac{\partial \theta}{\partial z} \right)^2 \right]$$

$$\text{rescale } z = \sqrt{\frac{K_{22}}{K}} \xi$$

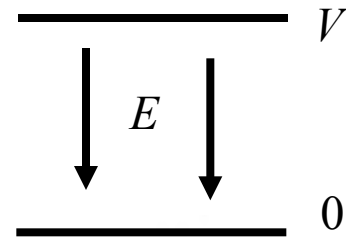
$$F = \frac{1}{2} \sqrt{K K_{22}} \int dx dy d\xi (\nabla \theta)^2 \rightarrow \text{Euler Lagrange eqn.: } \nabla^2 \theta = 0 \quad \underline{\text{electrostatics!}}$$

$$\nabla^2 \theta = 0 \quad \text{has an electrostatic analogue} \quad \theta \rightarrow V \quad \frac{\partial \theta}{\partial z} \rightarrow E$$

$$\nabla^2 V = 0$$

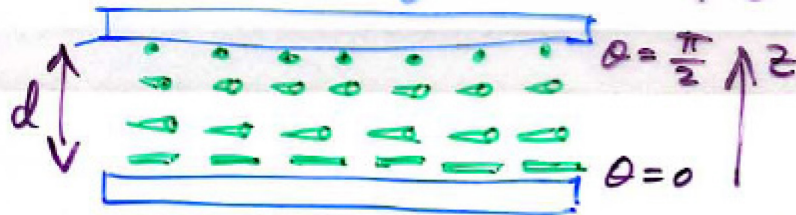
If the voltage is constant in a plane, then the electric field is constant.

parallel plate capacitor



Planar Structures

twist layer (1D)



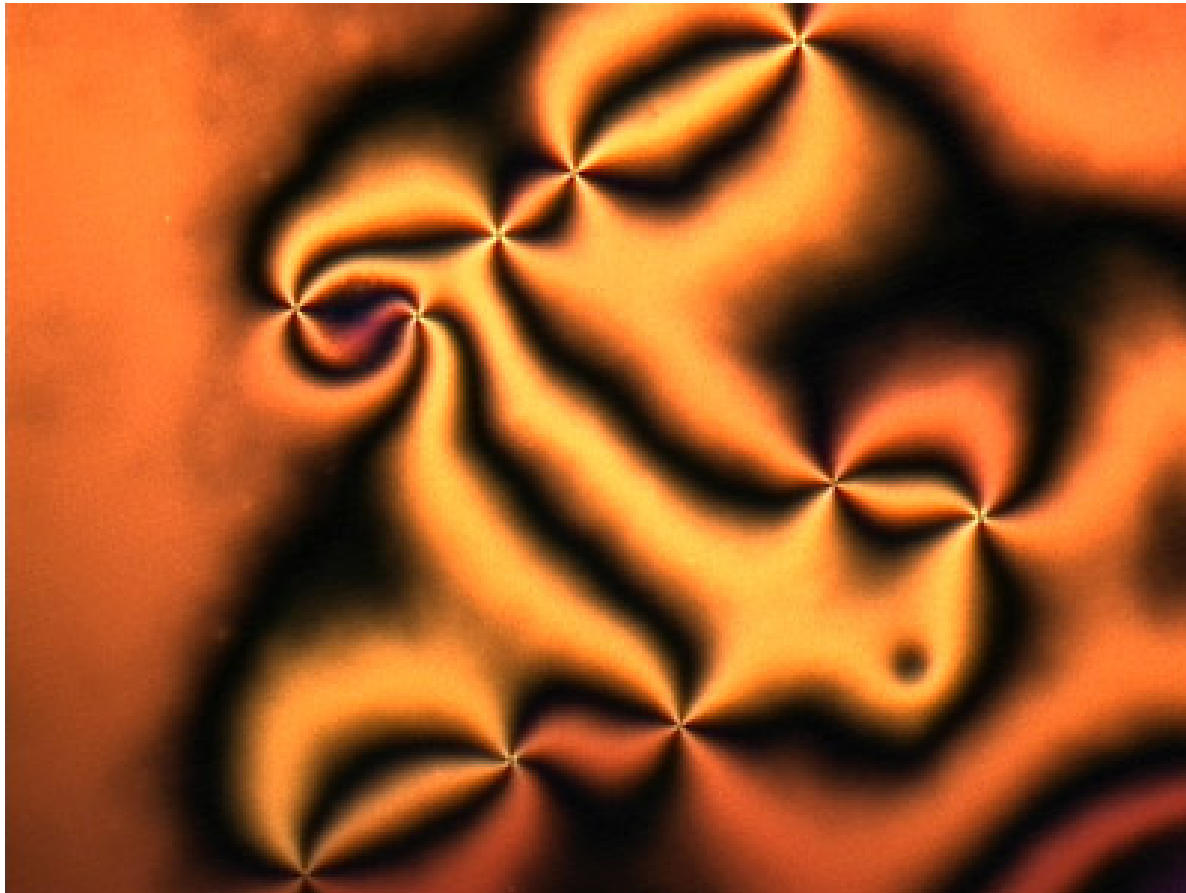
$$\nabla^2 \theta = 0 \Rightarrow \frac{\partial \theta}{\partial z} = \text{const.} = \frac{\pi}{2d}$$

$$\frac{F}{\text{area}} = \int_0^d dz \frac{1}{2} \kappa_{22} \left(\frac{\pi}{2d} \right)^2 = \frac{\pi^2}{8d} \kappa_{22}$$

Curvature distributed evenly minimizes energy

<http://dept.kent.edu/spie/liquidcrystals/textures1.html#ch13>

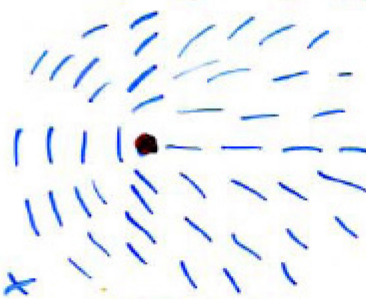
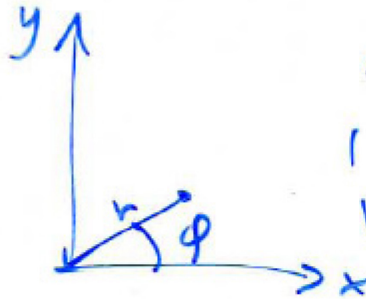
Liquid crystal disclinations seen through cross-polars



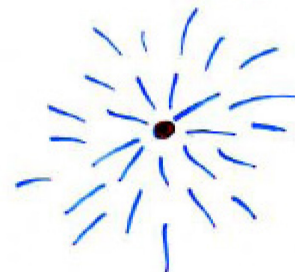
birefringence

Topology constrains m to be integer, or half-integer.

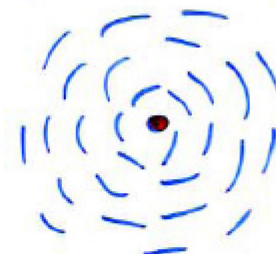
Disclinations (2D) Polar coordinates (r, ϕ) ; $\theta = (\pm)m\phi + \theta_0$



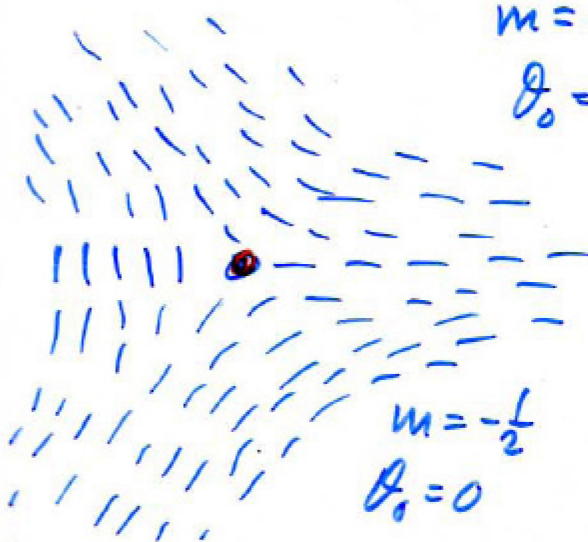
$m = +\frac{1}{2}$
 $\theta_0 = 0$



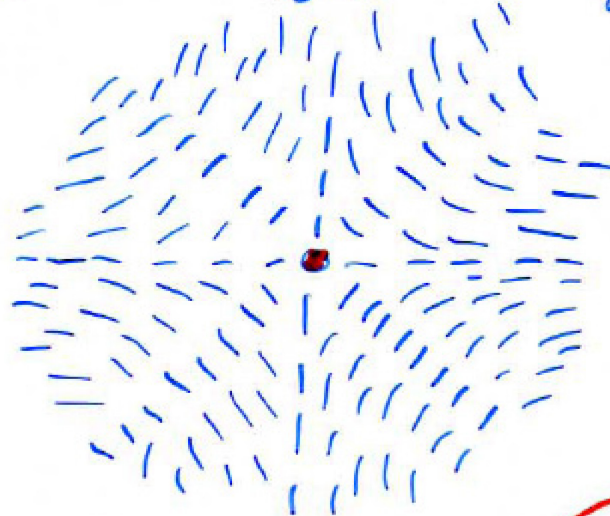
$m = +1$
 $\theta_0 = 0$



$m = +1$
 $\theta_0 = \frac{\pi}{2}$



$m = -\frac{1}{2}$
 $\theta_0 = 0$



$m = -1$
 $\theta_0 = 0$

[Invariant in z]

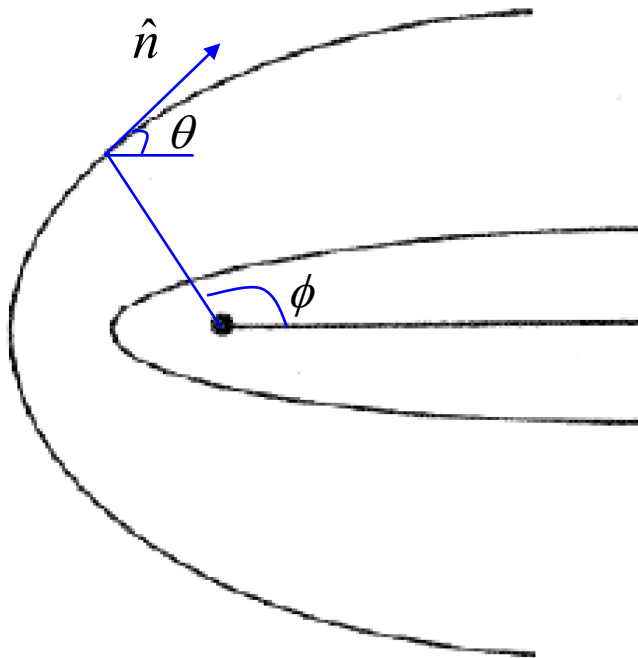
$$\frac{F}{\text{length}} = \int_{\epsilon}^R 2\pi r dr \frac{1}{2} k \left(\frac{m}{r} \right)^2 = \pi k m^2 \ln \frac{R}{\epsilon} + E_{\text{core}}$$

Energy / length

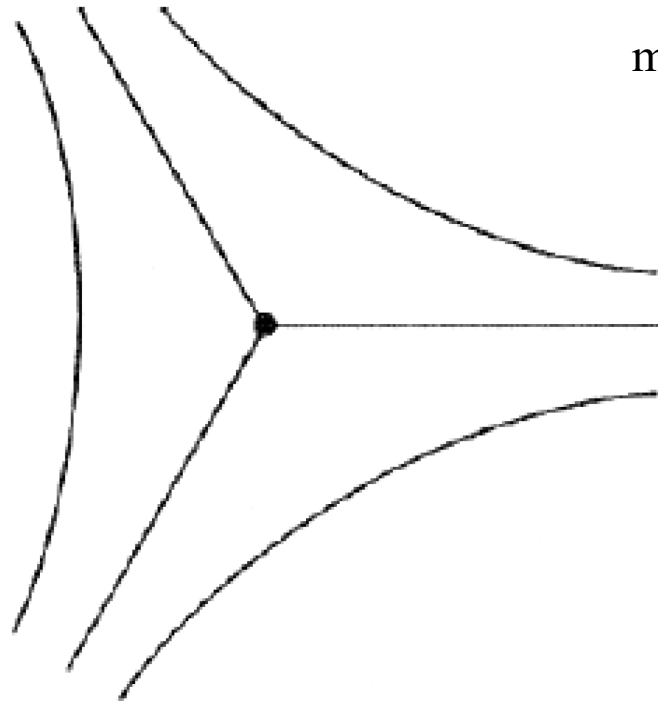
cutoff distance
"core" size

2D disclinations in nematic

$$\theta = \pm m \varphi + \theta_o$$



$m = +1/2$

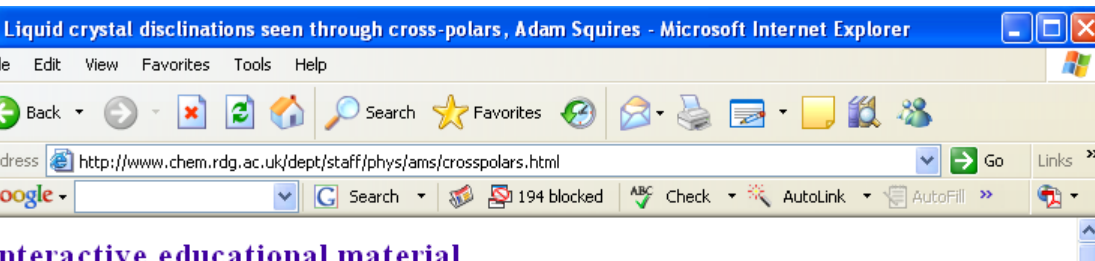


$m = -1/2$

<http://www.chem.rdg.ac.uk/dept/staff/phys/ams/crosspolars.html>

Liquid crystal disclinations seen through cross-polars

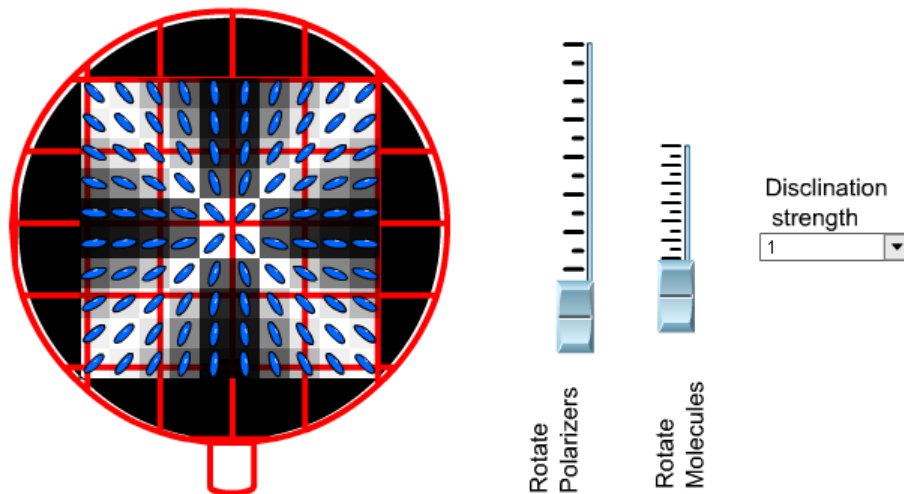
<http://dept.kent.edu/spie/liquidcrystals/textures1.html#ch13>



Interactive educational material

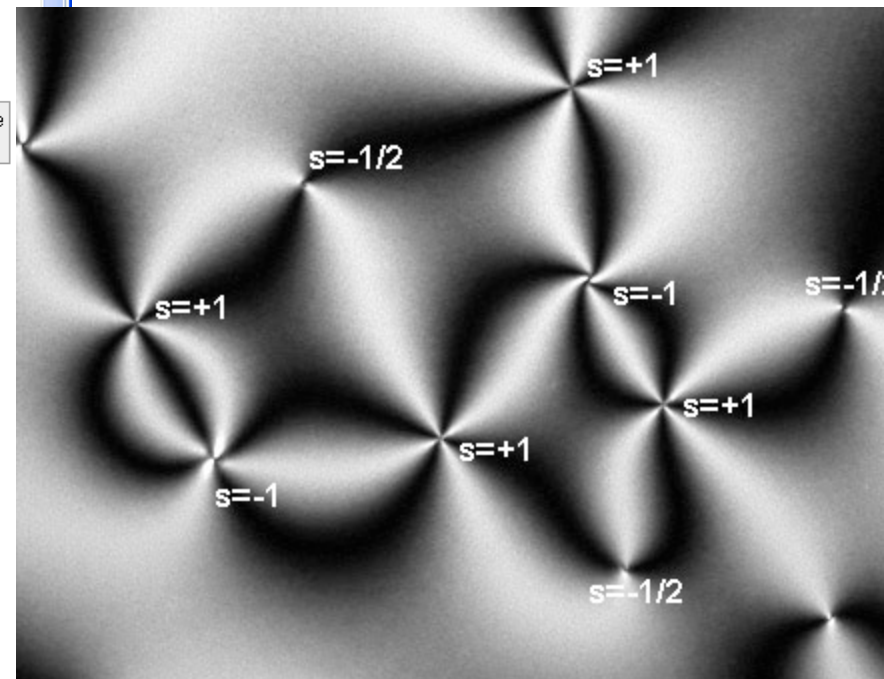
Liquid crystal disclinations seen through cross-polars

This interactive cartoon illustrates how the appearance of a liquid crystal disclination depends on the relative orientations of the molecules and the polarising filters.



Set the disclination strength to +1, and drag both sliders to their bottom position. This shows the familiar "maltese cross" pattern from structures with circular (or spherical) symmetry. You should convince yourself that areas where the molecules are aligned parallel / perpendicular to the polarisers appear darkest, whereas areas where the molecules are at 45 degrees to the polarisers are the brightest.

Rotate the polarisers through 360 degrees by dragging the appropriate slider upwards. The dark/light pattern should also rotate through 360 degrees, in the same direction. This is why the disclination is called "Strength +1". Rotate the molecules through 90 degrees using the other slider. You should see a continuum of defect structures, ranging from

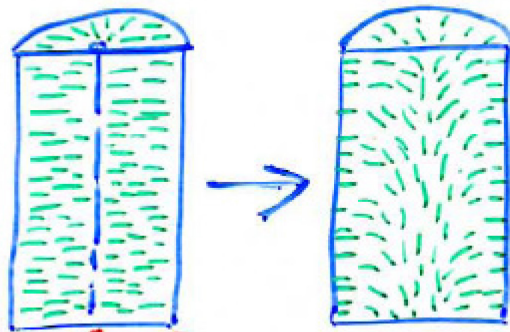


3D structures ($\hat{n}$ no longer planar)

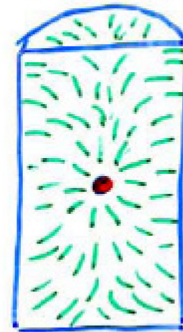
7

* point disclinations are possible

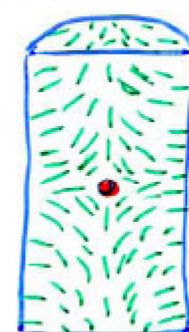
* Integer index ($m=1$) disclinations disappear!



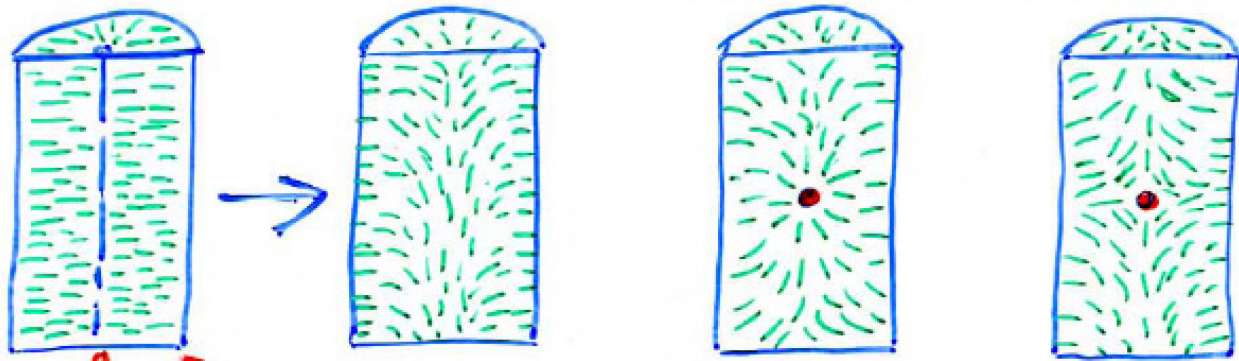
"escape" to the
third dimension



"hedgehog"



"negative
hedgehog"



"escape" to the
third dimension

"hedgehog"

"negative
hedgehog"

For the cylinder of radius r_0 , assume $n_r = \cos\theta$, $n_z = \sin\theta$
 $n_\phi = 0$:

$$\frac{F}{\text{length}} = \pi K \int_0^{r_0} \frac{dr}{r} \left[\cos^2\theta - 2r \sin\theta \cos\theta \frac{d\theta}{dr} + r^2 \left(\frac{d\theta}{dr} \right)^2 \right]$$

($K = K_{11} = K_{33}$) Euler Lagrange eqn: change to $x = \ln\left(\frac{r}{r_0}\right)$

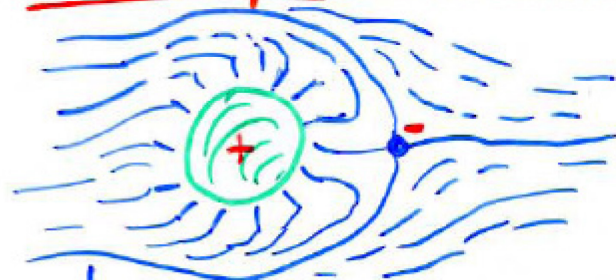
$$\frac{d^2\theta}{dx^2} = -\sin\theta \cos\theta$$

$$\text{Solution } \theta = 2 \tan^{-1}\left(\frac{r_0}{r}\right) - \frac{\pi}{2}.$$

$\frac{F}{\text{length}} = 3\pi K$ independent of r_0 . This is less than
the disclination energy, surely for $\ln \frac{r_0}{\epsilon} > 3$.

Nematic emulsions: A droplets of isotropic liquid⁸
suspended in a nematic:

Example: normal boundary conditions at interface



Pearl chain:

- + ⊕ + ⊕ + ⊕ + ⊕ + ⊕ + ⊕ + ⊕ + -

= hedgehog $\rightarrow$ needs a negative hedgehog nearby to cancel long range distortions.

Novel Colloidal Interactions in Anisotropic Fluids

Philippe Poulin, Holger Stark,* T. C. Lubensky, D. A. Weitz

SCIENCE • VOL. 275 • 21 MARCH 1997

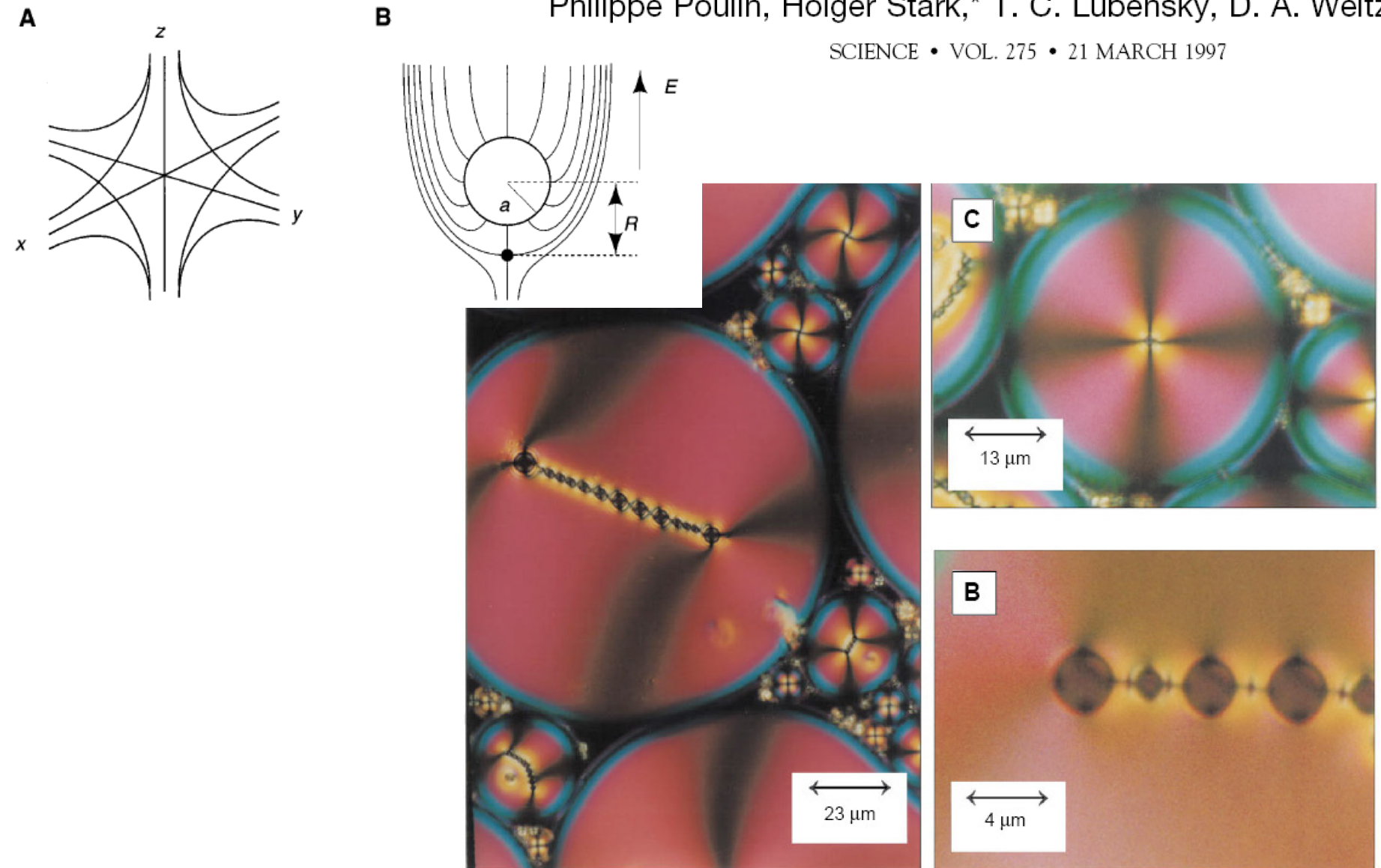
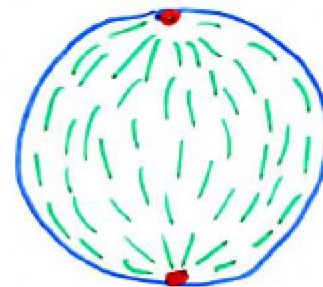
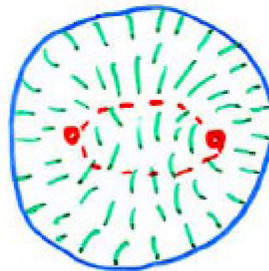
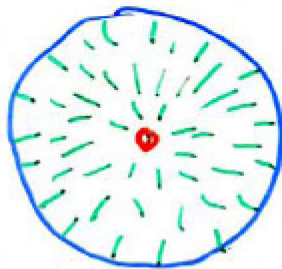


Fig. 1. (A) Microscope image of a nematic multiple emulsion taken under crossed polarizers. (B) A chain of water droplets under high magnification. (C) A nematic drop containing a single water droplet.

B. Droplets of nematic suspended in an isotropic matrix (solid or liquid): For $\perp$ or $\parallel$ boundary conditions, need disclinations in the structure of the droplet:



Field effects

① Anisotropy of susceptibility

Magnetic field: diamagnetic $\chi \sim 10^{-6}$ & negative.



π electrons in benzene rings are good at producing currents to oppose applied H .

Therefore rings tend to align parallel to H .

Electric field: Combination of electronic and dipolar effects:



large dipole dominates
response: $\hat{n} \parallel E$.

Magnetic free energy density:

$$f_H = \frac{1}{2} \chi_a H^2 \sin^2 \theta$$

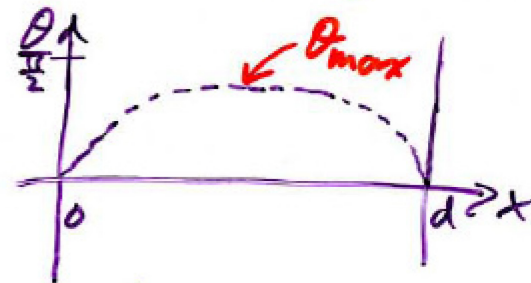
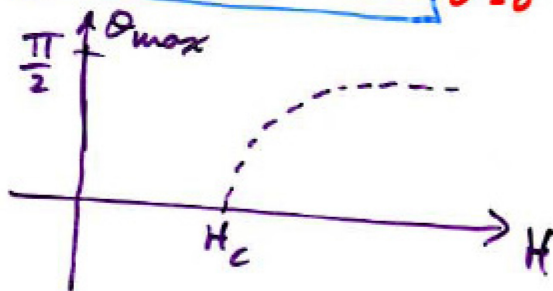
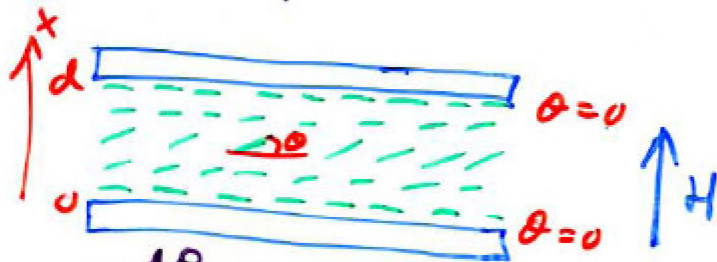


$$\chi_a = \chi_{||} - \chi_{\perp}$$

* Generally in a finite sample surface alignment and field alignment can compete. At low fields surface wins. At high field field wins, depending on sample dimensions.

Calculus of variations and Linear Stability Analysis. ^{CI}

Example: Frederike transition in splay-beer geometry



boundary condition:
 $\theta(0) = \theta(d) = 0$

$$\begin{aligned} \frac{F}{\text{area}} &= \int_0^d dx \left[\underbrace{\left(\frac{1}{2} k_{11} \cos^2 \theta + \frac{1}{2} k_{33} \sin^2 \theta \right)}_{A(\theta)} \left(\frac{d\theta}{dx} \right)^2 - \underbrace{\frac{1}{2} \chi_a H^2 \sin^2 \theta}_{C(\theta)} \right] \\ &= \int_0^d dx \left[A(\theta) \left(\frac{d\theta}{dx} \right)^2 + C(\theta) \right] \end{aligned}$$

For $H > H_c$ Need to solve the F.L. equation. Simplify for

now: $K_{11} = K_{33} = K$; $-2\left(\frac{1}{2}K\right)\frac{d^2\theta}{dx^2} - \frac{1}{2}\chi_a H^2 (2\sin\theta\cos\theta) = 0$

$$\frac{d^2\theta}{dx^2} + \frac{1}{\xi_H^2} \sin\theta\cos\theta = 0, \quad \xi_H = \frac{1}{H} \sqrt{\frac{K}{\chi_a}}$$

Solutions are elliptic integrals. Note: $\xi_{H_c} = \frac{d}{\pi}$

