

Low-Dimensions (I) [Le Hur]

- History: e^- $\xrightarrow{\text{classical motion}}$ $\xrightarrow{\text{quantum single-particle}}$
 (Thomson) ~ 1897 (Drude) ~ 1900 (Sommerfeld) ~ 1928

- Free Fermi gas: $\{c_k, c_k^\dagger\} = \delta_{kk}$, $\{c_k, c_k\} = \{c_k^\dagger, c_k^\dagger\} = 0$
 $C_V = \frac{\partial U}{\partial T} = \frac{\pi^2}{2} \frac{k_B T}{T_F}$; $\frac{T_F}{C_V} = \frac{3\mu_B^2}{2}$

- Fermi liquid: interacting e^- explained by quasiparticles, which has same quantum numbers as e^- .

interactions $\Rightarrow$ $\begin{cases} \text{mass renormalization } m \rightarrow m^* \\ \text{finite lifetime } \tau(\epsilon) \end{cases}$

valid if $\tau(\epsilon) \gg \hbar/\epsilon$.

- For 3D, phase space gives $\frac{1}{T_{ee}} \propto \sum_{\epsilon', \omega} \int d^D q \overbrace{|V(\omega, q)|^2}^{\sim \text{const.}} \overbrace{n_\epsilon (1 - n_{\epsilon - \omega})}^{\propto T} \overbrace{n_{\epsilon'} (1 - n_{\epsilon' - \omega})}^{\propto T}$
 $\propto T^2$

- Realizations of 1D system: carbon nanotubes, organic chain, edge states, optical lattices.

- Green function approach

free e^- : $G_R^{(0)}(\vec{k}, \omega) = \frac{1}{\omega - \xi_{\vec{k}} + i0^+}$

Fermi liquid: $G_R(\vec{k}, \omega) = \frac{1}{\omega - \xi_{\vec{k}} - \Sigma(\vec{k}, \omega)} = \frac{\xi_{\vec{k}}}{\omega - \xi_{\vec{k}}^* + i\Gamma_{\vec{k}}} + G_{inc}(\vec{k}, \omega)$
 $[\Gamma_{\vec{k}} \propto \text{Im}\{\Sigma(\vec{k}, \omega)\}]$

- Generally, $\text{Im} \Sigma_R = -\frac{2}{(2\pi)^{DH}} \int_0^\epsilon d\omega \int d^D q \text{Im} G_R(\epsilon - \omega, \vec{k} - \vec{q}) \text{Im} V_R(\omega, \vec{q})$

In 3D, $(\text{Re} \Sigma_R)_{\text{non-analytic}} \propto \epsilon^3 \ln|\epsilon|$

2D, $(\text{Re} \Sigma_R) \propto \epsilon/|\epsilon|$

And the dominant contribution comes from small angle.

- In 1D, energy conservation = momentum conservation.

$$\text{Im} \{ \Sigma(k \rightarrow 0, E, T) \} \sim - \left(\frac{v_F}{v_F} \right)^2 \max(E, T)$$

$$\text{Re} \{ \Sigma(E) \} \sim \left(\frac{v_F}{v_F} \right)^2 \ln |E| \Rightarrow \Sigma(k, E) \rightarrow \Sigma(k=0, E)$$

- Thus, for $T=0$, e^- Green function has power law decay.

for $tT \gg 1$, e^- Green function has exponential decay, $T_F^{-1} \propto \left(\frac{v_F}{v_F} \right)$

- In 1D we can map fermions to hard-core bosons:

$$c_j = e^{i\pi \sum_{j' < j} n_{j'}} b_j, \quad [b_i, b_j] = [b_i^\dagger, b_j] = [b_i^\dagger, b_j^\dagger] = 0 \text{ for } i \neq j, \quad \{b_i^\dagger, b_i\} = 1$$

hard-core bosons

- Write $b_j = \sqrt{p_j} e^{i\theta_j}$, $p(x) \sim p_0 + \frac{1}{\pi} \partial_x \phi$

$$\Rightarrow \begin{cases} \mathcal{H}_{\text{kin}} \sim \int dx \frac{p_0}{2M} (\nabla \theta)^2 \\ \mathcal{H}_{\text{int}} \sim \int dx U (\nabla \phi)^2 \end{cases}$$

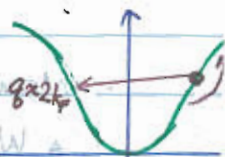
$$[\mathcal{H}_{\text{kin}} = \int dx \frac{1}{2M} (\nabla b^\dagger b)^2]$$

$$[\mathcal{H}_{\text{int}} = \int dx U (b^\dagger b - p_0)^2]$$

Define right- and left- fermion, $\begin{cases} p_+ + p_- = p(x) \\ p_+ - p_- = \frac{1}{\pi} \partial \theta(x) \end{cases}$

- Luttinger Paradigm

- Particle-hole pair excitations: $p_+(q) = \sum_k a_{k+q}^\dagger a_k$
 $p_-(q) = \sum_k b_{k+q}^\dagger b_k$



$$\omega(q) = |q| \sqrt{\left(v_F + \frac{g_1(q)}{2\pi} \right)^2 - \left(\frac{g_2(q)}{2\pi} \right)^2}$$

g_1 : forward scattering
 g_2 : backward scattering

- Effective Hamiltonian:

$$\mathcal{H} = \frac{V}{2} \int_0^L dx \left[\frac{1}{g} (\partial_x \phi)^2 + g (\partial_x \theta)^2 \right]$$

- For charge mode, $v_g = v_F$ and $g \approx 1 - \frac{U}{\epsilon_F}$

For spin mode, $v_s = 1$ and $v_s = v_F$

Hence we have spin-charge separation.

REMARK: At this stage we have considered spin-charge separation at Hamiltonian level only, i.e. $\mathcal{H}_{\text{Hubbard}} \sim \mathcal{H}_s + \mathcal{H}_c$

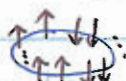
- In 1D there is no spin order. Seen from large- s expansion:

$$S_i^z = (s - a_i^\dagger a_i), \quad S_i^- = \sqrt{2s} a_i^\dagger, \quad S_i^+ = \sqrt{2s} a_i$$

$$\Rightarrow \langle S_i^z \rangle = s - \frac{1}{N} \sum_q \frac{e^{i\mathbf{E}(\mathbf{q})}}{e^{i\mathbf{E}(\mathbf{q})} - 1}$$

$$\sim \frac{1}{\beta J} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2} \text{ is IR divergent}$$

- Alternatively, consider "cost" of domain wall



$$F_0 = E - TS = E_0, \quad F_1 = E_0 + J - k_B T \ln N$$

$\Rightarrow$ domain wall preferred $\Rightarrow$ no ordering.

- The excitations in 1D are "spinons"

$$\langle S(\vec{x}) S(\vec{0}) \rangle \sim \frac{\cos(2k_F x)}{x^{1+g}}$$

Magnons are split into spinons
($S_z \sim 1$) ($S_z \sim 1/2$)



- Momentum-resolved tunneling

▲ Wire long compared with λ_F .

▲ Boost in momentum $\delta k_x \propto eBd = q_B$

▲ Measurements show 2 peaks in spectral function.



- Charge Fractionalization

▲ e^- is NOT good eigenstate: $[H, \Psi^\dagger] \neq E \Psi^\dagger$

▲ The exact eigenstates are chiral:

$$L_\pm(x, t) = \exp[-i\sqrt{\pi} N_\pm \Theta_\pm(x, t)] \quad \Theta_\pm = \theta \mp \phi/g$$

▲ Inject $N_l = e^-$ into the system, so that there are] more e^- at $+v_F$ than $-v_F$. Then we have conservation law: