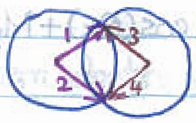
Renormalization Group on Fermions (III) [Shankar]

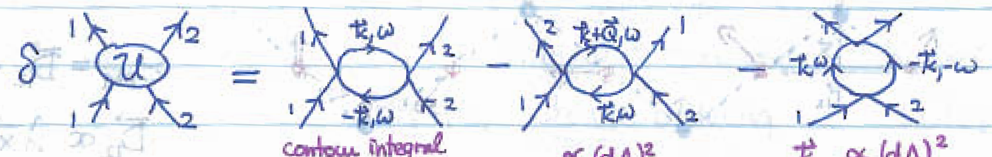
Recall in 2D we have 2 marginal interaction terms:



$u(\theta_1-\theta_2)$

and

$v(\theta_1-\theta_3)$

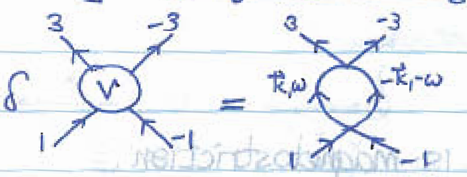


contour integral cancels

$\propto (d\Lambda)^2$

$\vec{k} \;\propto (d\Lambda)^2$

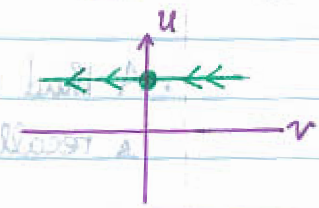
Similarly, out of the 3 diagrams for v, only one contributes



$$\propto \int \frac{v(\theta-\theta_1)\, v(\theta_3-\theta)\, d\theta\, d\omega\, dk}{(i\omega-\varepsilon(k))(-i\omega-\varepsilon(-k))}$$

$$\propto \int \frac{d\Lambda}{\Lambda} \int v(\theta-\theta_1)\, v(\theta_3-\theta)\, d\theta$$

$$\Longrightarrow \frac{dv}{dt}(\theta_1-\theta_3) \propto -\int v(\theta_3-\theta)\, v(\theta-\theta_1)\, d\theta$$



Expand $v(\theta) = \sum_m V_m e^{im\theta}$, $\frac{dV_m}{dt} = -cV_m^2$

$$\Longrightarrow V_m = \frac{V_m^0}{1-V_m^0 t} \approx \frac{1}{t} \ln(k_F/\omega_D)$$

This give rise to Cooper instability.

Now we have the theory at fixed pt. Go on to solve for physical quantity of the theory. e.g.,

$$\chi(q,\cancel{\omega}^{\,0}) = \langle \rho(q,\cancel{\omega}^{\,0}), \rho(-q,-\cancel{\omega}^{\,0}) \rangle$$

$$\square = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_{-\Lambda}^{\Lambda} \frac{d\theta\, d\omega}{(i\omega-\varepsilon(k))(i\omega-\varepsilon(k+q))}$$

$$\approx \int_{-\infty}^{\infty}\int_0^{2\pi}\int_{q\cos\theta}^{0} \frac{dk\, d\theta\, d\omega}{q\cos\theta}$$

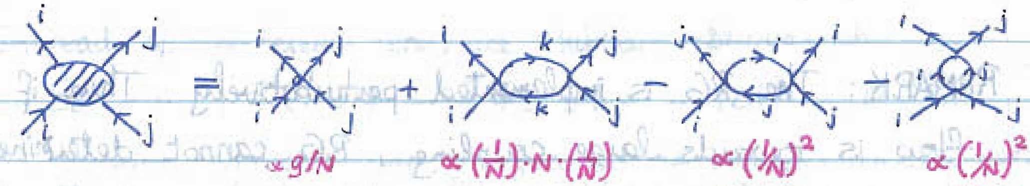
$$\square = u_0 \chi_0^2$$

$$\Longrightarrow \chi(q,\omega=0) = \frac{\chi_0}{1-u_0\chi_0}$$

NOTE: Diagrams such as  does not contribute in narrow bandwidth ($\Lambda \ll k_F$) limit.

Now, another tool: <u>large N expansion</u>

Consider $\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i \not{\partial} \psi_i + \frac{g}{N} \sum_{ij} (\bar{\psi}_i \psi_i)(\bar{\psi}_j \psi_j)$



We can relate large-N and narrow bandwidth via $N \sim \frac{k_F}{\Lambda}$

REMARK: The conclusion continues to hold if the Fermi surface is not circular. Cooper instability is presense as long as $\varepsilon(k) = \varepsilon(-k)$

Mathematically, change variable $(k, \omega, \theta) \longmapsto (\epsilon, \omega, \theta)$

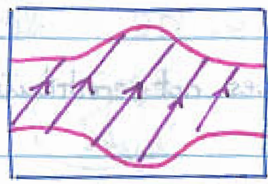
$$\Longrightarrow S_0 = \int \bar{\psi}(\epsilon,\omega,\theta)\,(i\omega - \epsilon)\,\psi(\epsilon,\omega,\theta)\, J(\theta)\, d\theta\, d\omega\, d\epsilon.$$

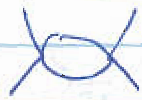
The only changes is that $u(\theta_1,\theta_2) \neq u(\theta_1 - \theta_2)$

For non-circular Fermi surface, shape of Fermi surface can chang

One can compute the 1-loop self energy (~ ) can recompute Fermi surface order to order.

However, one also need to worry about Nesting :

 $\longleftrightarrow$ $W(k_1, k_2; k_1+(\pi,\pi); k_2+(\pi,\pi))$

 $\Longrightarrow$ $\frac{dW}{dt}(\theta_1, \theta_2) = \int W \cdot W \, d\theta$

This give rise to CDW instability.

REMARK : The RG is implemented perturbatively. Thus if the flow is towards large coupling, RG cannot determine the properties of the resulting state. However, practically there will be a temperature associate with such instability, and these temperature can be very small and hence unimportant.

REMARK : Since all interactions happen on a thin shell, Fermi liquid in any dimension has the generic form

$$S_1 = \int_{-\infty}^{\infty} \int_A \int_{-\Lambda}^{\Lambda} \bar\psi(\omega,k,\theta)\,(i\omega-k)\,\psi(\omega,k,\theta)\, dk\, d\Omega\, d\omega$$

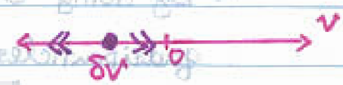
internal labels

However, something different can happen if Fermi "surface" has a different co-dimension, e.g. Fermi line in 3D

At Gaussian (zero-coupling) fixed pt, the above scale as $s^8/s^9 = s^{-1}$. Thus 4-fermion interaction is still irrelevant in Gaussian fixed point. But at strong coupling different co-dimension model may flow to different fixed points.

To handle the problem, we may do expansion in $d = 1+\varepsilon$ dimensions. From this we get, e.g.

$$\beta = \frac{d\nu}{dt} = -\epsilon\nu - \nu^2$$



By scaling, the singular part of energy scales as:

$$E(\delta\nu, h) = s^{-(2+\epsilon)} E(\delta\nu s^{\epsilon}, h s^{1+\epsilon})$$

Hertz – Millis approach:

Instead of 4-fermion term, use Hubbard – Stratonovich:

$$S_0 \mapsto \int \bar{\psi}(\dots)\psi + \bar{\psi}\bar{\psi}\Delta + h.c. + |\Delta|^2 = e^{S(\Delta)+|\Delta|^2}$$

For fermion WITHOUT gap, integrate out ψ generates singular behaviors

$$e^{S} \mapsto e^{\int \bar{\Delta}(\gamma + (q^2+\omega^2)^{3/2})\Delta \, d\omega \, d^{2+\epsilon}q + \int u \bar{\Delta}\Delta\bar{\Delta}\Delta}$$

The scaling seemingly is different from conventional way. But ultimately these will agree.

Lesson: why we need RG:

(1) It allow us to understand why (qualitatively) universal classes exist

(2) Traditional methods (e.g. high/low temperature series) faces singularities at critical points. But for RG, critical pt is a natural starting point for expansion (as β-function $= 0$ at fixed pt).

(3) RG allows one to obtain continuum theory in a non-perturbative way from a lattice model.