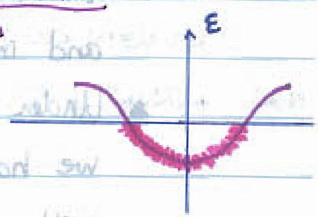
Renormalization Groups for Fermion (II) [Shanker]

- We shall first consider (1+1)-D.
  - ▲ NOTE that 1D is special since there is no alternative pathway. Two particles cannot pass through each other without collisions.
  - ▲ Take $\mathcal{H} = \underbrace{-\sum_n (\psi_n^\dagger \psi_{n+1} + h.c.)}_{\mathcal{H}_0} + \underbrace{U \sum_n \psi_n^\dagger \psi_n \psi_{n+1}^\dagger \psi_{n+1}}_{\lambda \mathcal{H}_1}$
  - ▲ In momentum space, $\mathcal{H}_0 = -\sum_k \psi_k^\dagger \psi_k \cos k$.

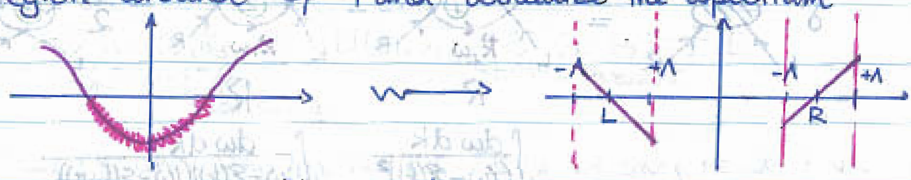


$\Longrightarrow$ the system is a perfect metal, with gapless excitation.

  - ▲ Instead, if $U \to \infty$, there are 2 degenerate ground state
    either $\cdots \circ \times \circ \times \circ \times \circ \cdots$
    or $\cdots \times \circ \times \circ \times \circ \times \cdots$

    And the gap is seen to be U : $\cdots \times \circ \times \cdots \longmapsto \cdots \times \times$
  - ▲ Hence we expect a phase transition somewhere $\cdots$ $U=0$ metal ——— U^*
  - ▲ The simplest way to consider mean-field :
    $\langle \psi_n^\dagger \psi_n \rangle = (-1)^n \Delta$

    Solve for spectrum and minimize w.r.t. Δ.
  - ▲ Meanfield predicts phase transition precisely at $U=0$. i.e., arbit[rary] perturbative U drives the system into CDW.
  - ▲ Now consider RG, we may, for low-energy, consider only region around ε_F, and linearize the spectrum



$$\Longrightarrow \mathcal{H}_0 = -\sum_\alpha \int_{-\Lambda}^{\Lambda} dk\, \psi_\alpha^\dagger(k) \psi_\alpha(k)\, k$$

$$\Longrightarrow Z_0 = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{S_0} \quad ; \quad S_0 = \sum_\alpha \int_{-\infty}^{\infty} \int_{\Lambda} d\omega dk\, \bar{\psi}_\alpha(\omega,k)(i\omega - k)\psi$$

  - ▲ Under $\Lambda \mapsto \Lambda/s$, $S_0 \longmapsto \sum_\alpha \int_{-\infty}^{\infty} \int_{-\Lambda/s}^{\Lambda/s} d\omega dk\, \bar{\psi}(i\omega - k)\psi$

    Change variable $k \mapsto k' = sk$; $\omega' \mapsto \omega' = s\omega$

$$\Longrightarrow S_0 \longmapsto \sum_\alpha \int_{-\infty}^{\infty} \int_{-\Lambda}^{\Lambda} \frac{d\omega' dk'}{s^2}\, \bar{\psi}\left(\frac{\omega'}{s}, \frac{k'}{s}\right) \left(\frac{i\omega - k}{s}\right) \psi\left(\frac{\omega'}{s}, \frac{k'}{s}\right)$$

Finally let $\psi'(\omega',k') = s^{-3/2}\,\psi\!\left(\frac{\omega'}{s},\frac{k'}{s}\right)$

Then S_0 is invariant.

▲ Now add in a general 4-fermion interaction:

$$S_4 = \int d1 \int d2 \int d3 \;\bar\psi_\alpha(4)\bar\psi_\beta(3)\psi_\gamma(2)\psi_\delta(1)\; U_{\alpha\beta\gamma\delta}(4321)$$

where $\int d1 \equiv \int_{-\infty}^{\infty} d\omega_1 \int_{-\Lambda}^{\Lambda} dk_1$, etc.

and implicitly we have conservation "1" + "2" = "3" + "4".

▲ Under the transformation $(\Lambda, k, \omega, \psi) \longmapsto (\Lambda/s, k', \omega', \psi')$,

we have $S_4 \longmapsto S_4'$, where the changes occur just on $U \mapsto U'$,

with $U'(k',\omega') = U\!\left(\frac{k'}{s},\frac{\omega'}{s}\right)$; $U'(4',3',2',1') = U\!\left(\frac{4'}{s},\frac{3'}{s},\frac{2'}{s},\frac{1'}{s}\right)$

▲ Henceforth we use the shorthand $k \equiv (k,\omega)$

▲ Taylor expand, $U(k) = U_0 + U_1 k + U_2 k^2 + \dots$ [assumed short-ranged]

$$U'(k') = U(k'/s) \implies \begin{cases} U_0' = U_0 \\ U_1' = U_1/s \\ U_2' = U_2/s^2 \\ \quad\vdots \end{cases}$$

Hence only U_0 is important.

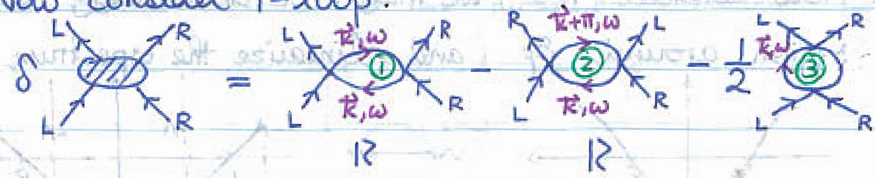
▲ Next, claim that if $\alpha=\beta$ or $\gamma=\delta$ then the term vanishes

Reason: $\psi_L(2)\psi_L(1) = -\psi_L(1)\psi_L(2) \implies U_{LL??} = 0$, etc.

▲ Thus the only object we need to consider is: U_{RLRL}.

▲ And we argued that at tree level $\underbrace{U}_{\equiv U^{(0)}_{LRLR}} \longmapsto U' = U$.

▲ Now consider 1-loop:



$$\int \frac{d\omega\, dk}{(i\omega-\varepsilon(k))^2} \qquad \int \frac{d\omega\, dk}{(i\omega-\varepsilon(k))(i\omega-\varepsilon(k+\pi))} \qquad \int \frac{d\omega\, dk}{(i\omega-\varepsilon(k))(-i\omega-\varepsilon(k))}$$

First: $\;\cong 0$ (pole cancels) (alternatively, need two $\,{}^{L}_{L}\!\times^{R}_{R}$, impossible to match)

Second: $\;\cong \int \frac{dk}{|E|} = \frac{d\Lambda}{\Lambda}$

Third: $\;= -$②

In ①, the sign of

In ②, sign (L/R) of internal leg is determined uniquely, while in ③ there is two choices of species. This cancels the 1/2.

▲ Conclusion : $\frac{dU}{d\tau} = \beta(U) = 0$

$\Longrightarrow$ The coupling does not flow. $\Longrightarrow$ system remains gapl

▲ But then where is the transition?

Answer: by Umpklapp term $\bar{U} = (k_1-k_2)(k_3-k_4)\bar{\psi}_R\bar{\psi}_R\psi_L\psi_L$

This term is irrelevant when expand around Gaussian fixed

but at intermediate coupling it becomes LESS irrelevant.

▲ The non-flow of coupling is related to the system bei

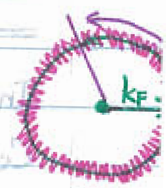
a Luttinger liquid.

- Now consider system in $(2+1)$-D.

▲ $\mathcal{H} = \int d^2K\ \psi^\dagger(\vec{K})\left(\frac{K^2}{2m}-\mu\right)\psi(\vec{K})$

$\approx \int d^2k\ \psi^\dagger(\vec{k})\ k\ \psi(\vec{k})$ $\quad \left[\frac{K^2}{2m}-\frac{K_F^2}{2m} \approx \underbrace{\frac{K_F}{m}}_{=V(=1)}\underbrace{(K-K_F)}_{=k}\right]$

$\Longrightarrow S_0 = \int dk\,d\omega\,d\theta\ \bar{\psi}(\omega,k,\theta)(i\omega-k)\psi(\omega,k,\theta)$



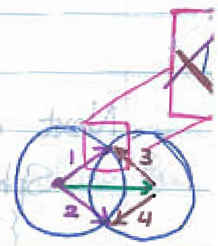
▲ The scaling $(\Lambda, k, \omega, \psi) \longmapsto (\Lambda/s, k', \omega', \psi')$

is the same as in 1D.

▲ Again only the constant part of 4-fermion

term can be important.

i.e., need only consider $U = U(\theta_1,\theta_2,\theta_3,\theta_4)$



▲ But in 2D and limit of very thin shell, only

(almost) forward scattering is possible

$\Longrightarrow \underbrace{U(\theta_1,\theta_2,\theta_3,\theta_4) = U(\theta_1,\theta_2)}_{\text{only "almost" forward scattering}} \underbrace{= U(\theta_1-\theta_2)}_{\text{rotation invariant}} = F(\theta)$

▲ This explains why in Landau theory we only need ONE f-func

$\delta n(k)\,\delta n(k')\,f(k,k')$, and why we can stop at 2^{nd}-order.

* However, we have another possibility :

This is the BCS channel