Renormalization Groups on Fermions (I) [Shanker]

- 3 "theories" of Fermi liquid : Landau, $\underbrace{\text{Landau-er}}_{\text{Landau's students}}$, $\underbrace{\text{Landau-est}}_{\text{modern}}$

- Renormalization is possible whenever the physics is described by a partition function Z.

- Example : Bosonic oscillator

$$H = \omega_0 a^\dagger a \quad ; \quad \mathcal{U} = e^{-iHt/\hbar}. \quad Z = \mathrm{Tr}\, e^{-\beta H}$$

Now, $e^{-\beta H} = (e^{-\frac{\beta}{N}H})^N \approx (1-\epsilon H)^N$ $\quad [\epsilon = \frac{\beta}{N}]$

Introduce partition of identity : $\begin{cases} I \propto \int dz d\bar{z} \, |z\rangle\langle \bar{z}| e^{-\bar{z}z} & , a|z\rangle = z|z\rangle \\ \langle \bar{z}'|z\rangle = e^{\bar{z}'z} & , \langle z|a^\dagger = \langle z|\bar{z} \end{cases}$

$$\Rightarrow Z = \prod d\bar{z}(\tau)\, dz(\tau) \exp\left[\sum\left(\left(\frac{\bar{z}(\tau+\epsilon)-\bar{z}(\tau)}{\epsilon}\right) z\epsilon - \epsilon \bar{z} z \omega_0\right)\right]$$

$$= \int \mathcal{D}\bar{z}\, \mathcal{D}z \; e^{\int_0^\infty \bar{z}\left(\frac{\partial}{\partial\tau} - \omega_0\right) z\, d\tau}$$

$$= \int \mathcal{D}\bar{z}(\omega)\, \mathcal{D}z(\omega) \; e^{\int \bar{z}(\omega)(i\omega - \omega_0) z(\omega) d\omega}$$

$$\Rightarrow \langle \bar{z}(\omega_1)\, z(\omega_2)\rangle = \frac{\int \mathcal{D}\bar{z}\mathcal{D}z\; \bar{z}z\, e^{-(\ldots)}}{Z} = \frac{\delta(\omega-\omega')}{i\omega - \omega_0}$$

$$\langle \bar{z}(4)\, \bar{z}(3)\, z(2) z(1)\rangle = \langle \bar{4}\, 1\rangle\langle \bar{3}\, 2\rangle + \langle \bar{4}\, 2\rangle\langle \bar{3}\, 1\rangle$$

- Example : Fermionic oscillator

$$H = \psi^\dagger \psi\, \omega_0 \quad ; \{\psi, \psi^\dagger\} = 1 , \quad \{\psi,\psi\} = \{\psi^\dagger, \psi^\dagger\} = 0.$$

Again we want resolution of identity, but we need:

$\psi|\chi\rangle = \chi|\chi\rangle \quad , \quad \psi^2|\chi\rangle = \chi^2|\chi\rangle = 0 \quad \Longrightarrow \chi$ Grassmann variable.

$$\Rightarrow Z = \int \mathcal{D}\bar{\psi}(\omega)\, \mathcal{D}\psi(\omega) \; e^{\int \bar{\psi}(\omega)(i\omega-\omega_0)\psi(\omega)\, d\omega}$$

$$\Rightarrow \langle \bar{\psi}(\omega_1)\, \psi(\omega_2)\rangle = \frac{\delta(\omega-\omega')}{i\omega - \omega_0}$$

$$\langle \bar{4}\; \bar{3}\; 2\; 1\rangle = \langle \bar{4}\, 1\rangle\langle \bar{3}\, 2\rangle - \langle \bar{4}\, 2\rangle\langle \bar{3}\, 1\rangle$$ [Wick's thm]

- More complicated example : $H = \sum_i \psi_i^\dagger \psi_i \omega_i + (\psi_i^\dagger \psi_i)(\psi_j^\dagger \psi_j)$

$$\Longrightarrow Z = \int \mathcal{D}\bar{\psi}(\omega)\, \mathcal{D}\psi(\omega) \exp\left[\sum_i \int \left(\bar{\psi}_i (i\omega - \omega_0)\psi_i - \bar{\psi}_i \bar{\psi}_j \psi_i \psi_j\right) d\omega\right)$$

• Now consider renormalization

$$Z(a,b) = \int \mathcal{D}x\, \mathcal{D}y\; e^{-S(a,b;x,y)}, \quad \langle f(x,y)\rangle = \frac{\int dx\,dy\, f(x,y)\, e^{-}}{Z}$$

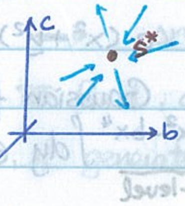
Obviously, $\langle f(x)\rangle = \dfrac{\int dx\, f(x) \left[\int dy\, e^{-S(a,b;x,y)}\right]}{\int dx \left[\int dy\, e^{-S(a,b;x,y)}\right]}$

$$= \frac{\int dx\, f(x)\, e^{-S_{eff}(a',b',c',\dots;x)}}{\int dx\, e^{-S_{eff}(a',b',c',\dots;x)}}$$

which is the basis of renormalization

Given many variables, we can continue this process:

$$(a,b,\dots) \longmapsto (a',b',c',\dots) \longmapsto (a'',b'',c'',\dots) \longmapsto \dots$$



$s^* \longmapsto s^*$: fixed pts.
flow-away direction: relevant
flow-toward direction: irrelevan
non-flow direction: marginal

• Example: Consider a lattice stat-mech model: $S = S(\varphi(i))$ depending on parameter, $\langle \varphi(i)\varphi(j)\rangle \sim e^{-|i-j|/\xi}$ as $|i-j|$

$\sim \dfrac{1}{|i-j|^{d-2+\eta}}$

Now, $\langle \varphi(i)\varphi(j)\rangle = G(r) = \int G(k)\, e^{ik\cdot r}\, dk$

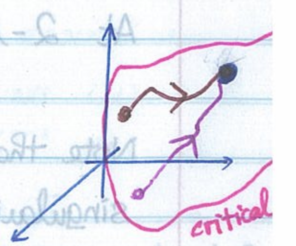
where $\langle \varphi(k)\varphi(k')\rangle = \delta(k-k')\, G(k')$

Long distance correlation depends on $G(k)$ near $k=0$.

By renormalization,

$$Z = \int \Pi\, d\varphi(k)\; e^{-S(\varphi(k))}$$

$$= \int \Pi_{k<\Lambda}\, d\varphi_<(k)\; \underbrace{\Pi_{k>\Lambda}\, d\varphi_>(k)\; e^{-S(\varphi_<,\varphi_>;\,a,b,\dots)}}$$



- Problem: Given H_0 known and gapless, consider $H = H_0 + \lambda H_1$, is H gapless

Renormalization approach: $Z_0 = \int \mathcal{D}\psi \, e^{-S_0}$, design RG s.t. $S_0 \mapsto S_0$

Then consider $Z = \int \mathcal{D}\psi \, e^{-(S_0 + \lambda S_1)}$, use RG $S_0 + \lambda S_1 \mapsto \ldots$

Determine if λS_1 is irrelevant or not.

- Issue: We need to perform the integral $\int dy \, e^{-S(a,b,\ldots;x,y)}$

Example: $S(a,b;x,y) = a(x^2+y^2) - b(x+y)^4$

For $b=0$, we have a Gaussian fixed pt., $(a=a_0, b=0) \mapsto (a=a_0, b=0)$

For $b \neq 0$, $e^{-S_{\text{eff}}} = \underbrace{e^{-ax^2-bx^4}}_{\text{tree-level}} \int dy \, e^{-ay^2} e^{-b(4x^3y + 6x^2y^2 + 4xy^3 + 6y^4)}$

Observe that $\dfrac{\int dy \, e^{-V(x,y;b,\ldots)} e^{-ay^2}}{\int e^{-ay^2} dy} = \langle e^{-V} \rangle_0$

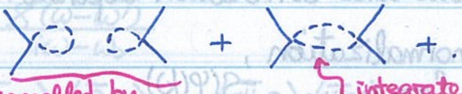
And $\langle e^{-V} \rangle_0 = e^{-\langle V \rangle_0 + \frac{1}{2}[\langle V^2 \rangle_0 - \langle V \rangle_0^2] + \ldots}$

At one loop (i.e. $\langle V \rangle_0$), $\langle y \rangle_0 = \langle y^3 \rangle_0 = 0$, and y^4-term contribute only a constant

$\Longrightarrow e^{-S_{\text{eff}}} = e^{-ax^2 - bx^4 - 6b\langle y^2 \rangle_0 x^2}$, with $\langle y^2 \rangle_0 = 1/a$

Diagramatically, we have

At 2-loop, we have



cancelled by $\langle V^2 \rangle$ & $\langle V \rangle^2$

integrate from, e.g. $\Lambda/2$ to Λ.

Note that the renormalization flow is analytic even if there are singularities in Green functions.