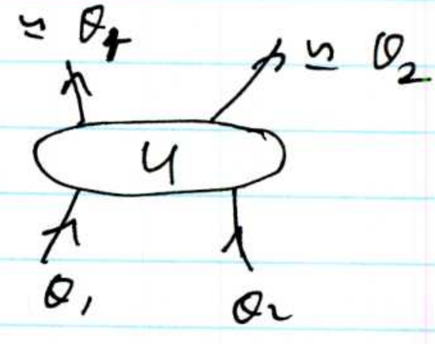
RG for fermions III Shankar 7/7/08.

<u>Recall:</u> For circular FS, at tree level we have just



I $u(\theta_1, \theta_2) = u(\theta_1 - \theta_2) = F(\theta)$

$\theta_3 \simeq \theta_1$ i.e $\theta_3 = \theta_1 \pm \Lambda/K_F$

$\theta_4 = \theta_2 \pm \Lambda/K_F$

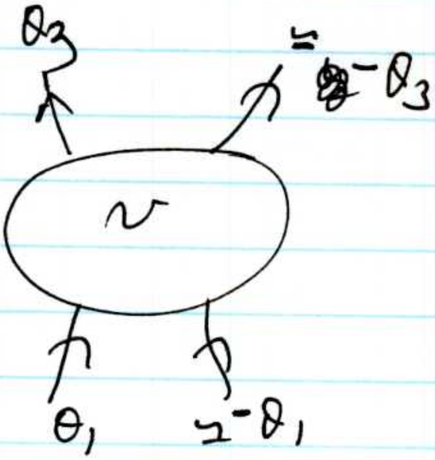
$u(\theta) = \sum u_m e^{im\theta}$

$\downarrow$ Landau.

i.e long as $\Lambda \neq 0$; non forward scattering is allowed, but u does not vary in this tiny range

II $v(\theta_1, \theta_3) = v(\theta_3 - \theta_1)$

BCS channel



again $\theta_2 \simeq -\theta_1 \pm \Lambda/K_F$

so BCS is not $P = 0$ but $P \simeq \Lambda/K_F$.

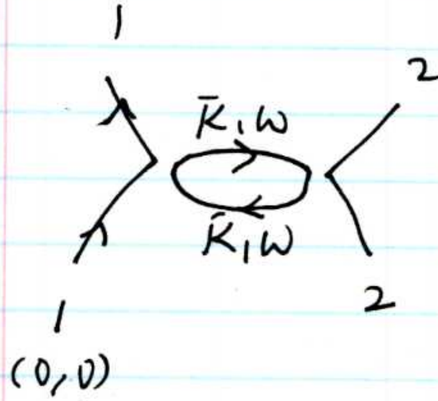
But v is evaluated at $\theta_2 = -\theta_1$; $\theta_4 = -\theta_3$.

Small wiggle room of order Λ/K_F; but v is assumed constant over that range.

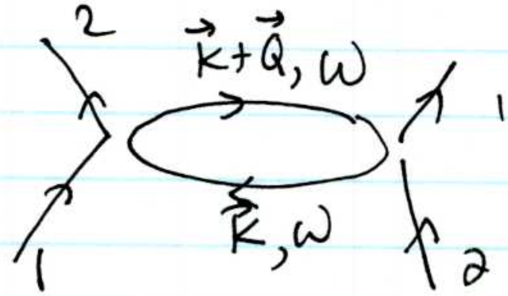
Need to go to 1 loop for u and v.

For u choose $\omega = k = 0$ at each leg.

u:

$\delta u =$ 

$= 0$ because poles on same ~~side~~ half plane, we need particle - hole

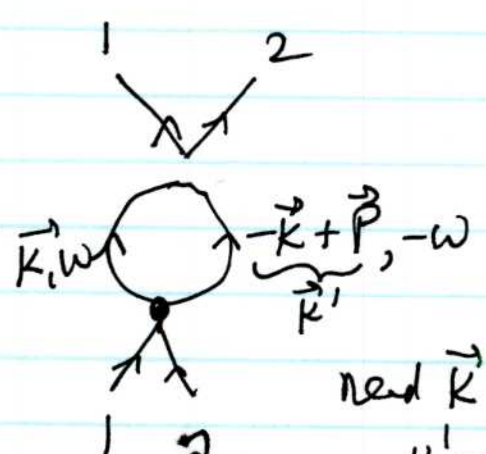
$-$ 

$Q \simeq k_F$

K lies in $d\Lambda$ of $-\Lambda$, needs to absorb Q and scatter to $d\Lambda$ of $+\Lambda$.

$\delta\theta = \frac{d\Lambda}{k_F}$, $dk = d\Lambda$

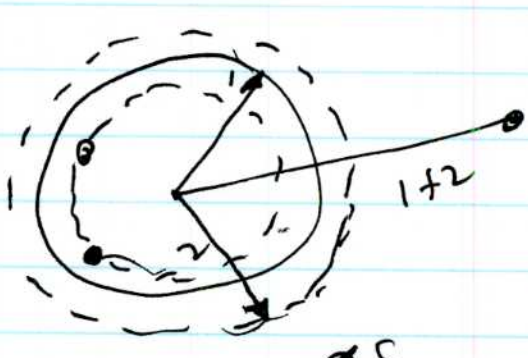
$du \sim \frac{d\Lambda^2}{\Lambda}$

$-\frac{1}{2}$ 

$P \simeq k_F$

need $\vec{K} \simeq 1$

$K' \simeq 2$



$\delta\theta$ is small

dK is small

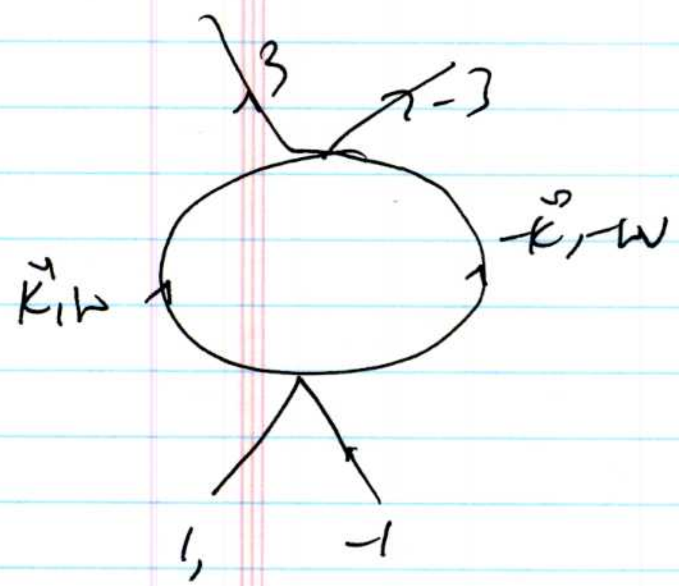
$du \sim \frac{(d\Lambda)^2}{\Lambda}$

$-\vec{K} + \vec{P} = FS$

2 choices for $-K$, $\delta\theta \simeq \delta\Lambda / k_F$

BCS coupling

$\delta V = \delta$  $= \underbrace{2S + ZS'}_{\frac{\delta\Lambda^2}{\Lambda}} + \underbrace{BCS}_{\text{conside}}$



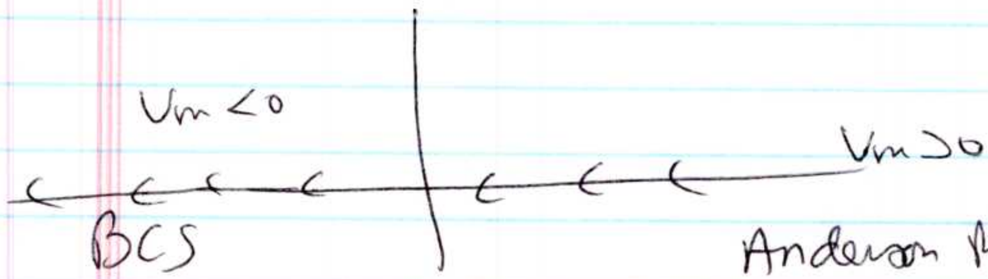
if $\vec{k}$ lies in shell so does $-\vec{k}$ for all θ

$$dV(\theta_3-\theta_1) = \int d\theta \int d\omega \int_{\Lambda\pm\delta\Lambda} dk \, \frac{v(\theta-\theta_1)\, v(\theta_3-\theta)\, d\theta \, d\omega \, dk}{(i\omega-k)(-i\omega-k)}$$

$$\frac{dV}{dt} \simeq -c\int v(\theta-\theta_1)\, v(\theta_3-\theta)\, d\theta$$

$$V(\theta) = \sum_m V_m e^{im\theta}$$

$$\frac{dV_m}{dt} = -c V_m^2$$



Anderson Morel reduction

$$V_m(t) = \frac{V_m(0)}{1 + t V_m(0)} \simeq \frac{1}{t} = \frac{1}{\ln \frac{k_F}{\omega_D}}$$

Let all $V_m \geq 0$ or +ve, or $T > T_c$.

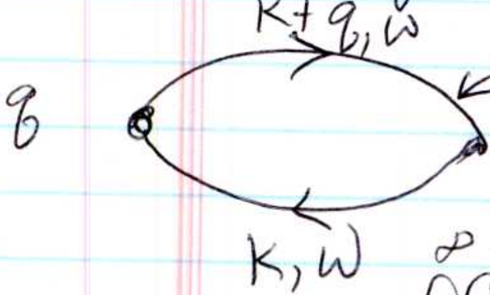
> Still need to solve theory with $U(\theta) \equiv F(\theta)$, and a small bandwidth Λ.

Consider $\chi(q,0) = \langle \rho(q,0)\, \rho(-q,0) \rangle$

compressibility

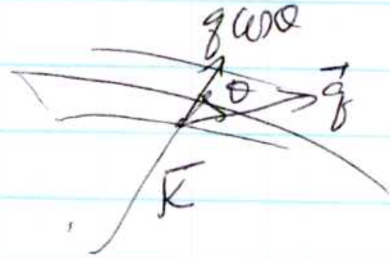
$\chi \cdot \langle \rho(q,0)\, \rho(-q,0) \rangle$

Free field.



lie in $\underline{|k| < \Lambda}$

$\underline{\text{not } |k| = \Lambda}$



$$\chi_0 = \int_{-\infty}^{\infty}\int_{-\Lambda}^{\Lambda}\int_0^{2\pi} \frac{d\omega\, dk\, d\theta}{[i\omega - E(k)][i\omega - E(k+q)]}$$

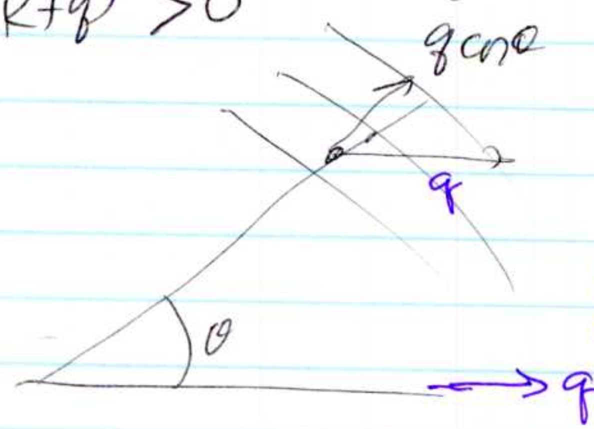
want $E(k) < 0$

$E(k+q) > 0$ (or vice versa)

$k < 0$

$k + q\cos\theta > 0$ ie

$$\int_{-q\cos\theta}^{0} dk$$



$\vec{q}$ choose q along x

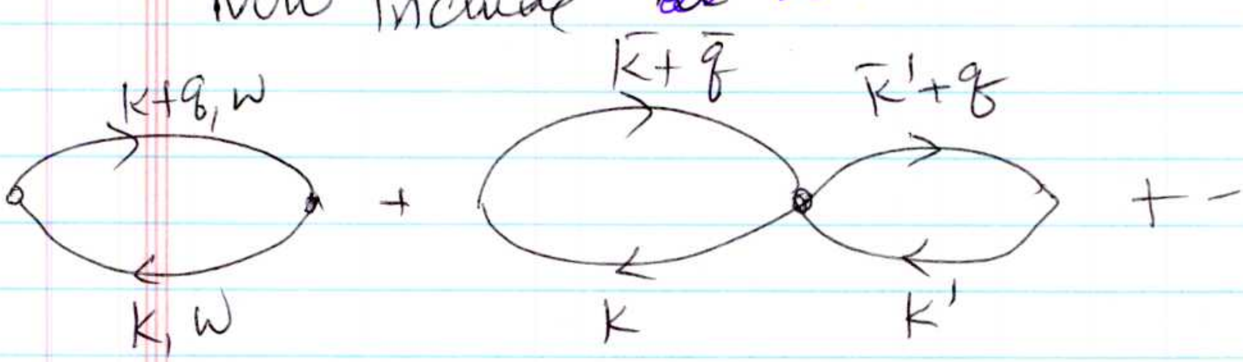
$$= \int_{-q\cos\theta}^{0} \frac{dk\, d\theta}{q\cos\theta}$$

$E(k+q) - E(k) = q\cos\theta$ ($v_F = 1$)

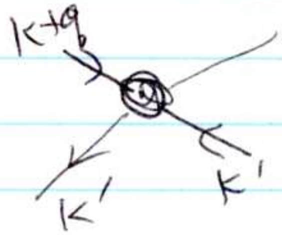
$$= \int d\theta$$

$\chi_0 =$ constant ($\propto$ density of states)

Now include ~~a~~ u:



$+ \; \cdots$



$= u(\theta_k - \theta_{k'})$

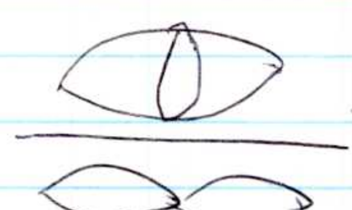
$$\int \underline{u(\theta_k - \theta_{k'})\, d\theta} \qquad \int \frac{d\omega\, dk}{(i\omega -)(\;)} = \text{const}$$

$\rightarrow u_0$

So $\chi = \dfrac{\chi_0}{1 - \chi_0 u_0}$

<u>Note:</u>

1. RPA is exact because as $\Lambda \to 0$ nothing else is possible



$= \dfrac{\Lambda}{k_F}$

2. Even if $\Lambda \to 0$ ~~the~~ $\int dk$ is order 1

3. Why u does not flow:  is Λ-indep (contribution is δ function at FS : because $\int_{-\Lambda}^{\Lambda} \delta(FS)\,dk = \Lambda$-independent

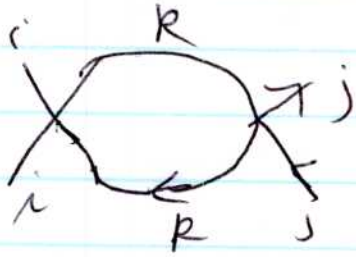
$\frac{1}{N}$ expansion

$$L = \sum \bar{\psi}_i \gamma \psi_i + \frac{g^2}{N} \sum_{ij} (\bar{\psi}_i \psi_i)(\bar{\psi}_j \psi_j)$$

$= $

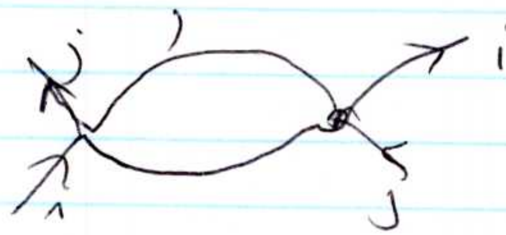
$\sim \frac{1}{N}$

$+$



$\frac{g^2}{N}\frac{g^2}{N} \cdot N \simeq \frac{1}{N}$

$+$



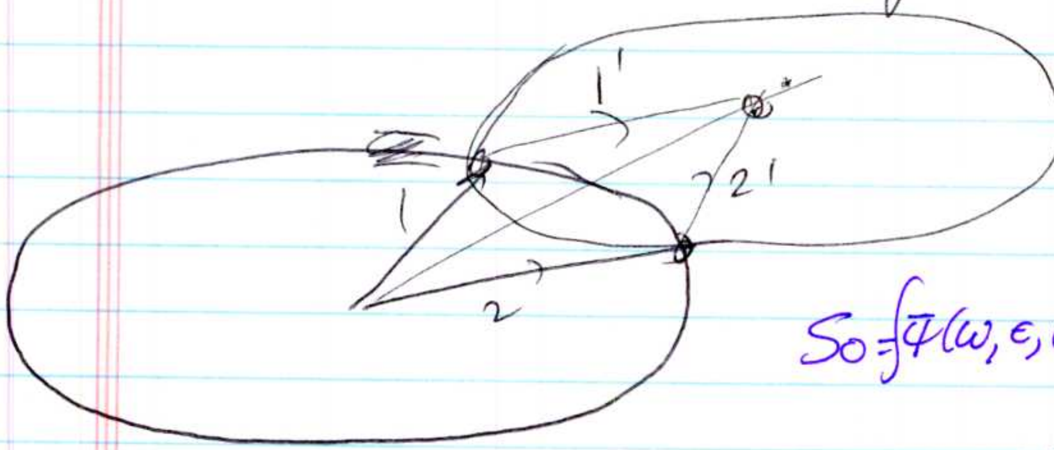
$\left(\frac{g^2}{N}\right)^2 = \frac{1}{N^2}$

$+$

Here

$N = \frac{k_F}{\Lambda}$

Non circular Fermi surface : Again Fermi liquid

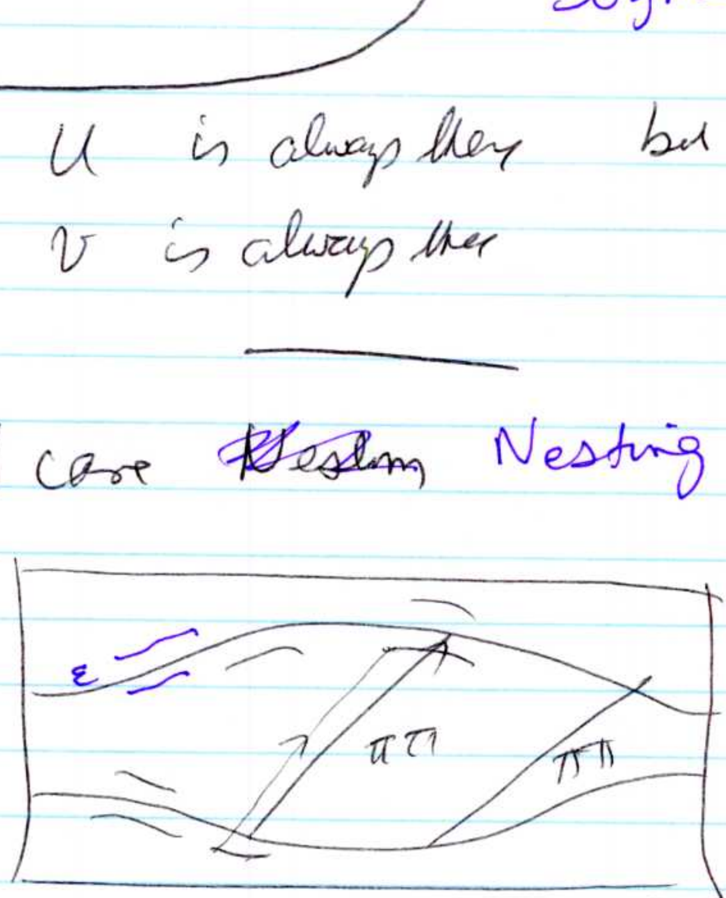


$S_0 = \int \bar{\Psi}(\omega,\epsilon,\theta)\,(i\omega-\epsilon)\,\psi(\omega\epsilon\theta)\,J(\theta)\,d\omega\,d\epsilon\,d\theta$

u is always there but $u(\theta_1,\theta_2) \neq u(\theta_1,-\theta_2)$

v is always there $v(\theta_1,\theta_3) \neq v(\theta_1,-\theta_3)$

Special case ~~Neslom~~ Nesting



$S_0 = \int \bar{\Psi}(\omega,\epsilon,\theta)\,(i\omega-\epsilon)\,\psi\;d\omega\,d\epsilon\,d\theta$

$E(K+\pi\pi) = -E(K)$
$\forall K$

So FSurface $(E=0)$ goes into itself

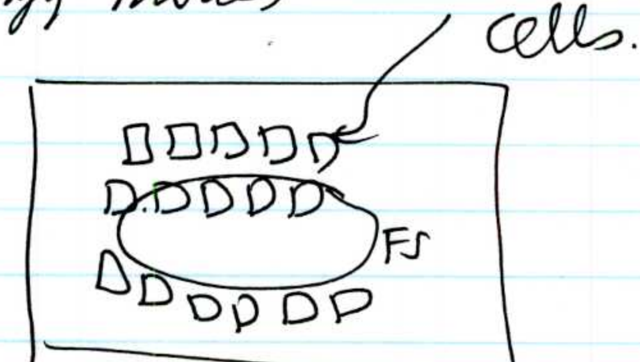
$E = 1 \rightarrow E=-1$ etc

new function $W = u(K_1, K_2, K_1+\pi\pi, K_2+\pi\pi)$

$\frac{dW}{dl} = \int WW \rightarrow$ CDW

Numerical approaches

It was suggested in RMP that we divide k space into cells and eliminate high energy modes



Can see if attraction emerges in a repulsive Hubbard model. This was carried out in detail by Zanchi and Schulz. Since then many others have done more of this (called NRG).

The ε- expansion (with Senthil)

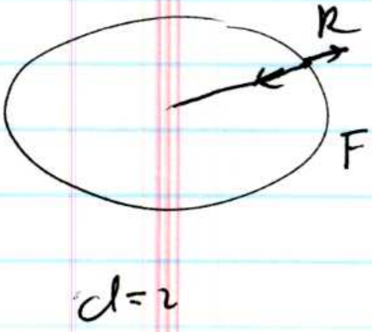
Fermion in any $d > 1$ is

$$S_0 = \int \bar{\psi}\,(i\omega - k)\,\psi(\omega, k, \theta)$$

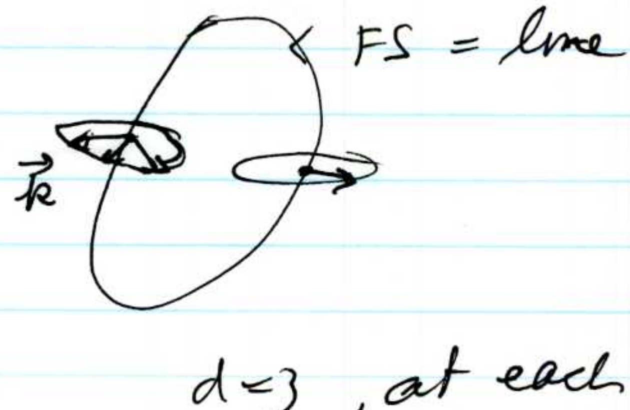
θ: angles on S^{d-1}

any quartic coupling is marginal because this is so in 1+1 dimensions

Let codimension of Fermion grow from 1 to 2 as $1+\varepsilon$. Begin in 1+1 d and in 2+1 dimension with a Dirac line in 3D



$d=2$

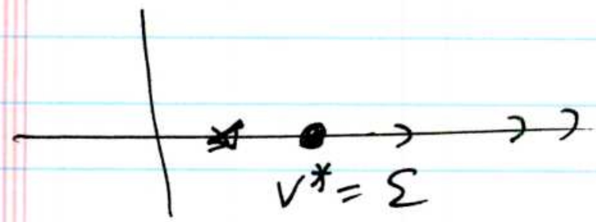


$d=3$, at each point in FS, $\vec{k} = 2$ dimens

$$S_0 = \int \bar{\psi}(w, \vec{k}, \theta)(iw - k)\psi(w\vec{k}\theta)\, d^{1+\varepsilon}k\, dw\, d\theta$$

$S_V =$ BCS term

$V \to V S^{-\varepsilon}$ in $2+\varepsilon$ at tree level



$V^* = \varepsilon$ at 1 loop.

$$E(\delta V, h) = S^{-(2+\varepsilon)} E(\delta V S^{\varepsilon}, h S^{1+\varepsilon}) \quad V S^{-\varepsilon}(1 + \varepsilon \log S)$$

Hertz millis.

Integrate out fermions.

$$S = \int \bar{\Delta}\left(r + (w^2+q^2)^{\varepsilon/2}\right)\Delta\, d^{2+\varepsilon}q\, dw$$

$$+ \int \bar{\Delta}\Delta\bar{\Delta}\Delta\, u_4 \ldots$$

$$E(r, u) \simeq S^{-(3+\varepsilon)} E(r_s, u_s)$$

Can they agree?

They do if you realize u_4 is singular and watch out for dangerous irrelevant behavior of u

Why do the RG

1. Qualitative understanding of Universality
2. Since β-function is analytic even at critical point, perturbative calculations of β allowed even at its zero.
3. Effective theory may be simpler (linear dispersion, few relevant or marginal couplings)
4. Unbiased treatment of competing instabilities
5. Defines nonperturbatively the renormalization in QFT.

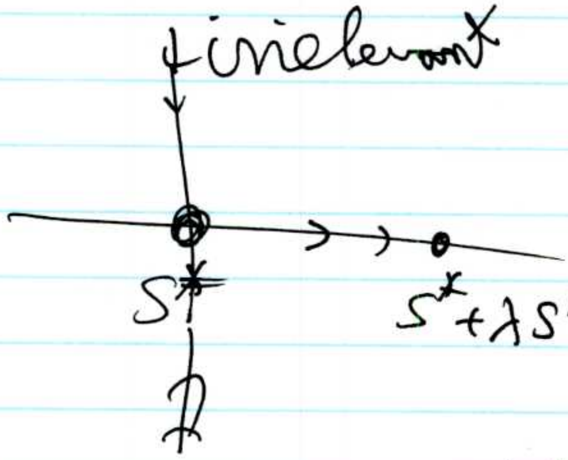
Big mystery of Renormalizability in QFT:

<u>Q:</u> How come all infinities in all Green's functions go away by choosing one or two $(m(\Lambda), g(\Lambda))$ in a cut-off dependent way.

<u>Note:</u> Removing ∞'s = removing Λ-dependence

<u>Answer.</u> We are dealing with a fixed point S^* with just one or two relevant directions. Consider just 1 relevant direction

<u>Note</u>



1) $m = \frac{1}{\xi}$ in field theory

2. $\xi = t^{-\nu}$ ν = critical exponent

$\xi(S^*) = \infty$, $m = 0$

Since $\xi = t^{-\nu} a$ $\quad$ a = lattice spacing usually $= 1$.

I can get any ξ I want by choosing t small enough or getting close enough to S^*.

Say $m_\pi = 100$ MeV

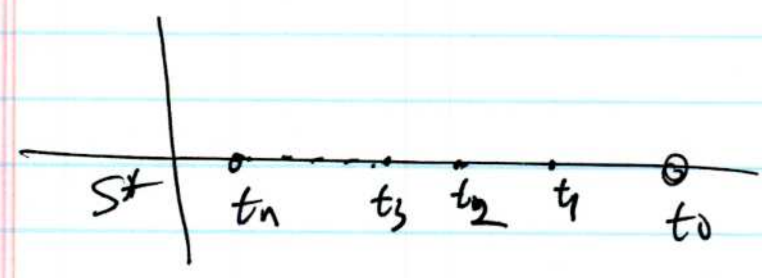
$\xi = 1$ Fermi

If $a = 1$F need

$\xi = 1$ F $\quad t = t_0$

if $a = 2^{-n}$ $\xi_n = 2^n$

i.e. $\xi(t_n) = 2^n$



For every lattice size $a_n = 2^{-n}$ (or $\Lambda = 2^n$)

I give a t_n such that $m_\pi = 100$ MeV

$\xi_\pi = 1$ Fermi

This is renormalization.

More precisely

$$t_n^{-\nu} = 2^n$$

$$t_n = 2^{-n/\nu} \qquad \text{But } 2^{-n} = a_n$$

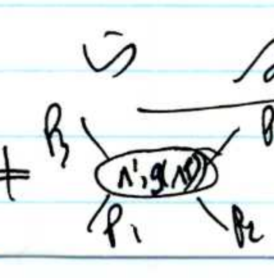
$$= (a_n)^{+1/\nu} \qquad\qquad \Lambda_n = \frac{1}{a_n} = \text{momentum cut-off}$$

$$= (\Lambda_n)^{-1/\nu}$$

$$\boxed{t(\Lambda) = \Lambda^{-1/\nu}}$$

Will all other quantities be Λ- independent as well? Yes because RG says t_n at lattice $a_n = 2^{-n}$ is effective coupling for the whole theory as long as we focus on long distance. But in QFT we don't let $p \to 0$, but let $\Lambda \to \infty$ so that any p is small p. i.e.

$\neq$



they differ by p/Λ terms. But $p/\Lambda \to 0$ as $\Lambda \to \infty$ $\forall p$.