1 July 08

RG for Fermion I. Notes / Shankar

RG will be applied to any problem that looks like a classical partition function and this covers almost everything

Eg. 1. Classical stat Mech

2. Quantum problems written as path integrals. Two examples Boson & Fermionic osscillators

Bosonic

$$\mathcal{H} = \omega_0 a^\dagger a$$

$$[a, a^\dagger] = 1$$

Use $$I = \int \frac{d\bar{z}\, dz}{2\pi i} |z\rangle\langle\bar{z}| e^{-\bar{z}z}$$

where $a|z\rangle = z|z\rangle \qquad \langle\bar{z}|a^\dagger = \langle\bar{z}|\bar{z}$

to write

$$Z = \mathrm{Tr}\, e^{-\beta H} \qquad (\text{for } \beta \to \infty,\ T=0)$$

$$= \int \Pi\, d\bar{z}(\omega)\, dz(\omega)\, e^{\int \bar{z}(\omega)(i\omega - \omega_0) z(\omega) \frac{d\omega}{2\pi}}$$

$$\langle \bar{z}(\omega_2^*)\, z(\omega_1) \rangle = \frac{2\pi\, \delta(\omega_2 - \omega_1^*)}{i\omega_2 - \omega_0} \equiv \langle \bar{2}\, 1 \rangle$$

$$\langle \bar{4}\, \bar{3}\, 2\, 1 \rangle = \langle \bar{4}\, 1 \rangle \langle \bar{3}\, 2 \rangle + \langle \bar{4}\, 2 \rangle \langle \bar{3}\, 1 \rangle$$

if there are many bosons

$$H = \sum_i a_i^\dagger a_i \omega_0^i$$

make that many z_i's and $\bar{z}_i$'s.

<u>Interactions</u>

$$H_{int} = \lambda\, a_i^\dagger a_j^\dagger a_k a_l$$

$$\to \lambda \int \bar{z}_i(\omega_1)\, \bar{z}_j(\omega_2)\, z_k(\omega_3)\, z_l(\omega_4)$$

$$\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

Fermi oscillators

$$H = \omega_0 \psi^\dagger \psi \qquad \{\psi, \psi^\dagger\} = 1$$
$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0$$

$$I = \int d\bar{\psi}\, d\psi \; |\psi\rangle\langle\bar{\psi}|\, e^{-\bar{\psi}\psi}$$

$$\psi|\psi\rangle = \psi|\psi\rangle \qquad \psi^2 = 0$$
(implies $\psi^2 = 0$)
$$\langle\bar{\psi}|\psi^\dagger = \langle\bar{\psi}|\bar{\psi}$$

$$Z = \int \pi\, d\bar{\psi}(\omega)\, d\psi(\omega)\, e^{\int \bar{\psi}(\omega)(i\omega - \omega_0)\psi(\omega)\frac{d\omega}{2\pi}}$$

$$\langle \bar{4}\, \bar{3}\, 2\, 1\rangle = \langle \bar{4} 1\rangle\langle \bar{3} 2\rangle - \langle \bar{4} 2\rangle\langle \bar{3} 1\rangle$$

$$\langle \bar{2}\, 1\rangle = \frac{2\pi\, \delta(\omega_2 - \omega_1)}{i\omega_2 - \omega_0}$$

$$H_{int} = \lambda\, \psi_i^\dagger\, \psi_j^\dagger\, \psi_k\, \psi_l$$
(all $\psi^\dagger$ to the left)

$$\rightarrow \lambda \int \bar{\psi}_i(\omega_4) \cdots \psi_l(\omega_1)\; \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

<u>Note</u> ψ, $\bar{\psi}$ are Grassmann #s. they are not "big" or small. There are rules for integration. See my RMP for details

What is it
Why do it
How to do it

Renormalization :

What) $Z(a,b) = \int dx\,dy\; e^{-S(x,y,a,b)}$

$$\langle f(x,y)\rangle = \frac{\int f(x,y)\, e^{-S}\, dx\,dy}{Z}$$

If we care only about x, then

$$\langle f(x)\rangle = \frac{\int f(x) \left[\int e^{-S(a,b,x,y)}\, dy\right] dx}{\int \left[\int e^{-S(a,b,x,y)}\, dy\right] dx}$$

$$= \frac{\int f(x)\, e^{-S_{eff}(a',b',\dots x)}\, dx}{\int e^{-S_{eff}(a'b'\dots x)}\, dx}$$

S_{eff} = effective action for x, that reproduces fully the contribution of y.

$a, b, \dots \rightarrow a' b' \dots$ is called

Renormalization

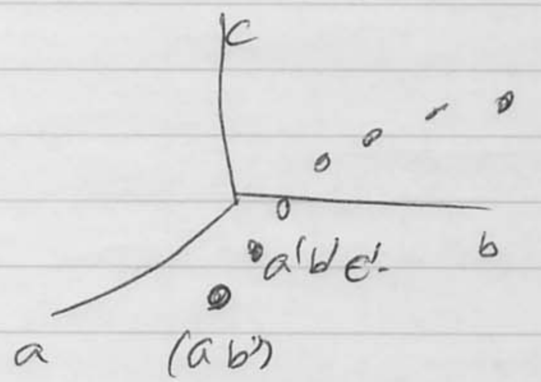
Eg: $S = a(x^2+y^2) + b(x+y)^4$

In real life

$$x = x_1 \cdots x_N$$
$$y = y_1 \cdots y_M$$

ie y's are eliminated the point moves in parameter space, new coupling arise.

$$(a\, b \cdots) \rightarrow (a' b' c' \cdots) \rightarrow (a'' b'' \cdots)$$



if $S_{eff} = S$ ie $(a', b'\, c' \cdots) = (a, b, c \cdots)$

we have a Fixed point

Near fixed point if you move away a bit and run the flow, the difference

1) can grow relevant perturbation
2) " shrink irrelevant "
3) Stay put marginal.

(Why RG)

In classical stat mech

$$G(r) = \langle \phi(r)\, \varphi(0) \rangle \underset{r\to\infty}{\sim} e^{-r/\xi}$$

in general, but at "critical point)

$$G(r) \underset{r\to\infty}{\to} \frac{1}{r^{d-2+\eta}}$$

Q η is same for many different systems when they become critical. How can this be?

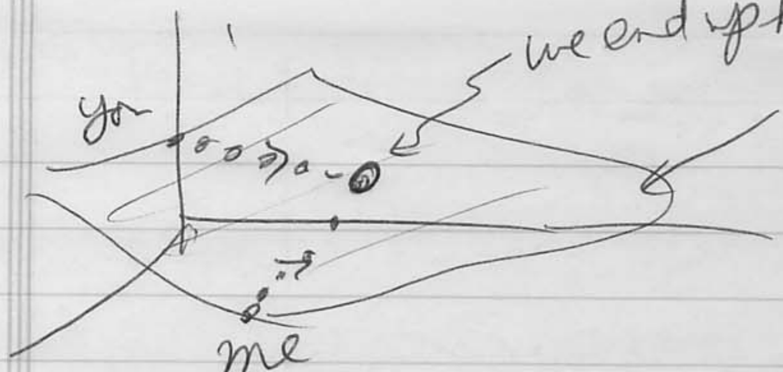
A: $G(r)$ for $\to\infty$ is controlled by $G(k)$ for ~~ooo~~ $k\to 0$

$G(k)$: $\langle \varphi(k)\, \varphi(k') \rangle = \delta(k-k')\, G(k)$

So $x = G(k)$ $k < \Lambda$ $\quad \Lambda =$ cut-off

$y = G(k)$ $k \geq \Lambda$

you and I can differ at beginning, but if all y's are eliminated our effective theories can end up being the same at critical point. So η ...



we end up here at Fixed point

all systems here are critical

i.e. as far as long distance physics goes our effective theories are same (as fixed point) though starting points are not.

Another case is a quantum problem

Let $H = H_0 + \lambda H_1$

Does H_1 make a difference?

RG approach

Let $H_0 \longrightarrow \int (d\Psi)\, e^{S_0}$

It will turn out $S_0 \to S_0$ under RG

now add H_1

$$\int (d\psi)\, e^{S_0 + \lambda S_1}$$

do the RG. If $\lambda \to 0$ under RG it doesn't matter

$\lambda \to \infty$ " " it matters

$\lambda \to \lambda$ marginal case

How

$$Z = \int dx\,dy\; e^{-a(x^2+y^2) - b(x+y)^4}$$

$$= \int dx\; e^{-ax^2 - bx^4} \quad \underline{\underline{\text{tree}}}\ \text{level}$$

$$\times \underbrace{\int dy\; e^{-ay^2 - b(4x^3y + 4xy^3 + 6x^2y^2 + y^4)}}_{\text{consider just this}}$$

$$\int dy\; e^{-V(x,y,a,b)}\, e^{-ay^2}$$

$$= \frac{\int dy\, e^{-V} e^{-ay^2}}{\int dy\, e^{-ay^2}} \quad \left(\int e^{-ay^2}\, dy \right)$$

ignore this guy which does not depend on x

$$\Downarrow$$

$$= \langle e^{-V} \rangle_0 \rightarrow \langle\,\rangle_0 = \text{avg. w.r.t. } \int dy\, e^{-ay^2}$$

$$\langle e^{-V} \rangle = e^{-\langle V \rangle + \frac{1}{2}[\langle V^2 \rangle - \langle V \rangle^2] + \cdots}$$

Consider $\langle V \rangle$

$$V = -b(4x^3y + 4xy^3 + 6x^2y^2 + y^4)$$

$$\langle V\rangle = -b(0 + 0 + 6x^2\langle y^2\rangle + \langle y^4\rangle)$$

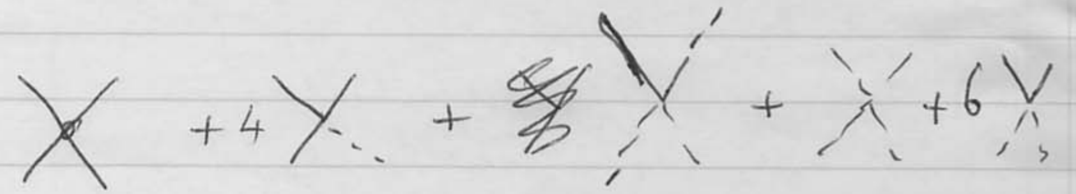
$$\langle y^2\rangle = \frac{\int y^2 e^{-ay^2}}{\int} = \frac{1}{a}$$

So

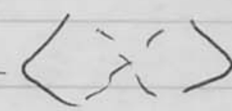
$$e^{-S_{eff}(x)} = e^{-ax^2 - bx^4}\, e^{-6x^2 b\langle y^2\rangle}$$

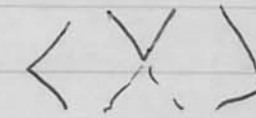
(forget $\langle y^4\rangle$)

$$a' = a$$
$$b' = b + 6b\langle y^2\rangle$$

$(x+y)^4 =$ 

x^4 $\quad 4x^3y \quad + 4xy^3 \quad + y^4 +$

$\langle\ \rangle = x^4 + 0 + 0 +$ 

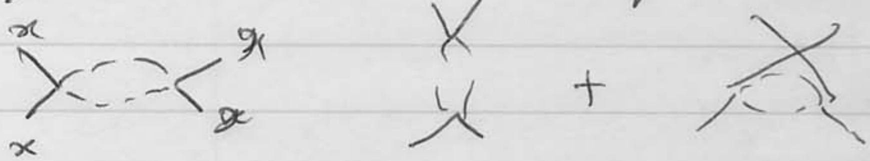
$+ 6$ 

$= x^4 +$  $+ 6$ 

$\langle y^4\rangle$ $\quad\uparrow \langle y^2\rangle = \frac{1}{a}$

At next order, you $\langle V^2\rangle - \langle V\rangle^2$

means connect graphs like



exactly as in φ^4 theory but with loop variables summed only over y's i.e. those being eliminated.

Thus in φ^4 theory if you are eliminating k below $\frac{\Lambda}{s}$ and Λ then

$\delta\lambda =$  $= \lambda^2 \displaystyle\int_{\Lambda/s}^{\Lambda} \frac{1}{k^2+m^2}\,\frac{1}{k^2+m^2}\, d^dk$

+ other diags

~~the diagrams~~

The diagrams for flow are free of <u>infrared</u> and <u>ultraviolet</u> divergences since they are cut off at both ends.