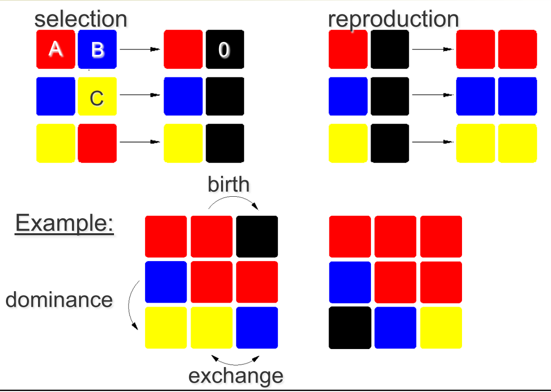
Spatial Models

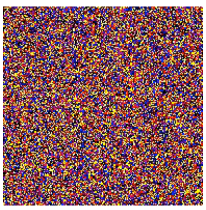
Local Interaction Rules



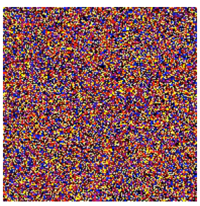
May-Leonard Model on a Lattice (N=L^2)

Add migration (ε)
$A0 \rightarrow 0A$
$AB \rightarrow BA$
...

macroscopic diffusion $D = \dfrac{\varepsilon}{2L^2}$



$D = 3\times10^{-5}$



$D = 3\times10^{-4}$

Stability of Biodiversity?

- For well mixed populations biodiversity is lost!
- Is there a critical value for the diffusivity D below which biodiversity is maintained?
- If yes, what is the spatial structure of the population and how to describe it?
- If yes, what is the nature of the transition?

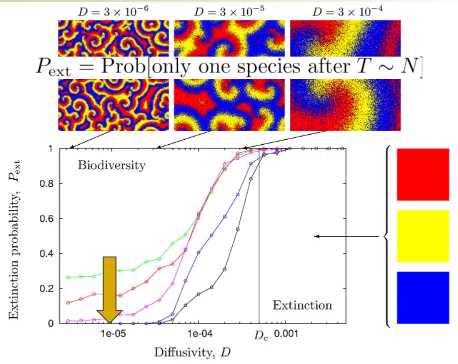
Extensivity: let T be the typical extinction time and N the size of the population (system)

$T/N \to \infty$ super-extensive / stable

$T/N \to O(1)$ extensive / neutral / marginal

$T/N \to 0$ sub-extensive / unstable

Diversity is lost above critical Diffusivity



T. Reichenbach, M. Mobilia and E. Frey, Nature (2007)

Loss of Biodiversity

- For large systems there is a well defined critical/threshold value D_c (μ,σ) for the diffusivity/mobility.
- Loss of biodiversity seems to be related to the size of the spatial structures (spirals) in the population.

Wolfram demonstration project (Martin Weidl)

Reaction-Diffusion equation

$$\partial_t \vec{a}(\vec{r},t) = D\nabla^2 \vec{a} + \vec{F}(\vec{a})$$
$$D = \frac{\epsilon}{2N}$$

$$\partial_t z \quad = \quad D\nabla^2 z + (c_1 - i\omega)z - c_2(1 + ic_3)|z|^2 z$$

Complex Ginzburg-Landau equation

Front propagation

$$\partial_t z(\vec{r},t) = D\Delta z(\vec{r},t) + (c_1 - i\omega) z(\vec{r},t)$$

$$\tilde{z}(\vec{k},t) = \int_{-\infty}^{\infty} d\vec{r}\ z(\vec{r},t) e^{-i\vec{k}\cdot\vec{r}} \qquad \tilde{z}(\vec{k},t) = \tilde{z}(\vec{k})\, e^{-i\Omega(\vec{k})t}$$

$$\Omega(k) = \omega + i(c_1 - Dk^2)$$

$$z(\vec{\xi} = \vec{x} - \vec{v}^* t, t) = \int \frac{d^2k}{(2\pi)^2}\, \tilde{z}(\vec{k})\, e^{i\vec{k}\cdot\vec{\xi} - i[\Omega(\vec{k}) - \vec{v}^*\cdot\vec{k}]t}$$

$$\left.\frac{d\Omega(k)}{dk}\right|_{k_*} = \frac{\operatorname{Im}\Omega(k_*)}{\operatorname{Im} k_*} \equiv v^*$$

Stochastic PDE

Look for a description in terms of local densities

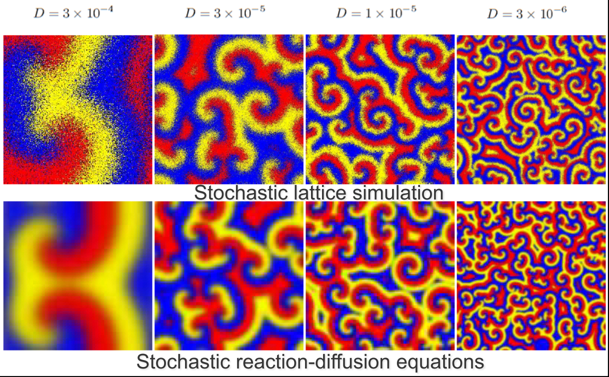
$$\partial_t \vec{a}(\vec{r},t) = D\Delta\vec{a}(\vec{r},t) + \mathcal{A}[\vec{a}] + \mathcal{C}[\vec{a}]\cdot\vec{\xi}$$

$$\mathcal{A} = \vec{F}$$
$$\mathcal{C}_A \quad = \frac{1}{\sqrt{N}}\sqrt{a(\vec{r},t)\left[\mu(1-\rho(\vec{r},t)) + \sigma c(\vec{r},t)\right]}$$
$$\langle \xi_i(\vec{r},t)\xi_j(\vec{r}',t')\rangle = \delta_{ij}\delta(\vec{r}-\vec{r}')\delta(t-t')$$

- stochastic partial differential equation
- particle exchange = diffusion
- reactions = drift & multiplicative noise ($\sim N^{-1/2}$)

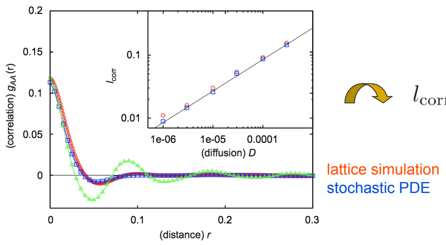
T. Reichenbach, M. Mobilia and E. Frey, PRL (2007)

How good is such a description?



Spatial Correlations

$$g_{ij}(\mathbf{r},0) = \langle a_i(\mathbf{r},t)a_j(\mathbf{0},t)\rangle - \langle a_i(\mathbf{r},t)\rangle\langle a_j(\mathbf{0},t)\rangle$$

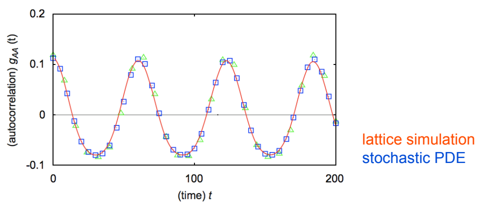


$l_{\mathrm{corr}} \sim \sqrt{D}$

raising the diffusion constant D increases the size of the spirals

Temporal Correlations

$$g_{ij}(\mathbf{0},t) = \langle a_i(\mathbf{r},t)a_j(\mathbf{r},0)\rangle - \langle a_i(\mathbf{r},t)\rangle\langle a_j(\mathbf{r},0)\rangle$$

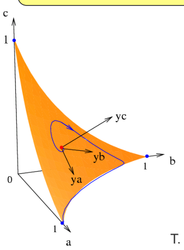


the rotation frequency is a function of the reaction rates μ and σ only

Complex Ginzburg Landau equation

$$\partial_t \vec{a}(\vec{r},t) = D\Delta\vec{a}(\vec{r},t) + \mathcal{A}[\vec{a}] + \mathcal{C}[\vec{a}]\cdot\vec{\xi}$$

$$\partial_t z \quad = \quad D\nabla^2 z + (c_1 - i\omega)z - c_2(1+ic_3)|z|^2 z$$

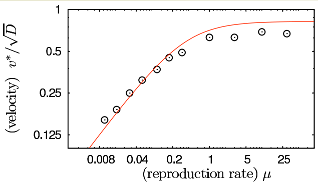


Project onto reactive manifold

Neglect the noise

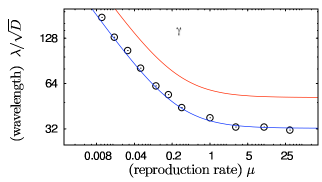
T. Reichenbach, M. Mobilia and E. Frey, J. Theor. Biol. 254, 368 (2008)

Compare CGLE and stochastic PDE's



spreading velocity

$$v^* = 2\sqrt{D}\sqrt{\frac{1}{2}\frac{\mu\sigma}{3\mu+\sigma}}$$

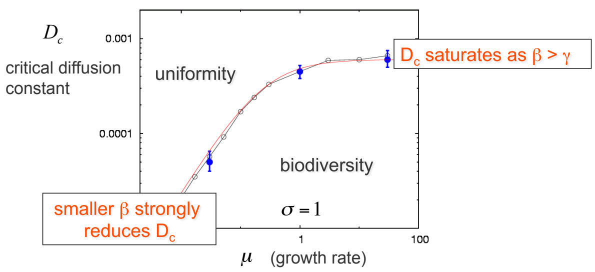


wavelength of spirals

$$\lambda = \frac{2\pi c_3\sqrt{D}}{\sqrt{c_1}\left(1-\sqrt{1+c_3^2}\right)}$$

State Diagram

The CGLE allows to calculate the state diagram



Conclusions

- Well-mixed populations: law of the weakest.
- Local interaction: pattern formation and biodiversity.
- There is a mobility-threshold above which biodiversity is lost.

Thanks to

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