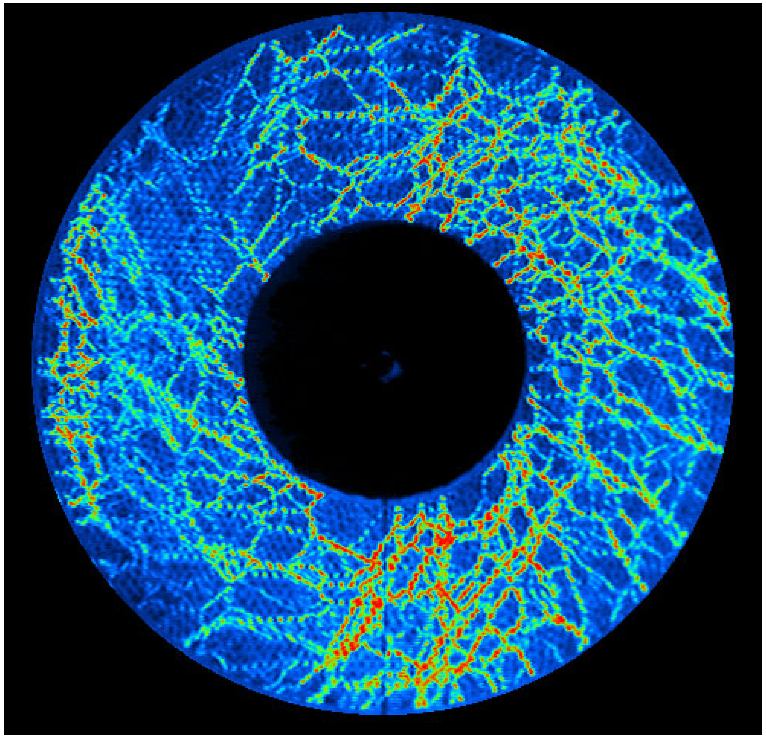

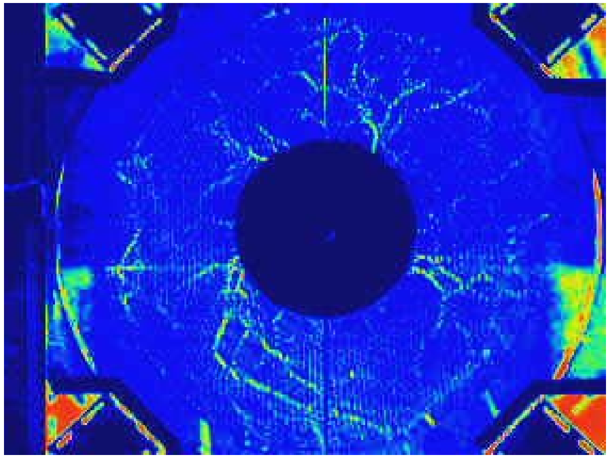

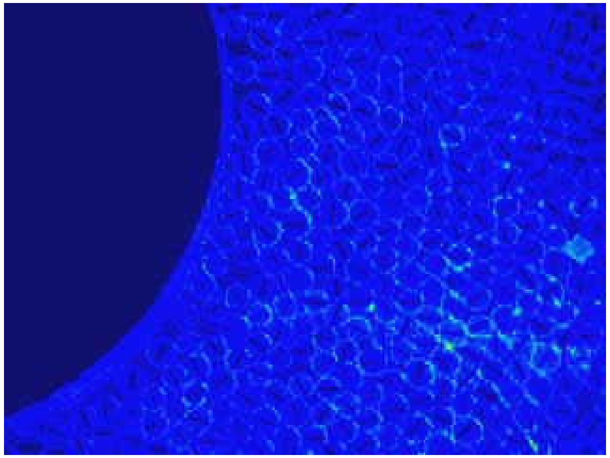

This false color image is taken from Dan Howell's experiments. This is a 2D experiment in which a collection of disks undergoes steady shearing. The red regions mean large local force, and the blue regions mean weak local force. The stress chains show in red. The key point is that on at least the scale of this experiment, forces in granular systems are inhomogeneous and itermittent if the system is deformed. We detect the forces by means of photoelasticity: when the grains deform, they rotate the polarization of light passing through them.

Howell, Behringer, Veje, *PRL* 1999 and Veje, Howell, Behringer, *PRE* 1999

Critical Scaling of a Sheared Granular Material at the Jamming Transition

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Daniel Valdez-Balderas, *University of Rochester*

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outline

- introduction - the *Jamming Phase Diagram*

- scaling ansatz for jamming at $T = 0$

- simulations in 2D at finite fixed shear strain rate

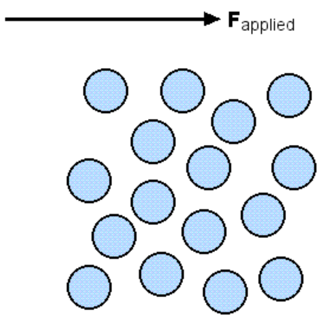
  large system with $N = 65{,}536$ particles
  critical exponents for viscosity and yield stress

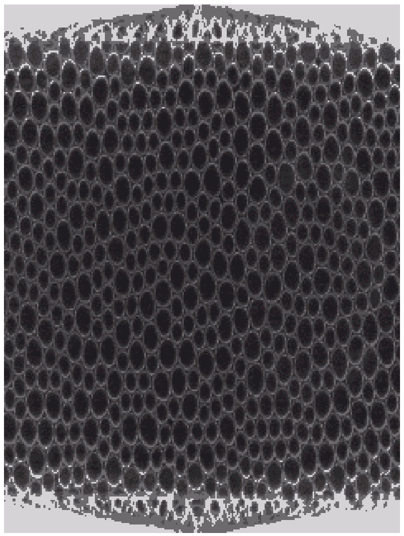
- quasi-static simulations in 2D of slowly sheared system

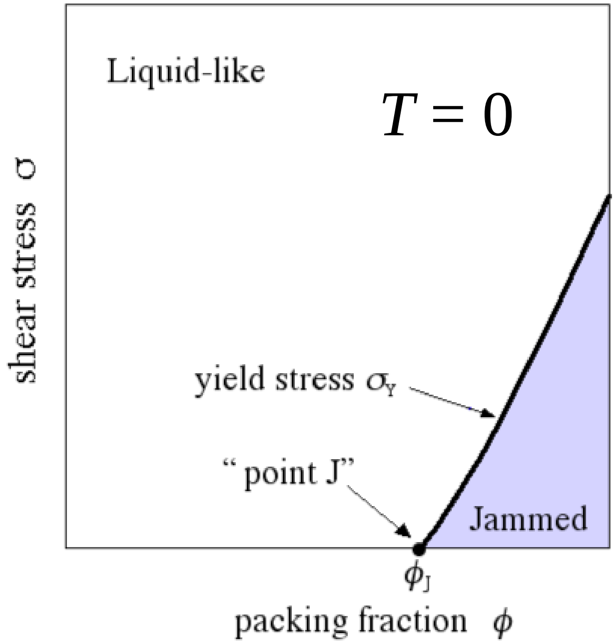
  finite size scaling with $N = 64$ to 1024 particles
  critical exponents for yield stress, correlation length, energy, pressure

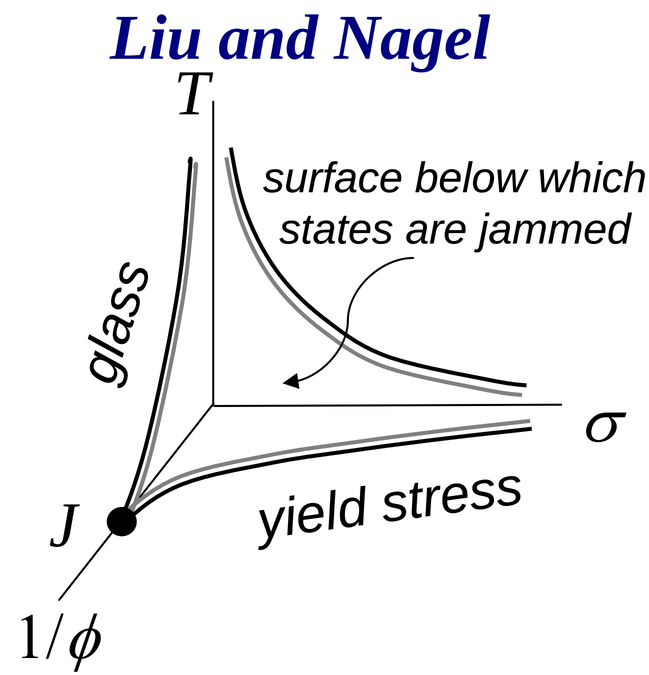
- conclusions

jamming phase diagram







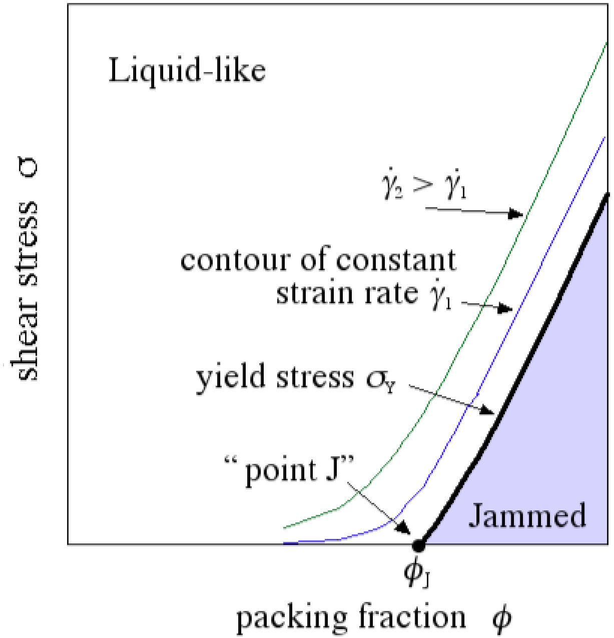


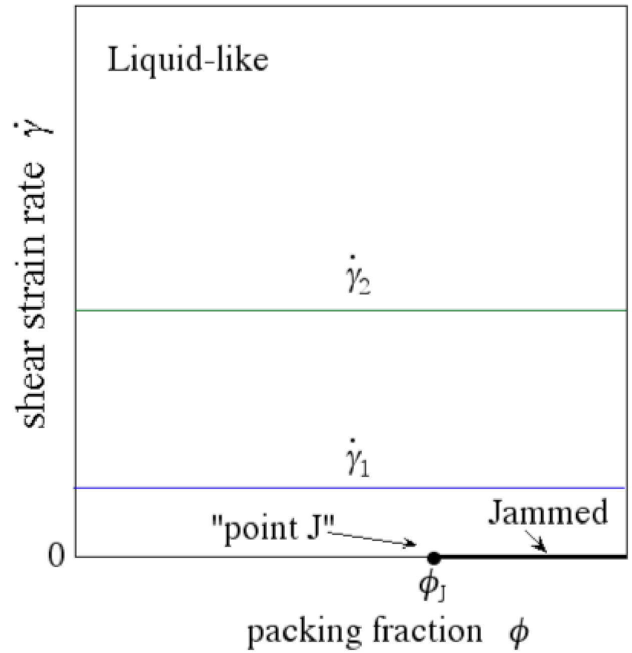
"*point J*" is a critical point
like in equilibrium transitions

critical scaling at *point J* influences
behavior at *finite* T and *finite* σ.

understanding $T = 0$ jamming at "*point J*" may have
implications for understanding the glass transition at finite T

jamming phase diagram $T = 0$ (Olsson & Teitel PRL 2007)





analogy to Ising model

shear strain rate $\dot\gamma \leftrightarrow h$ magnetic field

shear stress $\sigma \leftrightarrow m$ magnetization density

yield stress $\sigma_Y \sim (\phi - \phi_J)^\Delta$

Hatano, J Phys Soc Jpn 2008

scaling ansatz for jamming rheology

Control parameters: $\delta\phi \equiv \phi - \phi_J, \quad \dot{\gamma}, \quad L^{-1} \quad (L = \sqrt{N})$

Critical point J: $\delta\phi = \dot{\gamma} = L^{-1} = 0$

N = number particles

Scaling hypothesis (2nd order phase transitions):

At a critical point, all quantities that vanish or diverge do so as some power of a diverging correlation length ξ. When $\xi \to b\xi$, these quantities scale as a power of b, where b is an *arbitrary* length rescaling factor.

$$\delta\phi \sim b^{-1/\nu}, \quad \dot{\gamma} \sim b^{-z}, \quad L \sim b, \quad \sigma \sim b^{-\Delta/\nu}$$

ν is correlation length critical exponent

z is dynamic critical exponent

Δ is yield stress exponent

$$\sigma(\phi, \dot{\gamma}, L)\, b^{\Delta/\nu} = f(\delta\phi\, b^{1/\nu}, \dot{\gamma}\, b^{z}, L^{-1} b)$$

scaling law

$$\sigma(\phi, \dot\gamma, L) = b^{-\Delta/\nu} f(\delta\phi\, b^{1/\nu}, \dot\gamma\, b^{z}, L^{-1} b)$$

Thermodynamic limit: $L \to \infty$

choose $b = |\delta\phi|^{-\nu} \quad \Rightarrow \quad \sigma = |\delta\phi|^{\Delta} f(\pm 1, \dot\gamma\, |\delta\phi|^{-z\nu}, 0)$

$$\dot\gamma = 0 \quad \Rightarrow \quad \sigma_{\mathrm{Y}} \sim \delta\phi^{\Delta}$$

choose $b = \dot\gamma^{-1/z} \quad \Rightarrow \quad \boxed{\sigma = \dot\gamma^{\Delta/z\nu} f(\delta\phi\, \dot\gamma^{-1/z\nu}, 1, 0)}$

when plot $\dfrac{\sigma}{\dot\gamma^{\Delta/z\nu}}$ vs $\dfrac{\delta\phi}{\dot\gamma^{1/z\nu}}$

$\sigma(\phi, \dot\gamma)$ data should collapse to a single scaling curve

Olsson & Teitel PRL 2007

scaling of shear viscosity

$$\sigma = |\delta\phi|^{\Delta} f(\pm 1, \dot{\gamma}\,|\delta\phi|^{-z\nu})$$

$$\eta \equiv \frac{\sigma}{\dot{\gamma}} = |\delta\phi|^{\Delta - z\nu} g(\pm 1, \dot{\gamma}\,|\delta\phi|^{-z\nu})$$

viscosity exponent β

$$\text{as } \dot{\gamma} \to 0, \quad \eta \sim |\delta\phi|^{-\beta} \quad \Rightarrow \quad \beta = z\nu - \Delta$$

$$\boxed{z\nu = \Delta + \beta}$$

$$\sigma = \dot{\gamma}^{\Delta/z\nu} f(\delta\phi\,\dot{\gamma}^{-1/z\nu}, 1)$$

$$\Rightarrow \quad \boxed{\sigma = \dot{\gamma}^{\Delta/(\Delta+\beta)} f(\delta\phi\,\dot{\gamma}^{-1/(\Delta+\beta)}, 1)}$$

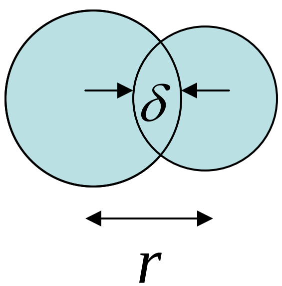
plot $\dfrac{\sigma}{\dot{\gamma}^{\Delta/(\Delta+\beta)}}$ vs $\dfrac{\delta\phi}{\dot{\gamma}^{1/(\Delta+\beta)}}$

model granular material

(O'Hern, Silbert, Liu, Nagel, PRE 2003)

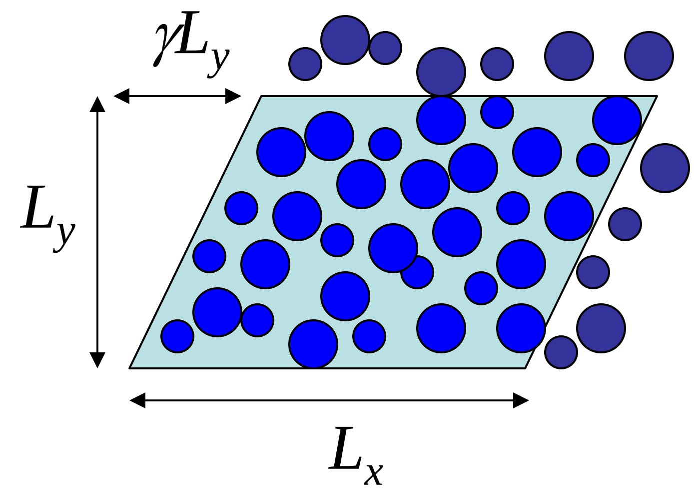
bidisperse mixture of soft disks in ***two dimensions*** at $T = 0$
equal numbers of disks with diameters $d_1 = 1$, $d_2 = 1.4$

frictionless, harmonic repulsive soft cores



$$V(r) \propto \delta^2$$

δ is particle overlap



Durian's overdamped foam dynamics

$$\frac{d\mathbf{r}_i}{dt} = -D \sum_j \frac{dV(r_{ij})}{d\mathbf{r}_i} + y_i \dot{\gamma} \hat{\mathbf{x}}$$

Lees-Edwards boundary Conditions for shear flow

pressure tensor

$$p_{\alpha\beta} = \frac{1}{L^2} \sum_{i>j} \frac{r_{ij\alpha} r_{ij\beta}}{r_{ij}} \frac{dV}{dr_{ij}}$$

shear stress $\sigma = p_{xy}$

pressure $p = \frac{1}{2}(p_{xx} + p_{yy})$

energy $u = \frac{1}{L^2} \sum_{i>j} V(r_{ij})$

simulation parameters

present work

fixed $\dot{\gamma}$, σ(t) fluctuates

N = 65,536

scaling collapse data to

$$\sigma = \dot{\gamma}^{\Delta/(\Delta+\beta)} f(\delta\phi\, \dot{\gamma}^{-1/(\Delta+\beta)})$$

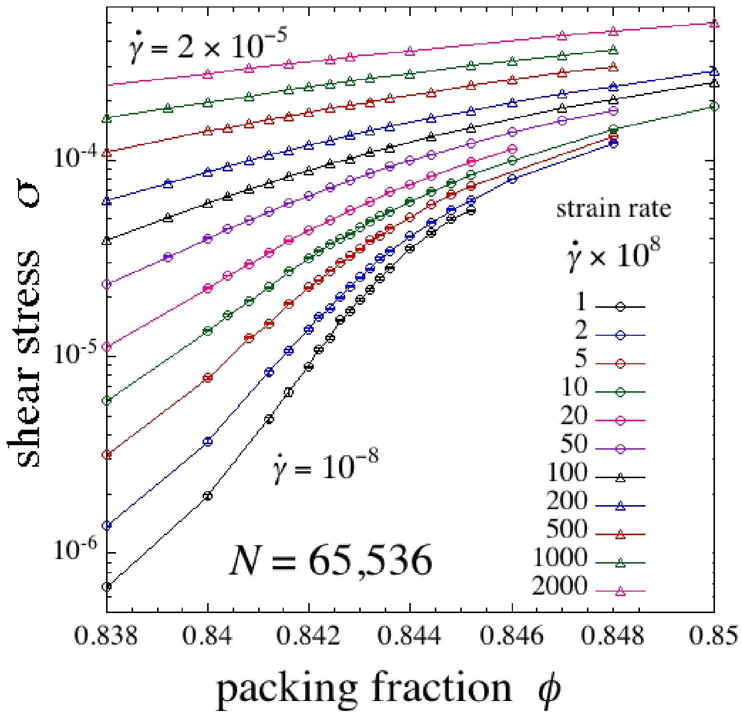
expand scaling function in polynomial for small values of its argument

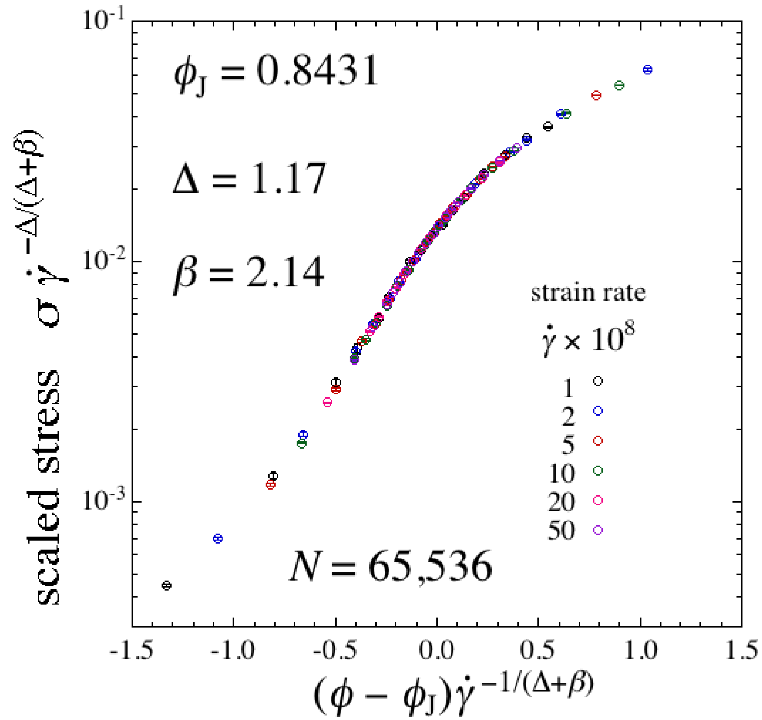
$$\sigma = \dot{\gamma}^{\Delta/(\Delta+\beta)} \sum_n c_n \left[(\phi - \phi_{\mathrm{J}})\, \dot{\gamma}^{-1/(\Delta+\beta)}\right]^n$$

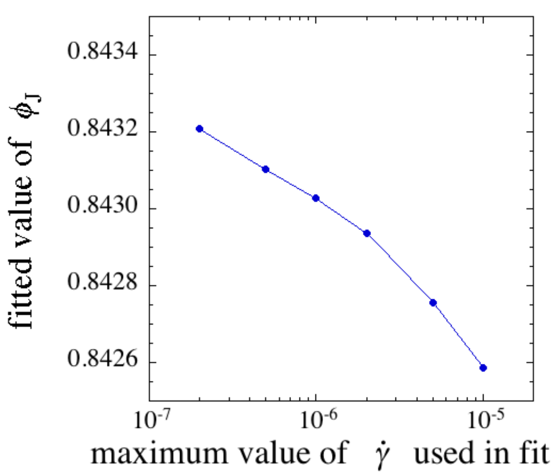
determine ϕ_{J}, Δ, β, c_n from best fit of data to this form

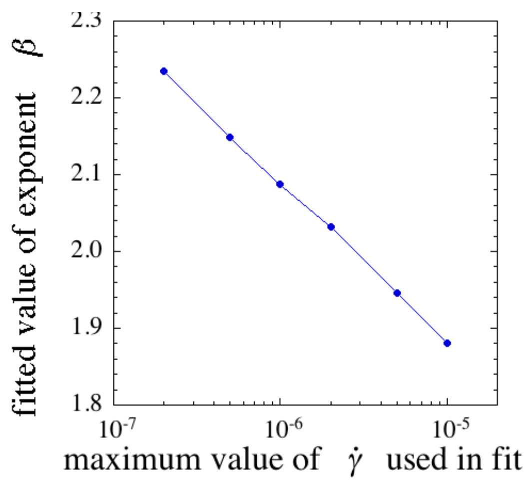
assumes finite size effects negligible, $\xi << L$ (can't get too close to ϕ_{J})

simulations with finite shear strain rate







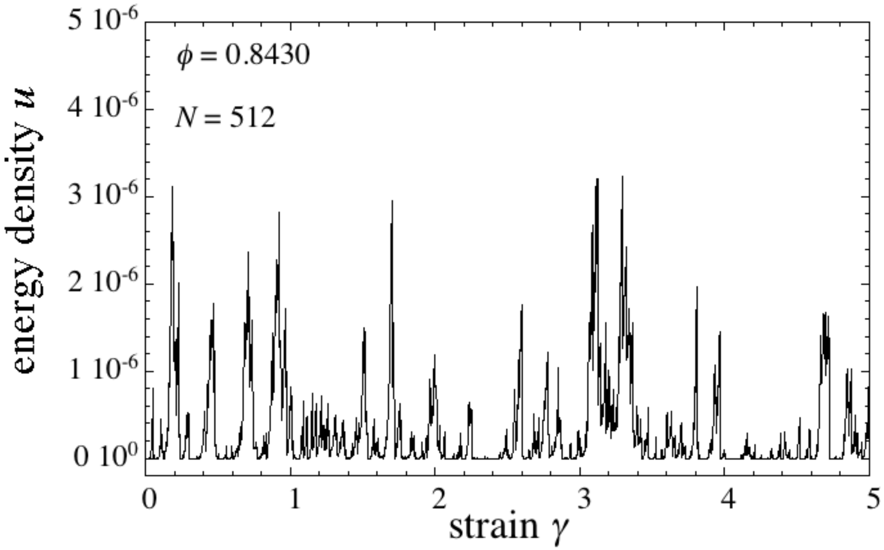


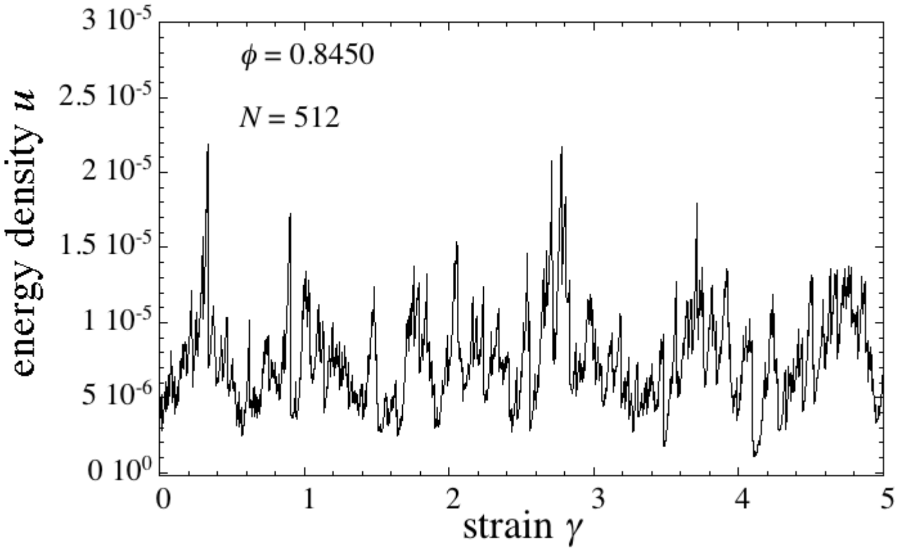
quasi-static simulations $\dot{\gamma} \to 0$

when the strain rate is sufficiently slow, system will always relax to be in an instantaneous local minimum of the interaction energy

Dynamics: increase strain in fixed steps $\Delta\gamma$, at each step use conjugate gradient method to relax the particle positions to local energy minimum

Maloney & Lemaitre, PRE 2006, $\phi > \phi_J$
Heussinger & Barrat, cond-mat 2009, at point J





$\Delta\gamma = 0.001$ $\gamma_{max} = 50$ to 150 depending on system size

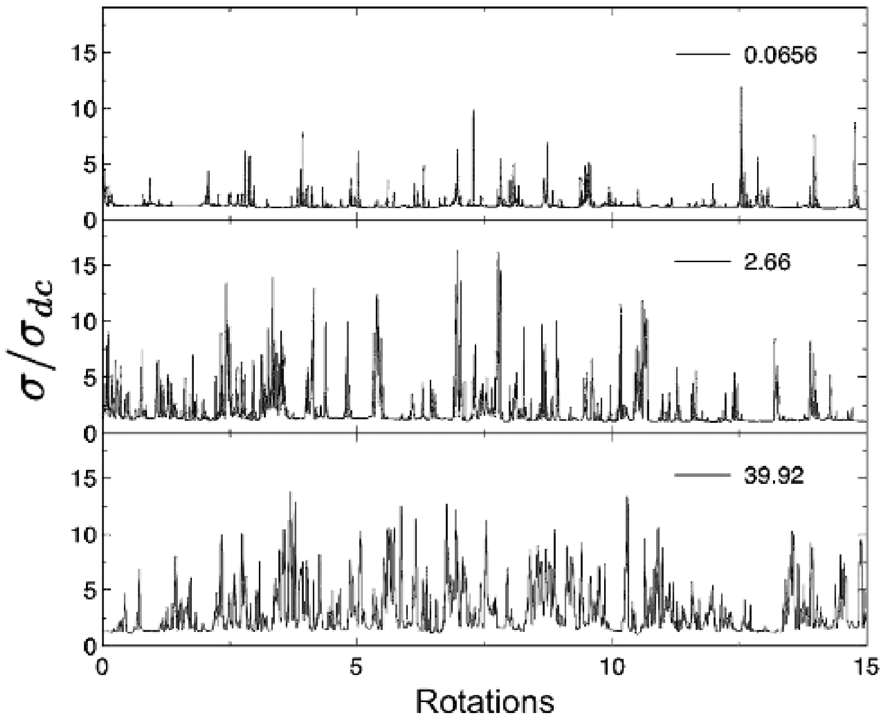


FIG. 2. Stress time series for several $\dot{\gamma}$ (mHz) for the smaller R_i.

Behringer, Bi, Chakraborty, Henkes, Hartley, PRL 101, 268301 (2008)

finite size scaling

$$L \equiv \sqrt{N}$$

$$\sigma = b^{-\Delta/\nu} f(\delta\phi\, b^{1/\nu}, \dot\gamma\, b^{z}, L b^{-1})$$

add system size L as new scaling parameter

choose $b = L, \quad \dot\gamma = 0$ for quasi-static limit

$$\sigma = L^{-\Delta/\nu} f(\delta\phi\, L^{1/\nu}, 0, 1)$$

Hatano, cond-mat 2008

exactly at ϕ_J $\quad \sigma \sim L^{-\Delta/\nu}$

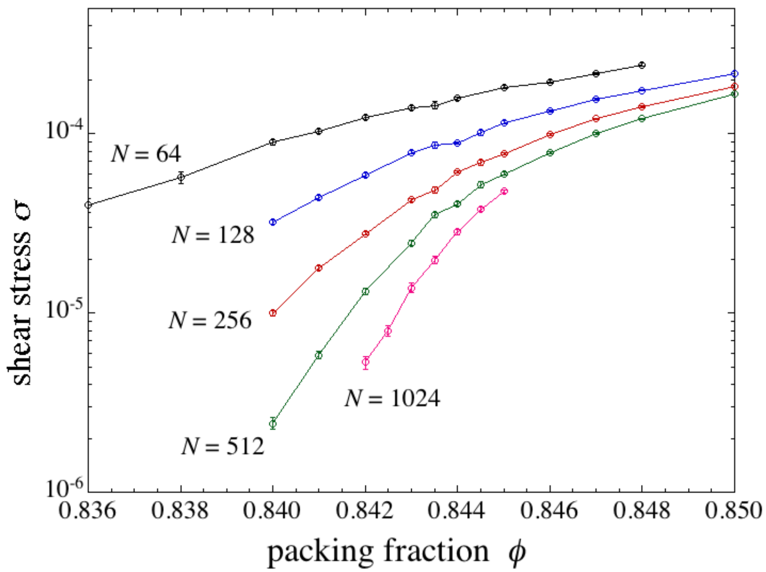
plot $\quad \sigma L^{\Delta/\nu} \quad \mathrm{vs} \quad \delta\phi\, L^{1/\nu}$

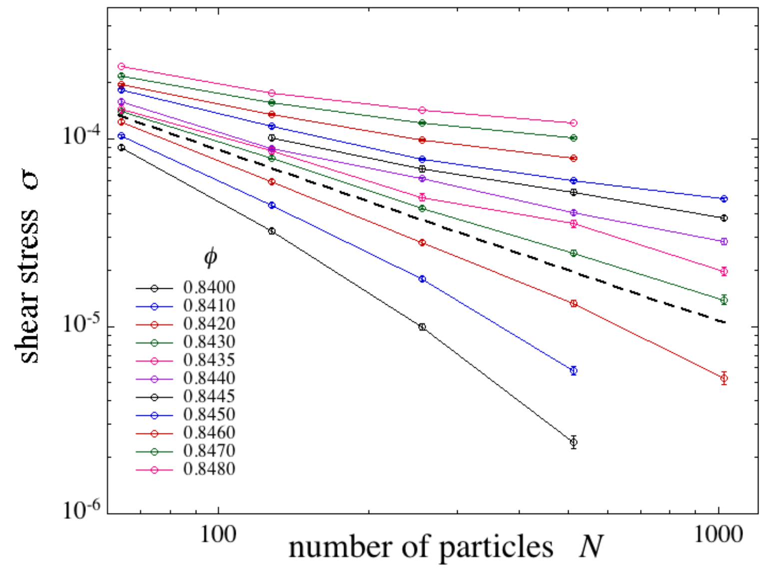
data collapse determines exponents Δ and ν

fit data to polynomial expansion of scaling function

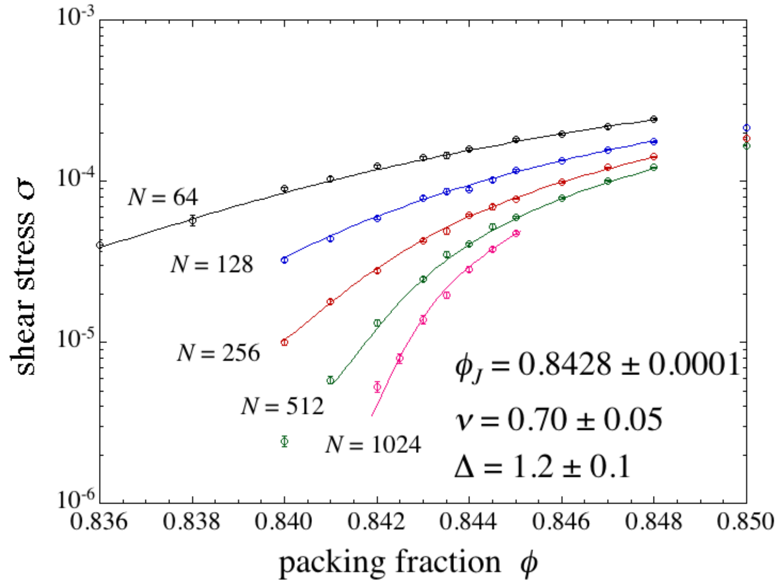
bonus of finite size scaling is that it gives the correlation length exponent ν *without having to explicitly compute the correlation length!*

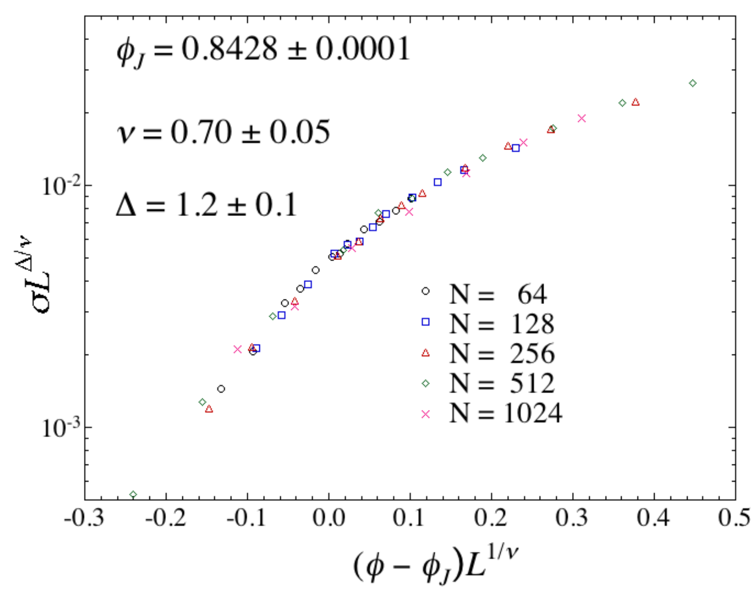
shear stress σ





$$\sigma = L^{-\Delta/\nu} f(\delta\phi\, L^{1/\nu}, 0, 1)$$



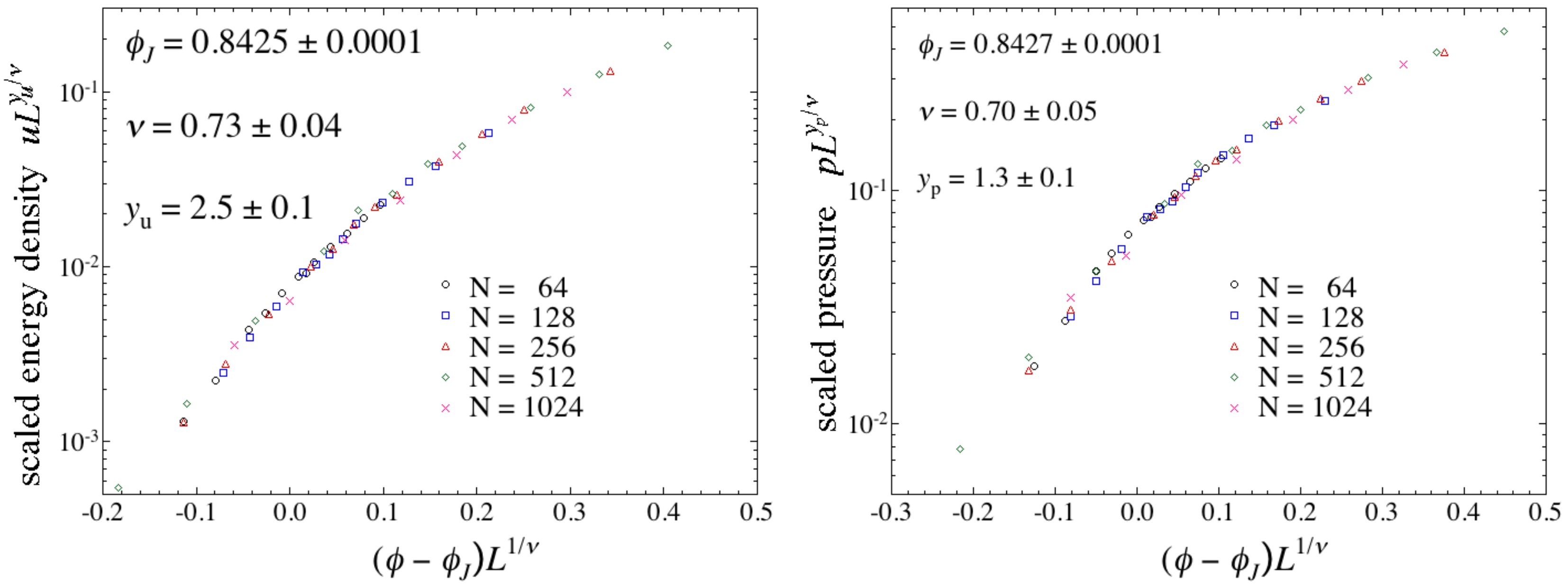


energy density *u*, *pressure p*

$$u \sim \delta\phi^{y_u} \;\Rightarrow\; u = L^{-y_u/\nu}\,\mathcal{U}(\delta\phi\, L^{1/\nu})$$

$$p \sim \delta\phi^{y_p} \;\Rightarrow\; p = L^{-y_p/\nu}\,\mathcal{P}(\delta\phi\, L^{1/\nu})$$

$$p = \phi\frac{du}{d\phi} \;\Rightarrow\; y_p = y_u - 1$$



effective temperature: Daniel Valdez-Balderas, P14-9, Wed 9:36 am

results

Finite Size Scaling with $N = 64$ to 1024

shear stress	energy density	pressure
$\phi_J = 0.08429 \pm 0.0001$	$\phi_J = 0.08425 \pm 0.0001$	$\phi_J = 0.08427 \pm 0.0001$
$\nu = 0.70 \pm 0.05$	$\nu = 0.73 \pm 0.04$	$\nu = 0.70 \pm 0.05$
$\Delta = 1.2 \pm 0.1$	$y_u = 2.5 \pm 0.1$	$y_p = 1.3 \pm 0.1$

Check if in scaling limit: Finite Size Scaling with $N = 128$ to 1024

shear stress	energy density	pressure
$\phi_J = 0.08431 \pm 0.0001$	$\phi_J = 0.08428 \pm 0.0001$	$\phi_J = 0.08429 \pm 0.0001$
$\nu = 0.78 \pm 0.07$	$\nu = 0.80 \pm 0.05$	$\nu = 0.79 \pm 0.07$
$\Delta = 1.2 \pm 0.1$	$y_u = 2.5 \pm 0.2$	$y_p = 1.3 \pm 0.1$

$\phi_J, \ \nu$ increase $\qquad$ $\Delta, \ y_u, \ y_p$ stay the same

Wyart, Nagel, Witten, Europhys Lett 2005: $\nu = 1/2$ soft modes of jammed solid

O'Hern, et al., PRE 2003 + Drocco, et al., PRL 2005: $\nu \sim 0.7$ numerical

Mike Moore (private communication): $\nu \sim 0.78$ mapping to spin glass

exponents and ensembles

we find

$$\Delta = 1.2 \pm 0.1, \quad y_u = 2.5 \pm 0.3, \quad y_p = 1.3 \pm 0.1$$

"classical values" for harmonic soft cores

$$\Delta = 1, \quad y_u = 2, \quad y_p = 1$$

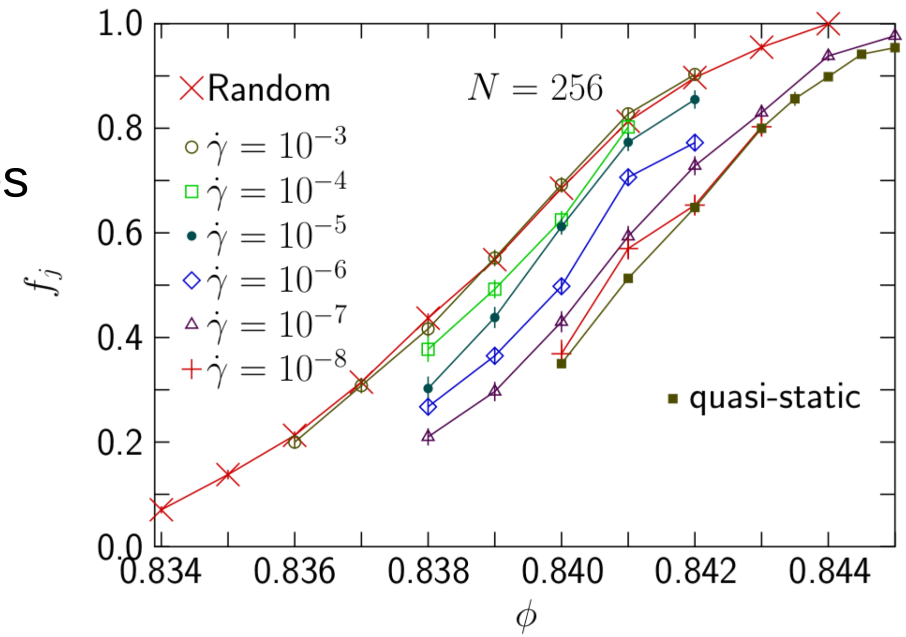
O'Hern, et al: $y_p = 1$
for harmonic soft cores

Why the difference?

Jammed ensemble depends on the preparation protocol

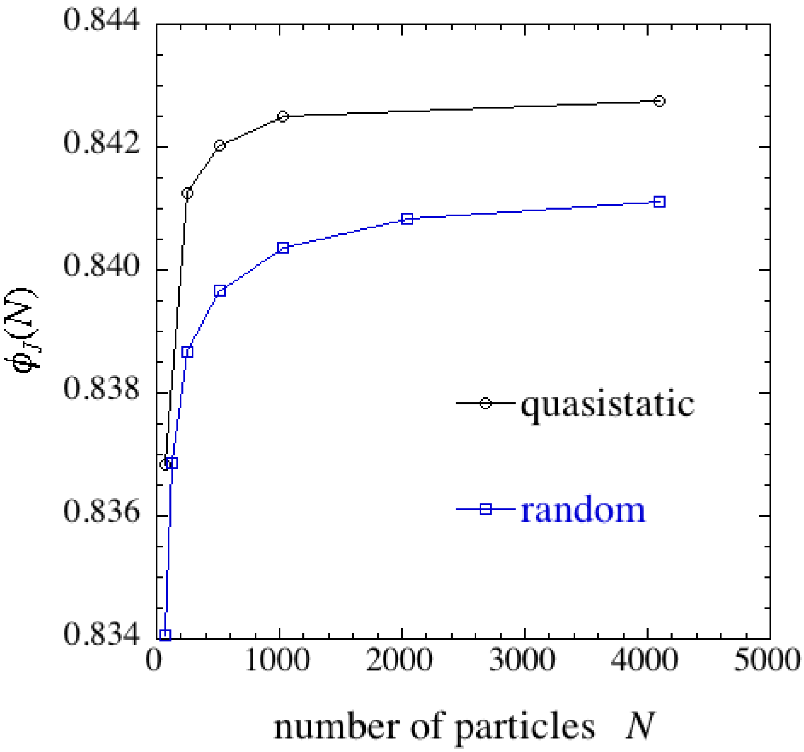
fraction of jammed states

Peter Olsson
J14-10



O'Hern et al. random protocol ≡ fast shearing

as shear rate decreases, system explores longer time scales, jamming density increases!



0.844
0.842
0.840
0.838
0.836
0.834
0
1000
2000
3000
4000
5000
$\phi_J(N)$
number of particles N
quasistatic
random

conclusions

quasi-static method looks promising!

need to study larger system sizes N

no definitive value for ϕ_J or ν yet, but we are getting there!

preparation protocol can make a difference