LECTURE 3 Appendix - A short review of critical scaling in the Ising model

Close to the critical point of the Ising model, the free energy density as a function of temperature and magnetic field, $f(T, h)$, scales as,

$$f(T, h) = b^{-d} f(tb^{y_t}, h^{y_h}) \tag{1}$$

where b is an arbitrary length rescaling factor, d is the dimensionality of the system, $t \equiv T - T_c$, and y_t and y_h are the scaling exponents associated with the two scaling fields, temperature and magnetic field, respectively [here we assume $d < 4$, where mean field theory breaks down]. The free energy density itself scales as b^d since one assumes that the total free energy $b^d f$ is scale invariant. The temperature scaling exponent is usually written as $y_t = 1/\nu$, where ν is the correlation length exponent.

One can now compute the magnetization density, m,

$$m = \frac{\partial f}{\partial h} = b^{y_h - d} f_h(tb^{y_t}, hb^{y_h}) \tag{2}$$

the magnetic susceptibility, χ,

$$\chi = \frac{\partial m}{\partial h} = b^{2y_h - d} f_{hh}(tb^{y_t}, hb^{y_h}) \tag{3}$$

and the specific heat, c,

$$c \sim \frac{\partial^2 f}{\partial t^2} = b^{2y_t - d} f_{tt}(tb^{y_t}, hb^{y_h}) \tag{4}$$

where subscripts on f denote a derivative with respect to the variable noted.

Critical Exponents

Since b is an arbitrary parameter (but since b is a length rescaling factor it must still be *positive*), one can choose it to be anything one wishes, so ...

I. Choose $b = |t|^{-1/y_t}$. Substituting this into the above equations we get,

$$m = |t|^{(d-y_h)/y_t} f_h(\pm 1, h/|t|^{y_h/y_t}) \tag{5}$$

$$\chi = |t|^{-(2y_h-d)/y_t} f_{hh}(\pm 1, h/|t|^{y_h/y_t}) \tag{6}$$

$$c \sim |t|^{-(2y_t-d)/y_t} f_{tt}(\pm 1, h/|t|^{y_h/y_t}) \tag{7}$$

In the above $+1$ refers to $T > T_c$ and -1 refers to $T < T_c$.

If we now set $h = 0$ in the above, we get the various standard critical exponents, expressed in terms of the scaling exponents y_t and y_h.

$$m \sim |t|^{(d-y_h)/y_t} \sim |t|^{\beta} \quad \Rightarrow \quad \beta = (d - y_h)/y_t \tag{8}$$

$$\chi \sim |t|^{-(2y_h-d)/y_t} \sim |t|^{-\gamma} \quad \Rightarrow \quad \gamma = (2y_h - d)/y_t \tag{9}$$

$$c \sim |t|^{-(2y_t-d)/y_t} \sim |t|^{-\alpha} \quad \Rightarrow \quad \alpha = (2y_t - d)/y_t \tag{10}$$

But this is not the only choice for b.

II. Choose $b = h^{-1/y_h}$ (we have in mind only $h > 0$). Substituting in for the magnetization we get

$$m = h^{(d-y_h)/y_h} f_h(t/h^{y_t/y_h}, 1) \tag{11}$$

If we take $t = 0$ in the above we get another standard critical exponent. At T_c we have,

$$m \sim h^{(d-y_h)/y_h} \sim h^{1/\delta} \qquad \Rightarrow \qquad \delta = y_h/(d-y_h) \tag{12}$$

So we have expressed the standard critical exponents α, β, γ, δ, all in terms of the two scaling exponents y_t and y_h.

Scaling Functions

Aside from giving the critical exponents, the scaling forms above give other important information. We can write Eq.(5) as,

$$\frac{m}{|t|^{(d-y_h)/y_t}} = g_\pm\left(\frac{h}{|t|^{y_h/y_t}}\right) \tag{13}$$

where $g_\pm(z) \equiv f_h(\pm 1, z)$. Thus if we measure magnetization $m(T,h)$ as a function of the two parameters T and h, but plot the data up as $m/|t|^{(d-y_h)/y_t}$ vs $h/|t|^{y_h/y_t}$, all our two dimensional data will collapse to a single one dimensional curve $g_\pm(z)$. This one dimensional function is called a *scaling function*.

Similarly we can rewrite Eq.(11) as

$$\frac{m}{h^{(d-y_h)/y_h}} = f_h\left(\frac{t}{h^{y_t/y_h}}\right) \tag{14}$$

Then if we plot our data as $m/h^{(d-y_h)/y_h}$ vs $t/h^{y_t/y_h}$, it again will collapse to a single one dimensional curve $f_h(z)$. This is another, equivalent, way to define a scaling function.

Finite Size Scaling

The above results applied to a system in the thermodynamic limit of infinite size. In practice, simulations always take place on a system of finite size. Let L be the length of a side of the system, with its volume L^d. Since the true critical point only exists for the infinite system, we can now regard L^{-1} as a new scaling field in the problem, on an equal footing as t and h. Decreasing $L^{-1} \to 0$ takes one to the critical point while increasing L^{-1} takes one away from the critical point. The scaling law of Eq.(1) then generalizes to,

$$f(T,h,L) = b^{-d} f(tb^{y_t}, hb^{y_h}, L^{-1}b) \tag{15}$$

where the scaling exponet of L^{-1}, which for consistency we could have called y_L, has the simple value $y_L = 1$ since b is a length rescaling factor and L clearly has dimensions of length.

The magnetization is similarly,

$$m = b^{y_h - d} f_h(tb^{y_t}, hb^{y_h}, L^{-1}b) \tag{16}$$

We can now choose $b = L$ and substitute into the above to get,

$$m = L^{y_h - d} f_h(tL^{y_t}, hL^{y_h}, 1) \tag{17}$$

For simplicity, let's consider the case of $h = 0$. The above then becomes

$$m = L^{y_h - d} \mathcal{M}(tL^{y_t}) \tag{18}$$

where $\mathcal{M}(z) = f_h(z, 0, 1)$ is a finite-size-scaling function.

A few important results follow from the above. In an infinite system, $m = 0$ exactly at T_c. However in the finite size system m at T_c remains finite. The above gives that $m(T_c, L) \sim L^{y_h - d} = L^{-\beta y_t}$. Thus m at T_c decays to zero algebraically with increasing system size L.

If one plotted mL^{d-y_h} vs T for different system sizes L, the curves for different L would all intersect at the common temperature T_c.

If one plotted mL^{d-y_h} vs tL^{y_t}, the data for different T and L would all collapse to a single one dimensional curve. Looking for such a data collapse enables one to try and determine the values of T_c and the scaling exponents y_t and y_h.

The above is all a quick summary. There are many details and elaborations one could add. The most common correction one might need to the above goes by the name "corrections to scaling". The above scaling forms in principal hold only asymptotically close to the critical point. In practice they may hold within some narrow finite window in the parameter space (t, h, L^{-1}) about the critical point - this is known as the *critical region*. One does not apriori know how big is the critical region. If one tries to scale data according to the above forms, but one is including data that lies outside the critical region, one will have troubles. One can often still make use of such data provided one incorporates *corrections to scaling* in the analysis. The discussion of such effects is fortunately beyond the scope of this quick review!