Driven Diffusive Systems: A Tutorial and Recent Developments

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Abstract. The first part of these lectures will be devoted to a general introduction to the statistical mechanics of driven diffusive systems. As models for non-equilibrium steady states, they provide fertile grounds for investigating surprising co-operative behavior absent from systems in thermal equilibrium. The distinctive differences between equilibrium and non-equilibrium steady states will be emphasized. The specific, unexpected properties of the latter will be presented in the context of several varieties of the driven Ising lattice gas. Despite their simplicity and two decades of investigations, many new surprises continue to surface. After a brief summary of the status of these systems, the more recent discoveries will be presented.

1. Introduction

Over a century ago, Boltzmann laid the foundations for equilibrium statistical mechanics, through the bold hypothesis that the probabilities to find an *isolated* system in any configuration are equal. Strengthened by the work of Gibbs, its superstructure is now so well established that it forms part of most core curricula in physics. The concepts and vocabulary of this framework are routinely applied, with considerable success, in a wide range of settings, from the local to the cosmological, in a large variety of disciplines. Of particular interest is the prediction of the existence of phase transitions, i.e., the presence of widely different macroscopic states of matter when subjected to different control parameters (such as temperature and pressure), even though the microscopic constituents and their mutual interactions are the same. One of the first attempts at establishing such a "micro-macro connection" is the simple model for the para-ferromagnetic transition, proposed nearly four score years ago by Lenz. Despite its simplicity, Ising was able to "solve" only a system in one dimension ($d = 1$), showing that there is *no* transition [1]. Another twenty years passed before Onsager demonstrated rigorously the existence of this transition in a $d = 2$ system [2] and computed some of its important properties. Transformed into a "gas" of particles on a lattices, with nearest-neighbor attractive interactions, it had been used to describe, for low temperatures, the co-existence of the familiar liquid and vapor phases, as well as phase segregation in binary alloys [3].

Though excellent approximation schemes can be applied to this "Ising model" in the physical dimension $d = 3$, an exact solution is yet to be found. Of course, most physical systems are far more complex than this model, so that it remained as a "toy for theoreticians" for many decades. Its status changed thirty years ago, however, thanks to a successful treatment of the idea of universality by renormalization group methods[4]. Now, we understand that its behavior in the neighborhood of the (second order) transition can be directly related to those in an entire class of complex physical systems.

Though successful, *equilibrium* statistical mechanics does have limitations. Specifically, non-equilibrium phenomena are much more abundant in nature. Even if they appear to be in *time-independent* states, non-equilibrium systems are beyond the powers of the Boltzmann-Gibbs framework. Unfortunately, the theoretical development of *non-equilibrium* statistical mechanics is still relatively primitive, so that our well developed intuition, based on equilibrium statistical mechanics, often leads us astray in attempts to understand such systems. For example, the application of typical energy and entropy arguments frequently fail dramatically. In these lectures, we will present a tutorial on driven diffusive systems, which form a small class of *non-equilibrium steady states*, summarizing briefly the intriguing features discovered, and our limited understanding developed, over the last twenty years. Before that, we believe it would be helpful to devote a section to clarify the difference between stationary states in equilibrium and non-equilibrium steady states. In a last section, some recent developments will be discussed.

2. Equilibrium *vs.* Non-equilibrium Steady States

Any description of a physical system begins with specifying the degrees of freedom to be considered. We will refer to these are the configurations of the system: $\mathcal{C}$. For simplicity, let us assume these are discrete, as is the case for the Ising lattice gas. We will also assume that there exists a "Hamiltonian" $\mathcal{H}(\mathcal{C})$ which provides the (internal) energy associated with $\mathcal{C}$. A statistical description requires $P(\mathcal{C}, t)$, the probability to find the system in configuration $\mathcal{C}$ at time t. The starting point of equilibrium statistical mechanics is the time-independent distribution: $P^*_{eq}(\mathcal{C})$. For an *isolated* system, where the total energy is fixed at $\mathcal{E}$, Boltzmann hypothesized that the probabilities of all configurations with this energy are *equal*, i.e.,

$$P^*_{eq,I}(\mathcal{C}) \propto \delta\left[\mathcal{H}(\mathcal{C}) - \mathcal{E}\right]. \tag{1}$$

From here, it is standard to derive a workable concept for temperature and show that, for a system in contact with a large (asymptotically infinite) thermal bath with temperature T,

$$P_{eq,T}^{*}(\mathcal{C}) \propto \exp\left[-\beta\mathcal{H}(\mathcal{C})\right], \tag{2}$$

where $\beta \equiv 1/k_B T$. If, on the other hand, we wish to extend our description to *non*-equilibrium states, then we must pay some attention to dynamics, i.e., the time-dependent distribution: $P(\mathcal{C},t)$.

The simplest context within which to discuss dynamics is the master equation

$$\partial_t P(\mathcal{C},t) = \sum_{\mathcal{C}'}\left[W(\mathcal{C}'\to\mathcal{C})\,P(\mathcal{C}',t) - W(\mathcal{C}\to\mathcal{C}')\,P(\mathcal{C},t)\right],$$

where $W(\mathcal{C}\to\mathcal{C}')$ represents the rate at which a configuration $\mathcal{C}$ changes to another: $\mathcal{C}'$. Note that this is just a continuity equation for the probability density and the sum on the right can be regarded as over net probability currents

$$K(\mathcal{C}'\to\mathcal{C}) = W(\mathcal{C}'\to\mathcal{C})\,P(\mathcal{C}') - W(\mathcal{C}\to\mathcal{C}')\,P(\mathcal{C}),$$

between configuration pairs $\mathcal{C}$,$\mathcal{C}'$. Like the postulate of Boltzmann, the linear dependence of J on P is also a hypothesis. As such, this equation can be written in matrix form

$$\partial_t P = \mathbb{L}P,$$

where $\mathbb{L}$ is known as the Liouvillian, in analogy with the Hamiltonian in $\partial_t\Psi = -i\mathbb{H}\Psi$. Note finally that, due to the built-in conservation law, $\mathbb{L}$ has at least one zero eigenvalue, associated with the *left* eigenvector $v(\mathcal{C}) = 1$.

For simplicity, let us focus on those systems which are well described by this master equation, with *time-independent* rates. Even with all these restrictions, it would be prohibitively difficult to *derive* the W's from the underlying dynamics (say, Newtonian) governing the evolution of both our system and its environments. Nevertheless, we can make progress by choosing W's which are based on sound principles. Thus, to insure that a system will reach an equilibrium distribution in the $t\to\infty$ limit, as needed in implementing Monte Carlo simulations, we must choose rates which satisfy "detailed balance." Specifically, if $P_{eq,T}^{*}(\mathcal{C})$ is the desired end-state, then $W(\mathcal{C}\to\mathcal{C}')$ can be any function of $\mathcal{C},\mathcal{C}'$ provided

$$W(\mathcal{C}\to\mathcal{C}')/W(\mathcal{C}'\to\mathcal{C}) = \exp\left[\beta\mathcal{H}(\mathcal{C}) - \beta\mathcal{H}(\mathcal{C}')\right].$$

For example, a favorite in simulation studies is the Metropolis rate: $W(\mathcal{C} \to \mathcal{C}') = \min[1, e^{\beta\mathcal{H}(\mathcal{C}) - \beta\mathcal{H}(\mathcal{C}')}]$. A less frequently noted, but equally important, consequence of detailed balance is $K^*(\mathcal{C}' \to \mathcal{C}) \equiv 0$ (where the superscript again denotes the stationary state), for any pair $\mathcal{C}, \mathcal{C}'$. In this sense, an equilibrium state is comparable to an electrostatic system, with time independent charge densities and *zero* currents.

By contrast, imagine coupling our system to more than one energy reservoir, such as *two* thermal baths at different temperatures. Then, we might expect, even when our system is in a stationary state, that there is constant energy flux *through* it. In other words, our system may constantly gain energy from the hotter bath and lose it to the colder one; at the same rates so that the net gain/loss is zero. To model such a situation by the master equation, the W's must be modified. Specifically, we can no longer expect rates to satisfy detailed balance, which is a consequence of microscopic reversibility. Indeed, the condition of detailed balance (or lack thereof) can be stated without reference to a Hamiltonian [5], namely,

$$\begin{aligned} & W(\mathcal{C}_1 \to \mathcal{C}_2)\, W(\mathcal{C}_2 \to \mathcal{C}_3) \cdots W(\mathcal{C}_n \to \mathcal{C}_1) \\ = \; & W(\mathcal{C}_1 \to \mathcal{C}_n) \cdots W(\mathcal{C}_3 \to \mathcal{C}_2)\, W(\mathcal{C}_2 \to \mathcal{C}_1) \end{aligned} \tag{3}$$

for *all* cycles $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \cdots \mathcal{C}_n\}$. So, one way to insure we have a non-equilibrium system is to postulate rates which *violate* detailed balance. Of course, it is impossible to investigate all such W's. The focus of these lectures is to study the effects of driving simple, Ising-like models into non-equilibrium steady states. Unlike the equilibrium case, there is no "end-state" to guide us in choosing the appropriate W's. On the contrary, the stationary distribution is expected to be very different from a naive Boltzmann-like factor (Eqn.2). One way forward is to introduce physically motivated W's which are minimal modifications of the familiar ones. Once a set of detailed-balance-violating rates are specified, Monte Carlo simulations can be carried out and properties of the stationary distribution P^* can be investigated.

If we stay within the framework of a master equation with time-independent rates, we are guaranteed at least one P^* which satisfies $\partial_t P^* = 0$, since we must have a right eigenvector of $\mathbb{L}$ with zero eigenvalue. Proving uniqueness is more involved, however – we will simply assume that this P^* is unique here. What distinguishes this stationary state from its equilibrium counterparts is that, in general, some of the currents K^* will be non-zero. As manifestations of detailed-balance-violation, these currents can be related directly to the differences between the left- and right-hand side in Eqn.(3) [6]. Of course, being part of a time independent states, the K^*'s must form current *loops*. As a result, these states are analogues of *magnetostatics*, and will be referred to as *non-equilibrium steady states*. To close this section, let us simply remark that, in the example of coupling to two thermal baths, the energy through-flux can be expressed in terms of $\mathcal{H}$ and these K^*'s [7].

3. Driven Diffusive Systems: a Primer

Motivated partly by the physics of fast ionic conductors [8] and mostly by the interest in non-equilibrium steady states, Katz, Lebowitz and Spohn [9] modified the well known Ising lattice gas by a deceptively minor addition of an external drive. As mentioned in the Introduction, many "surprises" surfaced and, despite the efforts of nearly twenty years, many of the puzzles remained unexplained. Further explorations continued, through a number of variations and generalizations of the proto model. Since the dynamics associated with the lattice gas is particle-number conserving – i.e., diffusive – these models are known as driven diffusive systems. In this section, a brief description of various microscopic models and our theoretical understanding of some of the unexpected phenomena will be presented, referring the readers to [10, 11] for a more detailed treatment.

3.1. Monte Carlo simulations of driven lattice gas models

3.1.1. *Single species, uniformly driven*

To simulate the simplest Ising lattice gas in equilibrium and its phase transition, we take a square lattice, with $L_x \times L_y$ sites and toroidal boundary conditions, with a particle or a hole occupying each site. A configuration $\mathcal{C}$ is specified by the occupation numbers $\{n_i\}$, where i is a site label and n is either 1 or 0. Occasionally, we also use spin language, defining $s \equiv 2n - 1 = \pm 1$. If we want to study critical properties, we use half-filled lattices: $\sum_i n_i = L_x L_y / 2$. The particles are endowed with nearest-neighbor attraction (ferromagnetic, in spin language), modeled in the usual way through the Hamiltonian: $\mathcal{H} = -4J \sum_{<i,j>} n_i n_j$, with $J > 0$. To simulate coupling to a thermal bath at temperature T, the Metropolis algorithm [12] may be used, i.e., the contents of a randomly chosen, nearest-neighbor, particle-hole pair are exchanged, with a probability $\min[1, e^{-\beta\Delta\mathcal{H}}]$, where $\Delta\mathcal{H}$ is the change in $\mathcal{H}$ after the exchange. Starting the system in some random configuration, updates with this method will lead to a set of configurations which characterizes $P^*_{eq,T}(\mathcal{C})$ and with which physical quantities such as particle correlations or specific heat can be measured.

Next, imagine that the particles are "charged" (as ions in a solid) and an external electric field ($\vec{E} = -E\hat{y}$) is applied. This field should favor/suppress "downwards/upwards" hopping particles. *Locally*, we may associate this drive with a potential, so that the Metropolis rate can be modified simply to $\min[1, e^{-\beta(\Delta\mathcal{H}-\epsilon E)}]$, where $\epsilon = (-1, 0, 1)$, for particle hops (against, orthogonal to, along) the drive. Note that it is possible to set E to ∞, with these rates, corresponding to disallowing all upward hops. Now, these rates can be imposed with the toroidal boundary conditions, so that there is no *global* potential which can be incorporated into $\mathcal{H}$. There is no detailed balance, as exemplified by the violation of (3) by any cycle involving, say, a particle traversing L_y vertically. In the long time limit, a non-trivial steady current is established, showing explicitly the presence of K^*. Further, we may regard the external field as another energy reservoir, so that, in the non-equilibrium steady state, our lattice gas tends to gain energy with downward jumps (more broken bonds, increasing $\mathcal{H}$). Similar considerations would lead to energy loss (to the thermal bath) associated with transverse jumps. Indeed, this energy through-flux can be measured. Now, the equilibrium ($E = 0$) system is known to undergo a second order phase transition at the Onsager critical temperature $T_O = (2.2692..)J/k_B$. When driven ($E \neq 0$), this system displays similar properties, i.e., a disordered phase for large T, followed by a continuous transition, at some $T_c(E)$, into a phase-segregated state for low T. However, on closer examination, this superficial similarity gives way to a series of surprises, at *all temperatures*. In particular, $T_c(E)$ appears to *increase* monotonically with E, saturating, for $E = \infty$, at about 40% higher than $T_c(0) = T_O$ [13, 14, 15]. Why should $T_c(\infty) > T_c(0)$ be surprising? In the above, we argued that, with large and finite E, energy is fed into our system with particle hops in the y-direction. This phenomenon is also observed in simulations. In this sense, E is "pumping" energy into the system and so, this reservoir can be regarded as "bath at a higher temperature." Indeed, at $E = \infty$, hops in y behaves precisely as a pure (biased) random walk – hop if down is chosen; no-hop if upwards is chosen, regardless of J. Such behavior is consistent with coupling to an infinite temperature bath! Now, if our system is coupled to *two* baths, one of which is tuned to be hotter and hotter, surely we need to *lower* T (the temperature of the other bath) for the system to order. Yet, simulations shows that, e.g., the system phase separates when E is raised while T is set at $1.1T_O$! When a similar phenomenon was discovered ten years later, it was popularized [16] by a catchy phrase: "freezing by heating." With more experience from studying non-equilibrium steady states, we noticed how easily a system would display "negative specific heat," and pointed out minimal conditions for it to occur [17]. Returning to the driven Ising lattice gas, we simply note that, though we now have some indications from approximate scheme for the increase of

$T_c(E)$ [18, 19]in this case, a set of convincing guidelines leading to the correct behavior are still lacking. In particular, it is easy to modify the drive slightly and have *no* ordered phase at all at large E! Apart from these intriguing aspects associated with $T_c(E)$, the lattice gas displays remarkable behavior at all T.

Far above criticality, the two-particle correlation, $G(\vec{r})$, is *not* exponentially suppressed, as in the case of the equilibrium Ising model. Instead, it is of dipolar form and decay as r^{-2} (in $d = 2$) [20, 21]. In Fourier space, this phenomenon manifests as a discontinuity in the structure factor, $S\left(\vec{k}\right)$, at the origin: $\lim_{k_x \to 0} S(k_x, 0) \neq \lim_{k_y \to 0} S(0, k_y)$ [22]. Indeed, this singularity is "generic," being associated with violation of detailed balance in driven diffusive systems in general, rather than occurring only at criticality[23]. It is also responsible for "shape dependent thermodynamics," namely, fluctuations of the particle density in a small sub-system depends on the shape [24]! Meanwhile, the drive breaks particle-hole symmetry, so that (truncated) correlations of *odd* number of particles need not vanish. The discontinuity singularity in S induces, in general more severe singularities in these higher correlations. In particular, the *three*-point function has been observed in simulations [45]. Near the transition, the divergence of S appears only in one direction, so that the discontinuity becomes infinite: $\lim_{k_x \to 0} S(k_x, 0) \to \infty$, with $\lim_{k_y \to 0} S(0, k_y) < \infty$. Reminiscent of properties associated with a Lifschitz point[26], this level of "strong anisotropy" implies that the momenta/lengths in the two directions scale differently: $k_y \sim k_x^{1+\Delta}$, with $\Delta > 1$. It demands that finite size scaling be performed with a series of rectangular lattices satisfying $L_y \sim L_x^{1+\Delta}$ [13, 14, 27]. Not surprisingly, these phenomena lead to critical properties definitely *outside* the Ising universality class. Though there are competing claims concerning the details of these properties, three sets of measurements [13, 14, 15] are entirely consistent with the predictions of the first field theoretic studies [28], the only *viable* theory for this system [30] to date. In a standard renormalization group approach, the upper critical dimension is identified as 5. Nevertheless, thanks to a hidden invariance, exponents can be calculated to all orders in an $\epsilon \equiv 5 - d$ expansion. In particular, the only deviations from a Gaussian like theory is the presence of strong anisotropy: $\Delta = 2$ in $d = 2$.

At low temperatures, phase separation occurs so that a single high density strip, aligned with the drive, co-exists with a low density strip. The interface between them is, unlike the equilibrium case, *smooth* [31]. When height-height correlations are measured, the usual q^{-2} divergence (in Fourier space) is found to be very much suppressed: $q^{-0.67}$[32].

Drawing from the experience from equilibrium models, we explored the possible connection between smooth interfaces and singularities in the orientation dependent interfacial energy [34]. Now, to induce interfaces with different (average) orientations in simulations, *shifted* periodic boundary conditions must be imposed: site $(i+h, L_y)$ rather than (i, L_y) being a nearest neighbor of $(i, 1)$. When such systems are studied, a host of new phenomena appeared. For example, instead of a single "vertical strip" of width $L_x/2$ in the "ordinary" driven lattice gas, a "tilted strip" prevails for small h, so as to accommodate the shift. Inspection shows that one of the two interfaces becomes much rougher while the other becomes much smoother, but their differences are yet to be quantified. Further, with moderate shifts $(h/L_y \sim 10)$, stable "multiple strips" form. On closer inspection, they actually connect, via the shift, into a single strip, with multiple winding around the torus [35]! The average strip width lowers to $L_x/2n$, where n is the winding number. To date, there is little quantitative understanding of either the anomalous interfacial properties or multiple connected strips.

Another, more natural, boundary condition to impose is that of an open system. Instead of having periodic boundary conditions along the drive direction, the "top" and "bottom" rows are decoupled, while particles are fed in at the top and removed them from the bottom. This would be a much better model for a current through a physical wire, for example. In our simulations [36], the microscopic dynamics in the bulk is left unchanged; only at the boundaries are they modified. Again, the system appears to display a "phase transition," though there is an inhomogeneous density profile even in the disordered phase. For low temperatures, phase segregation now involves "icicle" like structures rather than simple strips with interfaces parallel to the drive. Again, though qualitative arguments may be advanced to "explain" such structures, no quantitative theory exists at present. When the two types of boundary conditions are combined, even more complex phenomena are discovered [37]. It is even unclear how to characterize the low temperature state, which displays gigantic density fluctuations. Thus, large clusters (finite fraction of the system size) are observed to "break off" from their "parent domain" and drift through the system, reminiscent of icebergs calved from an arctic ice sheet! Needless to say, theoretical understanding of these systems are essentially non-existent.

Another well known aspect of the equilibrium Ising model is the role of anisotropic interactions: $J_x \neq J_y$. In particular, for square systems $(L_x = L_y)$, the ordered state will consists of a vertical (V) strip if $J_x < J_y$ and a horizontal (H) one for $J_x > J_y$ (both J's positive). In an effort to understand further the effects of a drive (which favors the V phase), we studied driven models with a range of α, where $J_x = J/\alpha; J_y = J\alpha$

[38]. While the equilibrium phase diagram is symmetric in the T-$\ln\alpha$ plane, it is drastically modified with increasing E. Indeed, $T_c(E)$ is not even monotonic for $\alpha > 1$ [38]. Though, it is not surprising that no H phase exist for large E, the effect it has on V was unexpected: $T_c(\alpha; E = \infty)$ is monotonically *decreasing* in α! In other words, though large α favors the V state in equilibrium *and* large E also favors this state for $\alpha = 1$, the combination of large α and E is *not*favorable. With further scrutiny, we discovered that, for large α, the V state tends to break up into a multitude of strips (Figure 1 in [39]). In a certain temperature range, for the system sizes we accessed, we observed a "stringy phase," where strips of high and low particle densities of a distribution of widths appear. While there is long range order in the y-direction (i.e., presence of strips, or "strings"), there appears to be no such order in the x-direction (i.e., absence of condensation into a single strip). Though there is an attempt to explain this stringy phase by the dynamics of metastability [40], we believe that the essential ingredients are still missing, since an *initially ordered*, single strip configuration is observed to break up into the stringy state (Figure 3 in [39]).

Beyond two dimensions, there are only a few studies of systems in higher dimensions. So far, no new surprises appear when the aspect ratios are not extreme. On the other hand, based on various motivations, a bi-layered lattice gas was investigated. The first study [41] consists of two identical driven Ising lattices, arranged one above the other in a bi-layer fashion. There are no inter-particle interactions between the planes, though particle hops between them are allowed. With infinite drive, *two* transitions are observed when T is lowered. At approximately the same $1.4T_O$, the disordered phase gives way first to single strips on both planes, as if a carbon-copy of a system in the low temperature state is made and stacked on top the original! When T is lowered further, a discontinuous transition occurs, so that the low temperature phase consists of homogeneous, but "opposite" densities on each plane. Characterizing the densities in each plane, we label the former "the strip (S) phase" and the latter, "the full-empty (FE) phase)." Since these two phases can be enhanced/suppressed by cross-plane inter-particle interactions ($J_\times$), we carried out an extended study to unravel this seeming mystery [42]. A phase diagram is found in the T - $J_\times/J$ plane. In the absence of E, this diagram is symmetric, since $\pm J_\times$ systems are related to each other by a simple (Ising) gauge transformation. Since there is no interface associated with the FE phase, the $J_\times = 0$ line is special and the ordered state is never S-like. However, the drive breaks the Ising symmetry and modifies the two phase-boundaries seriously. In particular, it enhances/suppresses the S/FE state so much that the S phase "invades" into a small region of $J_\times < 0$ half-plane (Figure

2 in [42]). Though it is unclear what is the precise role of interfacial energies, they must be sufficiently strong to favor the FE phase again for $J_\times = 0$ and low T . Thus, two transitions will occur when T is changed, for a range of (fixed) $J_\times$. Extending these studies, Chng and Wang [43] considered bi-layer systems with anisotropic interactions. In addition to verifying the qualitative features of the phase diagrams (e.g., "invasion" by the S into the FE region), these authors attempted to measure critical properties. The simulations turned out to be quite difficult technically and, while some exponents are measured, conclusive evidence concerning the universality class is still lacking. Below, we will discuss these issues further.

3.1.2. *Randomly driven or two temperature models*

It is difficult to realize toroidal boundary conditions in physical systems, let alone a uniform time-independent $\vec{E}$. However, we can imagine placing particles on the surface of a cylinder (which is consistent with periodic boundary) and changing the magnetic field through the cylinder, inducing a uniform $\vec{E}$ on the particles. To have a time-independent $\vec{E}$, we must increase the magnetic field linearly with time, a process which cannot be maintained indefinitely. On the other hand, it is easy to have an AC field, or a random field. Since AC fields necessarily involve a time scale (and likely lead to more complex, non-equilibrium *periodic* states), a simpler model to study would be the one with random drive. At large length and long time scales, such a random drive might be incorporated into an extra noise associated with hops along y. This is physical motivation for studying the "two-temperature model," [44] introduced first as a purely theoretical system [21]. To implement this model in simulations, we simply update particle-hole exchanges in x- (y-) direction according to $\min[1, e^{-\beta\Delta\mathcal{H}}]$ ($\min[1, e^{-\beta_y\Delta\mathcal{H}}]$). The concept of "extra noise" leads us to set $T_y > T$, i.e., $\beta_y < \beta$., although, with toroidal boundary conditions, this choice is hardly significant.

What distinguishes this system from the uniformly driven case is clearly the lack of a non-trivial particle current in the steady state. However, it is even more clear that there must be an energy flux *through* our lattice here. Remarkably, the superficial features of this system are similar to the uniform E case. For example, holding T_y at infinity (i.e., all particle-hole exchanges in y are accepted!) and lowering T, ordering into a single strip takes place at a temperature about 40% above T_O [44]! Particle correlations also appear to be quite similar, except that, of course, the Ising symmetry (particle⇔hole) is now restored. As a result, there must be a drastic difference in the *three*-point correlations, as observed in [45]. With closer examination, the critical properties of this system are *not* identical to those of the uniform E case [44]. Though there are recent claims that they belong to the same universality class [46], we believe that there is neither a viable theory that leads to such a conclusion, nor is there any compelling justification for two systems with drastically different symmetries to be members of the same universality class [30, 47]. Finally, the low temperature states in these systems also behaves quite differently. Specifically, instead of $q^{-0.67}$, the structure factor decreases as $1/|q|$ but appears to extrapolate to a gap at $q = 0$[32]! Although the $1/|q|$ behavior was predicted [33], there is still no understanding of the gap.

3.1.3. *Two species, driven in opposite directions*

A natural extension to driven Ising lattice gases is the introduction of more than one species of particles. In the equilibrium case, the simplest system with two interacting particle-species (and holes) is known as the Blum-Emery-Griffths model [48], displaying quite a rich variety of phenomena. What would be the effect of driving this system into a non-equilibrium steady state? An early study, motivated by micro-emulsions in an electric field [49], involved simulations of quite a complex model and showed several surprising features. To answer this question mores systematically, we began with a "base-line study," involving non-interacting particles (apart from excluded volume) being driven in *opposite* directions. For simplicity, we will again think of the drive as an "electric" field, and distinguish the particles by their "charge:" $+1$ or -1. The control parameters are simply (a) the size of the system: L_x, L_y, (b) the numbers of each species: N_+, N_-, and (c) the field strength: E. Much to our surprise, there exist phase transitions, even in the simplest case of a neutral system with overall particle density m ($N_+ = N_- = (m/2)\, L_x L_y$). A phase diagram in the E-m plane consists of lines of discontinuous, as well as continuous transitions, from a disordered, homogeneous state to a locked, inhomogeneous, stationary state [50]. For square systems, the latter is characterized by a high density "horizontal" strip, spanning L_x. As the aspect ratio increases ($L_x > L_y$), other ordered states appear: strips with non-trivial winding numbers [51]. Furthermore, for charged systems ($N_+ \neq N_-$), the spatial structure in the ordered state *drifts* "backwards." In other words, the inhomogeneity drift in a direction *opposite* to the one favored by the majority species [52]! The limit of $m \to 1$ is also interesting, as it models vacancy mediated dynamics in binary alloys [53].

Since the ordered state is due to the blockage of one species by the other (also known as a "jam"), the most natural question to ask is the following. If we allow the two species to exchange as well, will the jam continue to occur? To answer this question, we introduce a rate of "charge exchange," so that for every particle hole chosen to attempt exchange, γ pairs of $+/-$ particles are chosen. Of course, for $\gamma = 1$, there can be no jam! But, for small γ (up to 0.4), the jammed phase persists in some region in the E-m plane [54]. However, the discontinuous portion of the transitions retreats steadily, perhaps due to the fact that γ acts as a "softening" agent to blockages. Apart from gross features such as phase transitions, these systems display intriguing properties, even in the disordered state. Thus, particle correlations are not only long ranged, their associated structure factors are found to posses unusual distributions [55]. Finally, let us report on yet another intriguing aspect of this system. For a $d = 1$ model, it can be proved that the stationary state is always homogeneous [56]. Since this system reminds us of "1-lane" traffic (of fast cars and slow trucks, in a co-moving frame where the average speed is zero), this statement translates into the absence of a "traffic jam." Given that there is a jam in $d = 2$ systems, we are led naturally to explore "quasi-1-D" cases, i.e., multi-lane traffic. Again, we were surprised: a jam does occur even for the case of *two lanes* ($2 \times L$). To be precise, a macroscopic cluster scaling with L is always present, up to $L = 4000$[57]. Though there is a conjecture that the system disorders in the $L \to \infty$ limit [58], the belief is that the relevant L is so large (perhaps of the order of 10^{70} [59]) that it is irrelevant for all practical purposes.

Before closing this section, we should mention that, beyond the simple models presented above, there are numerous generalizations (e.g., long range hops, multi-species, multi-layers, quenched impurities, mixed dynamics for modeling reaction-diffusion, etc.) all displaying many more interesting phenomena. Many are motivated by physical systems, such as traffic [60] and gel-electrophoresis [61], as well as fast ionic conductors and vacancy mediated diffusion. Unfortunately, we are too limited here to report on all of them. Clearly, it would be worthwhile to continue explorations of these driven diffusive systems, while many more unexpected phenomena should be expected.

3.2. Theoretical understanding of non-equilibrium properties

The understanding of collective behavior in the long-time and large-scale limit has been well served by continuum descriptions, such as hydrodynamics and Landau-Ginzburg theories. Following the lines of, e.g., the φ^4 theory for the equilibrium Ising model, we formulate continuum theories for the driven lattices gases, in arbitrary dimension d. Instead of an attempt to derive, via coarse-graining of the microscopic dynamics [62], such theories, let us take a more phenomenological approach. We begin by identifying the slow variables for a Langevin equation, and letting the noise account for the fast degrees of freedom. Typically, the former are (a) conserved densities and (b) ordering fields which experience critical slowing down near T_c. For our driven Ising gas, both criteria are met by a single field, namely the local "magnetization" or excess particle density, $\varphi(\mathbf{x}, t)$. Here, $\mathbf{x}$ stands for x_i $(i = 1, ..., d; x_d = y)$, the last being the one-dimensional "parallel" subspace selected by E. Since φ is conserved, we begin with a continuity equation, $\partial_t \varphi + \nabla \mathbf{j} = 0$, where, in the absence of a drive, the current takes its Model B [63] form: $\mathbf{j}(\mathbf{x}, t) = -\lambda \nabla \frac{\delta \mathcal{H}}{\delta \varphi} + \eta(\mathbf{x}, t)$. Here, $\mathcal{H} = \int \{\frac{1}{2}(\nabla \varphi)^2 + \frac{\tau}{2}\varphi^2 + \frac{u}{4!}\varphi^4\}$ (with $\tau \propto T - T_c$; $u > 0$) is the Landau-Ginzburg Hamiltonian, while η models the thermal noise. The latter is assumed to be white, with zero mean and variance $< \eta_i \eta_j >= 2\sigma \delta_{ij} \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$. In the presence of E, the functional form of $\mathbf{j}$ must be modified. First, we should account for the non-vanishing mass transport through the system and write an additional "Ohmic" current $\mathbf{j}_E$. Due to the excluded volume constraint, $\mathbf{j}$ vanishes at densities 1 and 0, corresponding to $\varphi = \pm 1$. Writing $\mathcal{E}$ for the coarse-grained counterpart of E, the simplest choice is $\mathbf{j}_E = \mathcal{E}(1 - \varphi^2)\hat{\mathbf{y}}$. Next, the drive clearly induces serious anisotropies, so that all ∇^2 operators should be split into components parallel (∂^2) and transverse ($\nabla_\perp^2$) to $\mathcal{E}$. In particular, $\tau \nabla^2 \varphi$ must be replaced by $(\tau_{||}\partial^2 + \tau_\perp \nabla_\perp^2)\varphi$ and the noise variance should read $\sigma_i \delta_{ij}$, with $\sigma_1 = ... = \sigma_{d-1} \equiv \sigma_{||} \neq \sigma_d \equiv \sigma_\perp$. Had the original system been just an equilibrium Ising system with anisotropic diffusion, then detailed balance dictates $\tau_{||}/\tau_\perp = \sigma_{||}/\sigma_\perp$. However, for a *non-equilibrium* system like ours, this equality is not only unnecessary, it is *violated*. Indeed, a perturbative calculation of these coefficients leads to positive (but different) corrections to $\tau_{||}$ and $\sigma_{||}$, but *not* the others. Generically, however, we should expect $\tau_{||}/\tau_\perp \neq \sigma_{||}/\sigma_\perp$. Summarizing, we write down the full Langevin equation [28]:

$$\begin{aligned} \partial_t \varphi(\mathbf{x}, t) \; &= \; \lambda \{(\tau_\perp - \nabla_\perp^2)\nabla_\perp^2 \varphi + (\tau_{||} - \partial^2)\partial^2 \varphi - 2\alpha_\times \partial^2 \nabla_\perp^2 \varphi + \\ &\quad + \frac{u}{3!}(\nabla_\perp^2 + \kappa \partial^2)\varphi^3 + \mathcal{E}\partial \varphi^2\} - \nabla \eta(\mathbf{x}, t), \end{aligned} \tag{4}$$

with noise correlations

$$< \eta_i(\mathbf{x}, t)\eta_j(\mathbf{x}', t') >= 2\sigma_i \delta_{ij} \delta(x - x')\delta(t - t'). \tag{5}$$

As in the equilibrium case, both τ's should be positive far above criticality and a good description of the system exists using simple perturbation theory with the nonlinear terms. Even at the linear level, the discontinuity singularity can be understood from $\lim_{k_x \to 0} S(k_x, 0) \to \sigma_\perp/\tau_\perp$ and $\lim_{k_y \to 0} S(0, k_y) \to \sigma_{||}/\tau_{||}$ [22, 10]. Further, as T is lowered, the divergence of only $\lim_{k_x \to 0} S(k_x, 0)$ means that $\tau_\perp$ vanishes *but* $\tau_{||}$

does not. As a result, at T_c, the next non-vanishing term, $\nabla_{\|}^4 \varphi$, will compete with $\partial^2\varphi$, so that, at this tree level, $k_{\|}^2 \sim k_{\perp}^4$ and $\Delta = 1$. From this point, it is possible to follow standard renormalization group methods to arrive at a long series of conclusions [28]: $d_c = 5$, a new *non-equilibrium* fixed point, presence of a hidden symmetry ("Galilean" invariance), irrelevance of $u\nabla^2\varphi^3$, exponents at all orders in $d_c - d$, Gaussian-like critical behavior, etc. Three independent Monte Carlo studies [13, 14, 15] found that the data are entirely consistent with these predictions.

When an additional layer is added, simulations showed that there are two type of ordered states (S and FE), accompanied by two lines of second order transitions in the T-$J_\times/J$ plane. These two lines a first order transition line, separating the S from the FE, at a "bi-critical" point [42]. The original conjecture is that associated with the critical lines are two different universality classes. They key lies in the difference between the ordering fields. Defining φ_Σ and φ_Δ as the sum and difference of the magnetizations in the two layers

$$\varphi_\Sigma \equiv \varphi_1 + \varphi_2; \ \varphi_\Delta \equiv \varphi_1 - \varphi_2 \ , \tag{6}$$

we see that the ordering field is φ_Σ for the D/S transition, while φ_Δ becomes critical for the D/FE transition. Meanwhile, the remaining field remains massive and non-ordering. Now, φ_Σ is conserved and coupled to the drive, so that the critical behavior on the D/S line should be the same as for the single layer system. On the other hand, φ_Δ is not conserved and, since the Wilson Fisher fixed point is stable against all operators of non-equilibrium origins (in non-conserved dynamics) [64], transitions on the D/FE line should be in the class of a non-conserved Ising model. Unfortunately, despite extensive technical efforts in a careful simulations study of the D/FE line, the results are not clear cut [43]. As a result, a renormalization group analysis was carried out, in hopes of clarifying the apparent mystery [65]. To our surprise, this investigation turned out to be highly non-trivial, even at the one-loop level. The "starting" equations of motion are readily postulated [10]:

$$\partial_t\varphi_{1,2} = \lambda\nabla^2 \ (\delta\mathcal{H}/\delta\varphi_{1,2}) + \mathcal{E}\,\partial\varphi_{1,2}^2 \pm \lambda_\times \,(1-\varphi_1\,\varphi_2)\,[\delta\mathcal{H}/\delta\varphi_2 - \delta\mathcal{H}/\delta\varphi_1]\,.$$

Here, the Hamiltonian $\mathcal{H}$ would contain the effects of both the in-plane J and the cross-plane $J_\times$, while $\lambda_\times$ is the rate associated with cross-plane jumps. Adding and subtracting these leads to equations for φ_Σ and φ_Δ. Of course, appropriate noise terms and anisotropic splitting of differential operators must be incorporated. For the D-S transition, many of the critical properties are indeed unchanged: $d_c = 5$, the hidden symmetry, a single independent exponent Δ, etc. A new feature here is the presence of a "one-way coupling" of the critical φ_Σ field into the dynamics of the non-critical φ_Δ. Much of these features survive also at the *bicritical* point, where both fields are critical. However, the conserved φ_Σ essentially "enslaves" the non-conserved φ_Δ, so that the overall properties are dominated by φ_Σ. The situation is far more complex for the D-FE transitions, where only φ_Δ is critical. Being neither conserved nor directly coupled to $\mathcal{E}$, we can expect the usual $d_c = 4$ to remain. However, φ_Σ is, though non-critical, a conserved field and so, slowly varying. Thus, both φ_Σ and φ_Δ are "slow," and have effectively the same naive dimensions. This non-trivial interplay generates *new* operators as well as two dimensionless ratios of four dynamic coefficients. In the end, there are seven relevant couplings, so that 35 Feynman graphs are needed at the one-loop level! In this 7-dimensional space, there are many fixed points, most being unstable. While the stable fixed points are essentially in the class of model A [63] (statics in the equilibrium Ising class), there are novel features. Indeed, these fixed points form an entire *domain* in the plane of the two dimensionless ratios. As in the D/S case, there is a "one-way effect" on the non-critical due to the critical one. Though all the novelties affect only the *subleading* singular behavior, our system is still a bona-fide non-equilibrium system. Further, the presence of many unstable fixed points which are not Ising like may create serious crossover phenomena, and may very well be the source of the difficulties reported in [43]. The details are far beyond the limitations here. Let us refer the interested reader to [66].

For the randomly driven Ising gas (or the two temperature model), the Langevin equations cannot possibly be the same. Specifically, there is *no* particle current and *no* breaking of the Ising symmetry. On the other hand, detailed balance is still violated, so that the appropriate starting point would be Eqns. (4 and 5), except that the $\mathcal{E}\partial\varphi^2$ term is absent [67]. Since this term was crucial for arriving at $d_c = 5$ and the new fixed point, the randomly driven system cannot possibly belong to the same class as the uniformly driven case. When a renormalization group analysis is carried out, the hidden symmetry in the uniformly driven case is no longer present, so that exponents must be computed order-by-order in an ϵ-expansion. Fortunately, the critical dimension has dropped to $d_c = 3$, so that $d_c - d$ is only unity for all models in simulation studies. Again, there is generally acceptable agreement between simulations and theoretical predictions for the exponents [44]. A most intriguing feature is that the relevant fixed point here is actually the same as the one for an equilibrium Ising system with dipolar interactions [67], evolving under conserved dynamics. Of course, what we have is still a fully *non-equilibrium* system; only the *leading* singularities near criticality are the same. When subleading behavior, associated especially with dangerous irrelevant operators, are studied, the deviation from equilibrium characteristics will emerge. Clearly, a review of renormalization group studies is beyond the scope of these lecture note. The interested reader may find references in a recent summary [29].

Turning next to the two species model, the continuum approach leads us readily to a set of coupled, partial differential equations for the two conserved densities ρ_+ and ρ_-. The forms of their currents are postulated based on much the same ingredients: (a) mobility coefficients of the form $\rho_\pm (1 - \rho_+ - \rho_-)$, (b) gradients of pressures from a non-interacting free-energy, and (c) "Ohmic" terms for the drive. Written in terms of the hole and charge densities ($\phi \equiv 1-\rho_+-\rho_-$ and $\psi \equiv \rho_+-\rho_-$, respectively), the result is

$$\partial_t \phi = \nabla^2 \phi + \mathcal{E} \partial (\phi \psi) \tag{7}$$

$$\partial_t \psi = \phi \nabla^2 \psi - \psi \nabla^2 \phi - \mathcal{E} \partial (\phi (1 - \phi)) . \tag{8}$$

For simplicity, the noise terms are not shown. Even at this mean-field, deterministic level, these equations admit stationary solutions corresponding to both the homogeneous disordered phase and an inhomogeneous, ordered phase. Linear stability analysis from the disordered states indicates where the phase boundary might lie, in good qualitative agreement with simulation data. Further, we were able to find, analytically, non-trivial stationary states $(\phi (y) , \psi (y))$[68] by imposing periodic boundary conditions and constrains from overall densities:

$$\phi (0) = \phi (L) ; \; \psi (0 = \psi (L)) ; \; \int \phi = (1 - m) L ; \; \int \psi = 0. \tag{9}$$

Combining these with the technique of adiabatic elimination, the existence of both continuous ("second order") and discontinuous ("first order") transitions were predicted. Indeed, the continuous transition were found *after* the theoretical analysis [69]! In addition, by impose $\int \psi = qL$ instead of zero, we can prove that the inhomogeneous state must drift, in the direction opposing the majority species [52]. By an appropriate choice of $\mathcal{E}$, acceptable agreement between the theoretical density profiles and the observed can be achieved, while inclusion of a $O (\rho_+ \rho_- \phi)$ term gives excellent fits throughout [52]. Venturing further, we can describe systems with charge exchange by adding currents proportional to γ to these equations. Again, analytic inhomogeneous solutions can be found and both types of transitions persist for a range of γ [54]. Finally, by adding appropriate noise terms, we can understand the remarkable behavior displayed by the particle correlations and structure factors [55].

In this subsection, we have reported mainly the successes of the theories. Though these are many, we should not, however, lose sight of the majority of surprising phenomena yet to be understood. Beyond the puzzles we have discovered so far, we can be sure that these systems will continue to present us with new and astonishing displays.

4. Some Recent Developments

Driven diffusive systems continue to attract new investigations. Some focus on resolving existing puzzles while others reveal new intrigues. Too numerous to describe in detail, we limit ourselves to a brief description of just three recent developments.

4.1. Universality class for the "standard" driven Ising lattice

Since the modifications from the equilibrium Ising lattice gas were so deceptively minor, early simulation studies by Marro and his co-workers exploited, understandably, the same methods, e.g., isotropic finite size scaling, etc. [70]. Recognizing the importance of k_y scaling with a different power as k_x, Leung [13] carried out the first studies which rely on *anisotropic* finite size scaling. With limited resources of the time, the conclusion is that there is no inconsistency between the data and all field theoretic prediction in [28]. A later, more extensive study arrived at essentially the same conservative conclusion [71]. Since the two groups disagree on all fronts, from $T_c(\infty)$ to exponents, the controversy continues [71]. Meanwhile, the former group made a series of modifications of their methods, as well as a "rigorous derivation" of the Langevin equation showing that the Ohmic term is absent (but *only for* $E = \infty$!). As a result, the more recent publications claim that the critical properties of the uniformly driven lattice gas falls into the *same* class as the randomly driven one. In the above, we pointed out what the difficulties with such theories are [30]. In the past two years, a third group has mounted an extraordinary effort to clarify the various issues [15]. First, they clarified the roles of various observables carefully, a non-trivial but necessary task in view of the complications caused by detailed balance violation. For example, the two-point correlation function not only decays as a power law ($1/r^2$) at all temperatures above T_c, the amplitude is *negative* for transverse separations [20]. Worse, as $T \to T_c$, transverse and longitudinal lengths scale differently ("strong anisotropy"). Thus, it is crucial to come up with a definition of correlation length which is the least problematic, especially for finite size scaling analysis. Their choice was the most basic one, exploiting the structure factor $S\left(k_x, k_y\right)$. So, the "transverse" correlation length, e.g., is extracted from

$$\lim_{k_x \to 0} \frac{1}{S\left(k_x, 0\right)} \frac{\partial}{\partial\left(k_x^2\right)} S\left(k_x, 0\right).$$

For a finite system, the lowest few k_x's are used. Indeed, to explore finite size effects further, a general definition for ξ is employed, using $k_x = \ell\pi/L$ and $m\pi/L$ were chosen in defining

$$\xi_{m\ell}\left(L\right) \equiv \sqrt{\frac{L^2}{\left(\ell^2 - m^2\right)\pi^2}\left(\frac{S\left(m\pi/L, 0\right)}{S\left(\ell\pi/L, 0\right)} - 1\right)}.$$

Finally, since even ℓ's are insensitive to the onset of order into a single strip, the most detailed and extensive analysis is performed on ξ_{13}. The existence of the correlation length in the $L \to \infty$, usually taken for granted in equilibrium systems, is painstakingly checked. Apart from ξ other quantities studied include the (transverse) susceptibility $\chi \equiv S\left(2\pi/L_x, 0\right)$ and the Binder cumulant for the first (transverse) Fourier amplitude $\int \varphi(x, y) \exp\left[2\pi i x/L_x\right]$. Finite size scaling analysis is then carried out for extracting both the standard critical exponents and finite size corrections. The overall conclusion is that, using lattice which obey $\Delta = 2$, *all* results are in agreement with the predictions of the original field theory [28]. The details are quite involved and lengthy. The interested student is urged to study [72], in addition to the articles [15]. Most recently, these authors repeated this type of analysis with lattices obeying $\Delta = 1$ (an excellent approximation for the randomly driven case), in an effort to uncover the mysteries behind the claim that lattice gas driven both randomly and uniformly belong to the same universality class. In particular, the hope is to test, quantitatively, the conjecture that the "wrong" Δ induces extra singularities which produce effective exponents and mimic critical behavior of another class. Another possibility for the eventual resolution of these controversies is that, with the "wrong" thermodynamic limit (e.g., $L_x, L_y \to \infty$ with L_x^2/L_y fixed), the low temperature state of this system is *not* the expected single strip. If, for example, the state resembles more the "stringy phase" [38], then many of the results of the Marro group can be understood as a non-trivial cross-over phenomena. Clearly, the "final chapter" on this issue reminds to be written, while the ways this deceptively simple model can astonish us seems to be limitless.

4.2. Effects of differential mobility in biased diffusion of two species

Recall that, in the simplest model of two non-interacting species, driven in opposite directions, there are continuous and discontinuous transitions between disordered and jammed state [50]. Further, if the numbers of the species are unequal, then, in the ordered state, the jam drifts *opposite* to the majority species [52]. Motivated by modeling the behavior of cars and trucks, we turn to another natural generalization, namely, species with different mobilities [69]. Denoting $\mu_{\pm}$ as the mobilities, we arbitrarily choose $\mu_{+} \geq \mu_{-}$ and the focus on the ratio $\mu \equiv \mu_{+}/\mu_{-}$. To implement this ratio, we keep a list of the particles (and their locations). In one Monte Carlo step (MCS), particles are chosen from the positive/negative list randomly with the frequency ratio of μ. Employing just a half-filled, 40×40 system with various drive strengths, we again observe both type of transitions, for a range of μ up to 300. Here, perhaps the surprise is how little the phase boundary depends on μ. Nevertheless, there are novel features in both states. Due to $\mu > 1$, the homogeneous state now carries a non-trivial mass current. For the ordered state, the jam is observed to drift, much like the "charged" system in $\mu = 1$. However, the drift here is *in the direction* of the more mobile species. Using the more convenient "renormalized" MCS (RMCS defined as $\mu + 1$ MCS) for time, the drift velocity is found to depend on μ monotonically, saturating at about 1 lattice spacing per 1000 RMCS (see Fig. 1). On the theory front, the simplest modification is to multiply the currents by the factor $\mu_{\pm}$. The most immediate consequence is that the currents in the disordered state will differ by a factor of μ. This is indeed borne out, for a range of drive and μ. In the same vein, time dependence of all small fluctuations are controlled by complex eigenvalues, a signal of *drifting* decay. Beyond this trivial level, μ plays *no role* in both the linear stability analysis of the homogeneous state nor the non-trivial solutions for the inhomogeneous state! While the former might be argued as a good prediction for the minor μ-dependence of the phase boundaries, the latter is more disappointing: no drift for structures in the ordered states. In retrospect, these predictions can be expected. Both aspects involve setting some quantities to zero: the determinant of the linear stability matrix in the former and, for the latter, the entire left hand side of the evolution equations. Multiplication by constants ($\mu_{\pm}$) do not change zeroes. Of course, we could also argue that the drift is so small that zero is an excellent approximation! One possible way forward is to study the effect of the noise terms. Since differential mobility removes another symmetry between the species, the noise might renormalize the kinetic coefficients

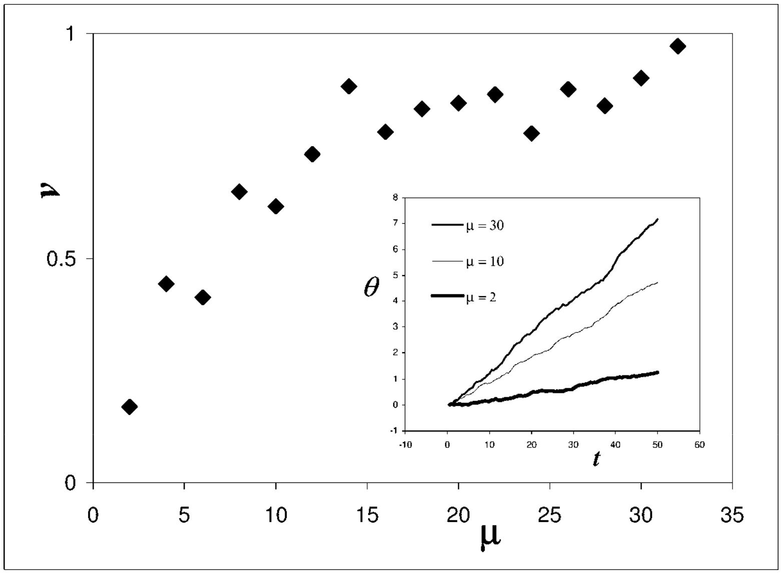

Figure 1. Velocity of the jammed structure, v (in units of lattice spacing per 1000 RMCS), as a function fo μ, the differential mobility. The inset shows, for three μ's, the center of mass position of the structure, $\theta \equiv 2\pi y/L$, drifting as a function of time, t (in units of 1000RMCS). All systems used are 40×40, so that $\theta = 2\pi$ corresponds to 40 lattice spacings.

differently and lead to a drifting solution. Hopefully, in the near future, we will be able to report only the existence of a drift, but also its direction.

4.3. Domain growth in a two-lane model with two opposing species

In the previous section, we presented the surprise associated with the absence/presence of an ordered state in biased diffusion of two species with charge exchange for one-/two- lane systems. In order to better understand the essential difference between them, we compared the time evolution of these systems, from an initial state of randomly distributed particles [73]. Specifically, we chose half filled systems with infinite drive and $\gamma = 0.1$. Comparing 1×1000 and 2×1000 cases, we found qualitative similarities up to 2500 MCS. The cluster size distribution in both cases display peaks (which move to larger sizes with time). After that time, there is not even a discernible peak in the one-lane case! Instead, the distribution turns monotonically decreasing and eventually settles down to the predicted, exponential form. The average cluster size is $O(1)$ (for this L as well as a range of them).

By contrast, in the two-lane case, not only does the average cluster sizes continue to increase (until a single macroscopic, $O(L)$ cluster [57] forms), the distribution displays dynamic scaling. To be precise, a cluster is defined as all particles (regardless of their charge) connected via nearest neighbor bonds and s as its total number of particles. Then, the "residence distribution," $p(s,t)$ (i.e., the probability of any site belonging to a cluster of size s at time t) is monitored. Consisting of two components, it displays two peaks, one at $s = 1$ and another moving to higher s with t. The first accounts for the "travelers" between the large clusters. In the language of traffic, these are the cars in a low density region, migrating from cluster to cluster. Focusing on the second component during intermediate times (before the formation of the single macroscopic cluster), we observed that the average cluster size, $\bar{s}(t)$, grows effectively as a power of t, which appears to approach 0.6 or possibly 2/3. Constructing a scaling plot, $\bar{s}(t)\, p(s,t)$ *vs.* $x \equiv s/\bar{s}$, we found excellent data collapse for $t \in [1K, 20K]$ MCS. On the theoretical front, there are two approaches. One attempts to capture the growth dynamics by focusing only on the *lengths* of the *particle*-clusters. Supposing there are n clusters, with label $\alpha = 1, ...n$, we denotes these lengths by ℓ_α, (approximately $s_\alpha/2$). The entire system is replace by a purely 1-D lattice of $L/2$ sites (ignoring the travelers), with n markers, located ℓ_α sites apart. When one cluster grows at the expense of its neighbor, the appropriate marker would move, much like a random walker. The "evaporation" of a cluster translates into coalescence of two walkers. If there are no interactions between the walkers, this system can be solved analytically[74] and predicts, e.g., $n \to t^{-1/2}$. Since $\bar{s}$ is essentially L/n, this naive model predicts a growth that is too slow. To account for the anomalously large growth, we introduce attractive interactions between the walkers, based heuristically on the properties of the "unjamming" transition. The resulting distributions and scaling plots are excellent approximations for the original system. Further details may be found in [73]. Finally, we have discretized the continuum equations (Eqns.7,8, modified to include charge exchange [54]) and track their evolution numerically. The resulting inhomogeneous structures display much the same properties of domain growth. Reminded by the $t^{2/3}$ behavior in the Burgers equation [75], we hope to find a similar analytic prediction for these equations [76].

5. Concluding Remarks

In these lectures, a small corner of the intriguing world of non-equilibrium statistical mechanics has been presented. The hope is that, with a clear distinction between equilibrium and non-equilibrium steady states, we can start searching for a sound theoretical foundation for the latter. We initiated this search by exploring a particularly interesting "small corner:" driven diffusive systems. Since their inception almost 20 years ago, many novel phenomena have been found and some were understood. Many more discoveries remains to be made and puzzles to be fathomed. These few paragraphs have been devoted to a brief review of simulation results of only two-dimensional systems and certain theoretical approaches. Indeed, we have not touched on the entire industry focused on one-dimension driven diffusive systems, for which analytic solutions are often available [77]. While interesting in their own ways, these systems lack *"transverse"* dimensions, which give rise to many of the remarkable behavior described here. They are truly "one-dimensional," in the sense of the common usage of that adjective.

Though we have included a few recent developments, the list is by no means comprehensive. As an example, the "base-line" study of driven two-species model is finally being extended to include inter-particle interactions. With the appearing of new types of transitions, even the phase diagram has not been completely mapped out [78]. Needless to say, much more effort, on both the simulations and the theoretical fronts, is needed before we can claim to have unraveled the secrets of driven diffusive systems. We hope that this short article will generate sufficient interest in a broad audience to join our venture.

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