

Reaction-diffusion processes and non-perturbative renormalisation group

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Abstract. This paper is devoted to investigating non-equilibrium phase transitions to an absorbing state, which are generically encountered in reaction-diffusion processes. It is a review, based on [1, 2, 3], of recent progress in this field that has been allowed by a non-perturbative renormalisation group approach. We mainly focus on branching and annihilating random walks and show that their critical properties strongly rely on non-perturbative features and that hence the use of a non-perturbative method turns out to be crucial to get a correct picture of the physics of these models.

1. Introduction

Our understanding of equilibrium critical phenomena has largely benefited from the success of perturbative renormalisation group (RG) techniques and from conformal symmetry properties in two dimensions. Unfortunately, this success story has not spread out to non-equilibrium systems, though the latter are far more common in nature. Indeed, the study of phase transitions between non-equilibrium steady states has taught us that critical phenomena or generic scale invariance turn out to be much richer far from thermal equilibrium — where detailed balance relations are violated — than in equilibrium statics or even near-equilibrium dynamics.

However, despite considerable efforts, the very ingredients settling the universality classes out of equilibrium remain on a fragile footing. The problem originates mostly in the absence of an effective free energy functional that would allow to straightforwardly classify the universal behaviours in terms of symmetries and interactions. And even when such a functional exists — which occurs for reaction-diffusion processes or for Langevin type dynamics — analytical progress turns out to be difficult. On the one hand, models with Langevin dynamics cannot be conformal invariant. On the other hand, the efficiency of standard RG approaches is hindered by the fact that first, critical dimensions, when they are identified, happen to lie far from the physically interesting ones and furthermore, no calculations are available above two-loop order, which prevent from resorting to powerful re-summation techniques. In this context, any theoretical tool that can overcome the previous pitfalls is valuable and the non-perturbative renormalisation group (NPRG) stands as a promising candidate [4]. Indeed, this method has lead to great successes for systems at equilibrium during the last decade, appearing as a well-adapted tool to tackle strong coupling problems [4, 5]. Moreover, the NPRG formalism has been recently extended to non-equilibrium systems [1] and has given rise to important progress in the field of reaction-diffusion

processes, unveiling genuinely non-perturbative effects [1, 2, 3], which we review in this paper.

This paper is organised as follows. In section 2, we give an overview of reaction-diffusion processes and recall their field theoretical formulation. In section 3, we briefly introduce the NPRG formalism generalised to non-equilibrium systems and we derive the flow equations related to reaction-diffusion processes, before focusing on specific models. Some results on the universal properties of directed percolation (DP) are first reviewed in section 4 and the remainder of the paper (section 5) is devoted to the study of branching and annihilating random walks (BARW), which split into two categories, namely the ‘odd’ and ‘even’ BARW.

2. Reaction-diffusion processes

Reaction-diffusion processes constitute simple models that allow to gain some insights in non-equilibrium critical phenomena [6]. They are defined by a set of microscopic rules that govern the dynamics of a species of particles A . These particles randomly diffuse at a rate D and undergo some reactions, typically birth $A \xrightarrow{\sigma_m} (m+1)A$ and death $kA \xrightarrow{\lambda_k} \emptyset$ processes. From these competing interactions emerges at long time a stationary state which nature depends on the reaction rates λ_k and σ_m . Either all particles eventually vanish, leaving an empty state where all stochastic fluctuations cease and which is therefore called ‘absorbing’. Or the density eventually saturates to a finite average value yielding an ‘active’ state where the dynamics constantly generates density fluctuations. The active and absorbing states are separated by a continuous phase transition [6].

A large range of the absorbing transitions encountered in reaction-diffusion processes fall into the DP universality class [7], which stands as the most prominent one. It has led to a famous conjecture in the early eighties by Janssen and Grassberger [8] stating that a continuous transition to an absorbing state driven by a one-component order parameter will generically fall into the DP universality class (provided there is no additional symmetry or quenched disorder). The DP model, which corresponds to the rules

$$A \xrightarrow{\sigma} 2A, \quad 2A \xrightarrow{\lambda} \emptyset, \quad A \xrightarrow{\mu} \emptyset, \quad (1)$$

therefore plays a paradigmatic role, as the counterpart of the Ising model for equilibrium statistical physics.

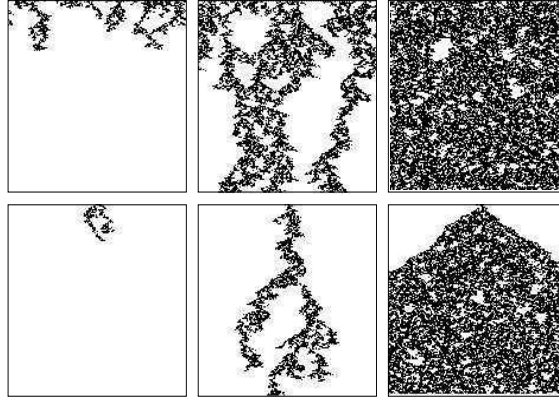
Reaction-diffusion processes naturally lend themselves to Monte-Carlo simulations, which have indeed largely contributed to our understanding of these processes (see [6, 9] for reviews). On the other hand, the simplest analytical approach is to devise a rate equation for the time-dependent average density $n(t)$, assuming the various reactions to occur proportionally to the concentration of reactants. For instance, for the DP processes (1), this yields:

$$\partial_t n(t) = (\sigma - \mu) n(t) - 2\lambda n(t)^2. \quad (2)$$

Eq. (2) entails a mean-field type of approximation since the density correlations are neglected — the joint probability of finding two particles at the same position has been factored. Eq. (2) is readily solved. It exhibits two stationary solutions $n_a = 0$

and $n_s = (\sigma - \mu)/(2\lambda)$, which stability depends on the sign of the ‘mass’ $\Delta = (\sigma - \mu)$. If $\Delta < 0$, the density decreases to the absorbing solution n_a , whereas it saturates to the active solution n_s for $\Delta > 0$. The explicit solution for the time-dependent density $n(t) = n_0 n_s / [n_0 + (n_s - n_0) \exp(-\Delta t)]$ shows that both asymptotic states are reached exponentially fast in time. The relaxation time Δ^{-1} diverges when $\sigma = \mu$ causing an algebraic decay of the density which corresponds to the critical state. The DP absorbing phase transition is illustrated in Fig. 1 [6], which represents the time evolution of particles on a one-dimensional lattice for increasing values of Δ (see caption).

Figure 1. Time evolution of particles on a one-dimensional lattice, starting from either a uniformly distributed initial configuration (top row) or a seed particle (bottom row), for the DP model (endowed with the dynamical rules (1)) from [6]. In this model, the stationary states (respectively absorbing on the left-hand side and active on the right-hand side) are reached exponentially fast in time, and are separated by a critical state (middle column) where both time and space correlation lengths diverge, resulting in an algebraic decay of the density.



Absorbing phase transitions can be characterised by a set of critical exponents, typically:

$$n_s \sim (p - p_c)^\beta \quad (3)$$

$$\xi_\perp \sim |p - p_c|^{-\nu_\perp} \quad (4)$$

$$\xi_\parallel \sim |p - p_c|^{-\nu_\parallel}, \quad (5)$$

where $z = \nu_\parallel/\nu_\perp$ embodies the dynamical exponent, which represents the anomalous scaling between space and time.

From rate equations such as (2), one can simply work out the — d -independent — values of the critical exponents at mean-field level (for instance for DP $\beta = 1$, $\nu_\perp = 1/2$ and $z = 2$ [6]).

Field theory

The mean-field picture holds as long as the density remains homogeneous — short-range correlated — enough in the system so that the role of space and time fluctuations is indeed negligible, which is generally justified in high dimensions.

However, the reaction-diffusion processes under scrutiny are so-called ‘diffusion-limited’, which means that the diffusion is not efficient enough to compensate the effect of reactions that locally alter the density distribution and hence invalidate the mean-field approximation. A finer analysis therefore requires a systematic incorporation of spacio-temporal fluctuations, which can be achieved through the construction of a field theory.

A field theory to describe reaction-diffusion processes can be derived following a well-known formalism due to Doi and Peliti [10]. The starting point is the master equation, which describes the time evolution of the probability distribution $P(\{n_i\}, t)$ of the occupation numbers n_i of the sites i of a lattice — assuming no occupation restriction. The idea is to write the change in the occupation numbers on each site by means of creation and annihilation operators. Upon introducing Fock states $|\{n_i\}\rangle$ to represent the configurations of the lattice, and a formal state vector $|\Phi(t)\rangle = \sum P(\{n_i\}, t) |\{n_i\}\rangle$, the master equation can be recast into a Schrödinger-like equation for the state vector $\partial_t |\Phi(t)\rangle = -H |\Phi(t)\rangle$. Then, resorting to coherent-state path integrals as in quantum mechanics, one can construct a functional integral $\mathcal{Z}[\phi, \hat{\phi}] = \int \mathcal{D}\phi \mathcal{D}i\hat{\phi} \exp(-\mathcal{S}[\phi, \hat{\phi}])$, which captures exactly (up to the continuum space limit) the microscopic stochastic fluctuations [10, 11]. The time-dependent statistical averages of observables — which are necessarily mere functions of the occupation numbers — can then be computed from $\mathcal{Z}$. For the general processes $A \xrightarrow{\sigma_m} (m+1)A$ and $kA \xrightarrow{\lambda_k} \emptyset$, this procedure yields [12]:

$$\mathcal{S}[\phi, \hat{\phi}] = \int d^d x dt \left\{ \hat{\phi}(x, t) (\partial_t - D \nabla^2) \phi(x, t) - \lambda_k (1 - \hat{\phi}(x, t)^k) \phi(x, t)^k + \sigma_m (1 - \hat{\phi}(x, t)^m) \hat{\phi}(x, t) \phi(x, t) \right\}. \quad (6)$$

The diffusion is encoded in the kinetic part and stands as the Gaussian (quadratic) theory which corresponds to the Brownian motion. All the reactions give rise to potential interaction terms. The action (6) can then root field theoretical treatments, and in particular NPRG methods. This action indeed underlies in the following the investigation of various reaction-diffusion processes, namely DP and BARW models. Before presenting these analyses (in sections 4 and 5), we first give a brief overview of the NPRG techniques.

3. Non-perturbative renormalisation group out of equilibrium

We do not intend to detail here the implementation of the NPRG, but rather set out its principle (and refer to [5, 4] for reviews). This formalism relies on Wilson’s RG idea [13], which consists in building a sequence of scale-dependent effective models, that interpolate smoothly between the short-distance physics at the (microscopic) scale $k = \Lambda$ and the long-distance physics at the scale $k = 0$, through progressively averaging over fluctuations. Rather than expressing — as in the original Wilsonian formulation — the flow of effective Hamiltonians for the slow modes, one can work out the flow of effective ‘free energies’ Γ_k for the rapid ones, following [14, 5]. Γ_k thus only includes fluctuation modes with momenta $|q| \geq k$. At the scale $k = \Lambda$, no fluctuation is yet taken into account and Γ_Λ coincides with the microscopic action $\mathcal{S}$, while at $k = 0$, all fluctuations are integrated out and Γ_0 is the analogue of the Gibbs free energy Γ at thermal equilibrium, in that it encompasses the long distance properties

of the system. To construct Γ_k , one needs to freeze the slow modes. This is achieved by adding a scale-dependent mass-like term to the original action [5, 1, 15]:

$$\Delta\mathcal{S}_k[\hat{\phi}, \phi] = \int_{x,t} \hat{\phi}(x,t) R_k(\nabla^2, \partial_t) \phi(x,t), \quad (7)$$

where the cutoff function R_k behaves as $R_k(q^2, \omega) \sim k^2$ (in Fourier space) for small momenta $|q| \leq k$ — so that the slow modes are decoupled — and R_k vanishes for large momenta $|q| \geq k$ — so that the rapid ones remain unaltered. The “partition functions” $\mathcal{Z}_k[j, \hat{j}] = \int \mathcal{D}\phi \mathcal{D}\hat{\phi} \exp(-\mathcal{S} - \Delta\mathcal{S}_k + \int j \phi + \int \hat{j} \hat{\phi})$ become therefore k -dependent. Γ_k is obtained through the Legendre transform of $\log \mathcal{Z}_k[j, \hat{j}]$:

$$\Gamma_k[\psi, \hat{\psi}] + \log \mathcal{Z}_k[j, \hat{j}] = \int j\psi + \int \hat{j}\hat{\psi} - \Delta\mathcal{S}_k[\psi, \hat{\psi}] \quad (8)$$

and is a functional of the conjugate fields $\psi = \delta \log \mathcal{Z}_k / \delta j$ and $\hat{\psi} = \delta \log \mathcal{Z}_k / \delta \hat{j}$. Note that the last term in Eq. (8) ensures that Γ_k has the proper limit at $k = \Lambda$: $\Gamma_{k=\Lambda} \sim \mathcal{S}$ [5, 15]. An exact functional differential equation governs the RG flow of Γ_k under an infinitesimal change of the scale $s = \log(k/\Lambda)$ [5, 1]:

$$\partial_s \Gamma_k = \frac{1}{2} \text{Tr} \int_{q,\omega} \partial_s \hat{R}_k \left(\hat{\Gamma}_k^{(2)} + \hat{R}_k \right)^{-1}, \quad (9)$$

where $\hat{R}_k$ is the symmetric, off-diagonal, 2×2 matrix of element R_k (in Fourier space) and $\hat{\Gamma}_k^{(2)}[\psi, \hat{\psi}]$ the 2×2 matrix of second derivatives of Γ_k with respect to ψ and $\hat{\psi}$. Obviously, Eq. (9) cannot be solved exactly and one usually devices an *ansatz* for Γ_k . However, as the approximations used do not rely on the smallness of a parameter, the approach remains non-perturbative in essence. In particular, it is not confined to weak-coupling regimes or to the vicinity of critical dimensions and is therefore suitable to overcome the usual perturbative RG schemes.

Since the critical physics corresponds to the long distance ($q \rightarrow 0$) and long time ($\omega \rightarrow 0$) limit, a sensible truncation consists in expanding Γ_k in powers of gradients [14] and time derivatives. Retaining only the leading order in derivatives, the most general *ansatz* for Γ_k related to the field theory (6) writes [1]:

$$\Gamma_k(\psi, \hat{\psi}) = \int d^d x dt \left\{ \hat{\psi}(x,t) [D_k[\psi, \hat{\psi}] \partial_t - Z_k[\psi, \hat{\psi}] \nabla^2] \psi + U_k[\psi, \hat{\psi}] \right\}. \quad (10)$$

The effective potential U_k encompasses all possible reactions. The kinetic renormalisation functions D_k and Z_k of the diffusive part account for the anomalous scalings of the fields and of time. Indeed, the anomalous dimension η of the fields and the dynamic exponent z are defined such that at criticality, $\psi \hat{\psi} \sim k^{d+\eta}$ and $\omega \sim k^z$ respectively [16]. It follows that $\eta_k = -\partial_s \ln D_k$ and $z_k = 2 + \eta_k + \partial_s \ln Z_k$, the critical exponents η and z corresponding to the (k -independent) fixed point values of η_k and z_k .

The flow equations for the potential U_k and the renormalisation functions Z_k and D_k have been established for generic reaction-diffusion processes in [1]. Different levels of approximation can be implemented. The simplest one, the so-called local potential approximation (LPA), consists in neglecting the kinetic renormalisations,

i.e. in setting $Z_k = D_k = 1$, upon which $z = 2$ and $\eta = 0$. The LPA generally enables one to get a reliable qualitative picture of the physics as well as a fairly accurate determination of the static exponent ν . This approximation can be refined by including (field-independent) kinetic renormalisation coefficients D_k and Z_k , which allows for non-trivial estimates of η and z . This approximation is referred to as leading order (LO). Finally, to get more accurate values of the exponents requires to incorporate the field dependence of the kinetic functions $Z_k[\psi, \hat{\psi}]$ and $D_k[\psi, \hat{\psi}]$ — referred to as next to leading order (NLO) — which becomes much more tedious numerically.

In order to study a specific model, the generic flow equations $\partial_s U_k$, $\partial_s Z_k$ and $\partial_s D_k$ must be solved for the functionals Γ_k respecting the symmetries of the problem, starting at scale Λ from the corresponding microscopic action (6). We focus in the next section on the DP universality class.

4. Directed Percolation

The action corresponding to the DP processes (1) can be deduced from (6) and writes:

$$S[\phi, \hat{\phi}] = \int_{x,t} \hat{\phi}(\partial_t - D\nabla^2)\phi - \lambda(1 - \hat{\phi}^2)\phi^2 + \sigma(1 - \hat{\phi})\phi\hat{\phi} - \mu(1 - \hat{\phi})\phi. \quad (11)$$

It can be conveniently rewritten expanding the response field around its stationary solution $\hat{\phi}(x, t) = 1 + \tilde{\phi}(x, t)$ and rescaling the fields according to $\tilde{\phi} \rightarrow \sqrt{2\lambda/\sigma} \tilde{\phi}$ and $\phi \rightarrow \sqrt{\sigma/2\lambda} \phi$, which leads to:

$$\mathcal{S}_{DP}[\phi, \tilde{\phi}] = \int d^d x dt \left\{ \tilde{\phi} [\partial_t - D\nabla^2 - (\sigma - \mu)] \phi + \sqrt{2\sigma\lambda} [\tilde{\phi}\phi^2 - \tilde{\phi}^2\phi] + \lambda(\phi\tilde{\phi})^2 \right\}. \quad (12)$$

This action turns out to be equivalent to the so-called Reggeon field theory [17], which has been studied in particle physics since the early seventies [18]. The critical exponents have been computed in that context to two-loop order. However, the upper critical dimension of this field theory is $d_c = 4\ddagger$, which is distant from the physical dimensions $d = 1$ and $d = 2$. Moreover, in spite of its simplicity, the DP model has no exact solution in $d = 1$ — at variance with the Ising model. Thus, the best estimates of the critical exponents of DP rely on numerical calculations and are given in table 1.

Providing analytical estimates of the DP exponents thus requires a method that is not confined to the vicinity of a critical dimension. This motivates the use of a non-perturbative approach — the NPRG — to fulfil this task, since the latter allows to span arbitrary dimensions. The generic flow equations introduced in section 3 can be used to compute the critical exponents of the DP universality class, provided the symmetries of the DP model are specified. The bare action (12) turns out to be invariant under the change:

$$\begin{cases} \phi(x, t) & \rightarrow -\tilde{\phi}(x, -t) \\ \tilde{\phi}(x, t) & \rightarrow -\phi(x, -t), \end{cases} \quad (13)$$

which is called the ‘rapidity’ symmetry. The effective potential $U_k(\psi, \tilde{\psi})$ and kinetic renormalisation functions $D_k(\psi, \tilde{\psi})$ and $Z_k(\psi, \tilde{\psi})$ — denoted X_k in the following —

$\ddagger$ Note that in this paper, d denotes the dimension of *space* — *i.e.* does not include the time direction.

Table 1. Critical exponents of the DP universality class, from NPRG calculations within different levels of approximation — LPA, LO in $d = 1, 2, 3$ [1] and NLO in $d = 2, 3$ [15]. The last column gathers the best numerical estimates, ensuing from Monte Carlo (MC) simulations [19, 20] and series expansions [21].

d		LPA [1]	LO[1]	NLO [15]	MC+series [19, 20, 21]
3	ν	0.584	0.548	<i>0.59</i>	0.581(5)
	β	0.872	0.782	<i>0.83</i>	0.81(1)
	z	2	1.909	<i>1.90</i>	1.90(1)
2	ν	0.730	0.623	<i>0.73</i>	0.734(4)
	β	0.730	0.597	<i>0.59</i>	0.584(4)
	z	2	1.884	<i>1.70</i>	1.76(3)
1	ν	1.056	0.888		1.096854(4)
	β	0.528	0.505		0.276486(8)
	z	2	1.899		1.580745(10)

must be invariant under the rapidity transformation (13). This in turn imposes that the generic term of the Taylor expansion of the X_k 's be of the form $a_{\alpha\beta}(\psi^\alpha\tilde{\psi}^\beta + (-1)^{\alpha+\beta}\tilde{\psi}^\alpha\psi^\beta)$, which only involves the two invariant combinations $x = \psi\tilde{\psi}$ and $y = \psi - \tilde{\psi}$. Thus, parameterising the X_k 's in terms of the invariants x and y ensures that they satisfy the rapidity symmetry constraints.

The flow equations for U_k , D_k and Z_k then have to be solved numerically. One can integrate the flows with the scale s starting from some initial bare rates [1]. For a fine tuned initial mass Δ , the (dimensionless) effective potential flows to a fixed function, which corresponds to criticality. The exponents are calculated in the vicinity of this fixed solution. This procedure has been performed at different levels of approximation, namely the LPA, the LO and the NLO [1, 15]. The results are displayed in table 1 and show that the estimates converge fairly rapidly as the approximation is enriched to values in good agreement with the best numerical estimates.

5. Branching and annihilating random walks

Initially introduced by Bramson and Gray [22], models of BARW can be seen as reaction-diffusion processes endowed with the generic rules:

$$A \xrightarrow{\sigma_m} (m+1)A, \quad kA \xrightarrow{\lambda_k} \emptyset, \quad (14)$$

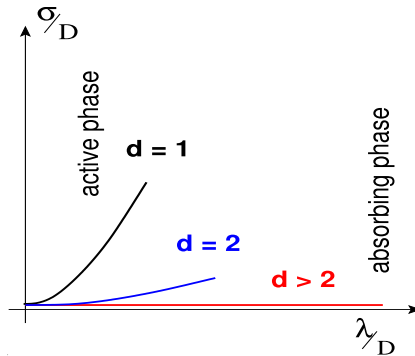
from which is excluded the spontaneous decay $A \rightarrow \emptyset$ (*i.e.* $k > 1$). The particles can therefore only disappear through a k -body annihilation. This restriction has drastic consequences since at mean-field level, it forces the system to reach an active state with mean density $n_s = [(m\sigma_m)/(k\lambda_k)]^{1/(k-1)}$ for any non-zero branching rate σ_m . A trivial transition occurs at $\sigma_m = 0$, where the model coincides with the pure annihilation model. The latter is well-controlled theoretically [11] and it predicts an algebraic decay of the density, which follows $n(t) \sim t^{-1/(k-1)}$ above the upper critical dimension $d_c(k) = 2/(k-1)$ and is slowed down by fluctuations below $d_c(k)$ [11]. However, early simulations [23, 24] have evidenced in low dimensions the existence of absorbing phase transitions at non-vanishing branching rates in these systems. These transitions have been found to belong to two different universality classes,

depending on the parity of m and k §. The corresponding models are therefore called ‘odd’ and ‘even’ BARW. Obviously, the effect of fluctuations cannot be neglected in low dimensions and need to be incorporated in a systematic way. In a seminal paper [12], Cardy and Täuber have derived the complete field theory for BARW and analysed both ‘odd’ and ‘even’ categories through perturbative RG. Their main results are summarised in the following. We mention here a straightforward outcome of their analysis, which is that for given k and m , all the reactions $k - 2, k - 4 \dots$ and $m - 2, m - 4 \dots$ are generated under renormalisation, the lowest m and k processes standing as the most relevant ones. Thus, investigating the generic processes $(A \xrightarrow{\sigma} 2A, 2A \xrightarrow{\lambda} \emptyset)$ and $(A \xrightarrow{\sigma} 3A, 2A \xrightarrow{\lambda} \emptyset)$ suffices to describe the critical behaviour of the odd and even BARW respectively. In both cases, the general idea advocated in [12] is to start from the pure pair annihilation process in the vicinity of its upper critical dimension $d_c = 2$ and to treat perturbatively a small branching rate $\sigma \sim \epsilon$. The problem is then to determine if fluctuations irretrievably destabilise the absorbing state — as suggested by mean-field — or if they allow for a non-trivial transition to take place. We first focus on the case of odd-BARW.

5.1. odd-BARW: phase diagram

For odd-BARW, the spontaneous decay $m = -1$ turns out to be generated under renormalisation through the combination $A \rightarrow 2A \rightarrow \emptyset$ at a renormalised rate μ_R depending on the bare rates λ and σ . The mass $\Delta_R = \sigma_R - \mu_R$ hence acquires corrections and the aim is to determine whether it can become negative for a non-zero bare σ and drive a transition to an absorbing state. If so, the transition will naturally belong to the DP class since then the renormalised action is identical to (12). Cardy and Täuber computed Δ_R following two different procedures which yielded consistent results [12] — that are sketched on figure 2. The conclusion is the following: fluctu-

Figure 2. Sketch of the transition lines of the BARW $A \xrightarrow{\sigma} 2A, 2A \xrightarrow{\lambda} \emptyset$ computed in [12]. For each dimension, the active phase lies above the transition line, the absorbing phase below.



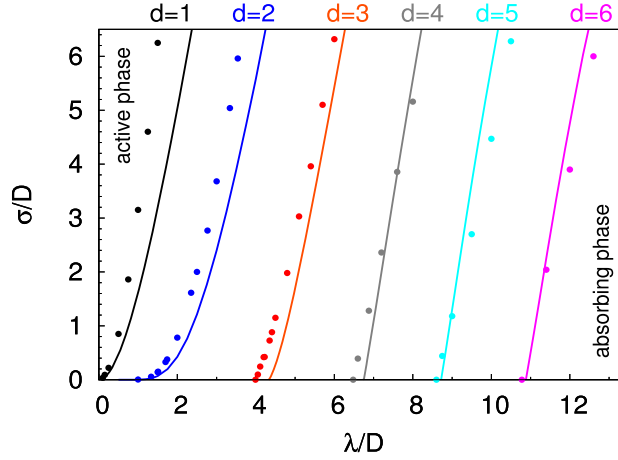
ations are strong enough to induce a non-trivial phase transition only in $d \leq 2$. The

§ Note that it has been argued in [25] that the parity conservation had no influence on similar reaction-diffusion systems and that the parity-wise distinction between odd and even BARW processes could be somewhat incidental, see also [26].

transition line is given by $\sigma_c = D[\lambda/(2D\pi\epsilon)]^{2/\epsilon}$ for $d < 2$ and is exponentially suppressed following $\sigma_c = D \exp(-4\pi D/\lambda)$ in $d = 2$ [12]. The perturbation theory breaks down above two dimensions. However, since the annihilation rate becomes irrelevant in $d > 2$ and since moreover directed random walks are known to be reentrant only for transverse dimensions $d < 2$ (that is the probability of intersection of two random paths vanishes above two transverse dimensions), Cardy and Täuber inferred that an absorbing state could no longer exist above two spatial dimensions [12]. In other words, their analysis suggests that the mean-field results are recovered for $d > 2$, that is the system is always in an active state above two dimensions.

We re-examined odd-BARW using NPRG. It should be stressed that critical rates, like critical temperatures, are non-universal quantities. Since the NPRG formalism allows to keep track, through the scale integration of the flow, of the initial microscopic (bare) action, it enables one to determine the critical rates for which the flow leads to a scale-invariant effective potential. We thus integrated the flow equations $\partial_s U_k$, $\partial_s Z_k$ and $\partial_s D_k$ at LO for different initial bare rates in dimensions 1 to 6 [2].

Figure 3. Phase diagrams of the BARW $A \xrightarrow{\sigma} 2A$, $2A \xrightarrow{\lambda} \emptyset$ in dimensions 1 to 6, from [2]. Lines present NPRG results, rescaled as explained in the text. Symbols follow from numerical simulations. For each dimension, the active phase lies on the left of the transition line, the absorbing phase on the right.



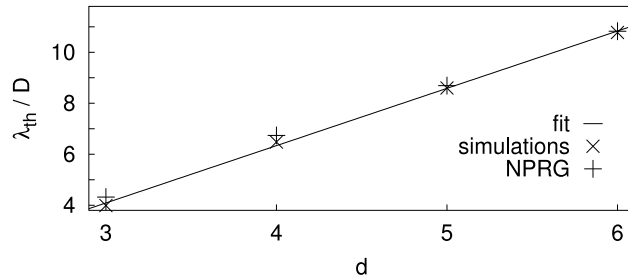
We also performed extensive numerical simulations, using an efficient algorithm introduced in [25] to back up our findings. For given values of λ/D , we generated on a lattice the stochastic time evolution of particles subject to the dynamics $A \xrightarrow{\sigma} 2A$, $2A \xrightarrow{\lambda} \emptyset$ and we searched for the critical rates σ/D which yield an algebraic decay of the density — in dimensions 1 to 6 [2].

Prior to comparing the obtained ‘discrete’ critical points $(\lambda/D, \sigma/D)$ with the ‘analytical’ ones ensuing from NPRG, the latter need to be rescaled by dimensional factors (C^{2-d}, C^2) to account for the continuous space limit underlying the field theory [2]. We emphasise that, however, matching a single point suffices to uniquely fix the value of the C parameter for all dimensions. Both numerical and (rescaled) analytical phase diagrams are displayed on figure 3 and show a remarkable agreement. Indeed,

let us recall that, at variance with universal quantities such as critical exponents, non-universal ones depend on all irrelevant operators (and microscopic details) and the accuracy of NPRG results was therefore unexpected.

Let us now analyse these phase diagrams. First, the transition lines of figure 3 are in accordance with the perturbative results in their region of validity, that is in the vicinity of the origin. Both numerical and analytical transition lines are indeed quadratic in $d = 1$ and exponential in $d = 2$ [2]. But away from the origin, the phase diagrams we obtained drastically differ from the perturbative results [12] sketched on figure 2. Indeed, we found a transition in $d = 3$ and in fact in all probed dimensions up to $d = 6$. The inactive phase appears to emerge only after a finite threshold $(\lambda/D)_{\text{th}}$ which is in essence a non-perturbative feature. It is indeed out of reach of any perturbative expansion around the origin. Furthermore, the transition lines seem to undergo a simple drift as the dimension grows. A closer investigation of the variation of the thresholds with the dimension reveals that they grow linearly with d (which has been checked up to $d = 10$) as shown on figure 4. This strongly suggests that an absorbing

Figure 4. Evolution of the thresholds $(\lambda/D)_{\text{th}}$ with the dimension, from [2].



DP transition occurs in all finite dimensions, or in other words that the mean-field result is recovered only in the $d \rightarrow \infty$ limit [2].

We emphasise that this study unveils a remarkable instance where fluctuations turn out to *qualitatively* invalidate the mean-field and even one-loop phase diagrams. Indeed, fluctuations do not only bring quantitative corrections to the critical rates but build up genuinely non-perturbative effects that affect the very existence of the transition. We now come to the case of even-BARW and show that non-perturbative features are even more crucial there since they entirely entail the physics of the model.

5.2. even-BARW: universality class

The models of even-BARW have concentrated much attention [23, 12, 26] as they embody the first example of an absorbing phase transition *not* belonging to the DP class, but to a new one — improperly, see [‡] — called ‘PC’ for parity conserving. For these models, a phase transition has been observed numerically in one dimension and characterised by non-DP exponents. To provide an analytical support to these findings, Cardy and Täuber have attempted a RG treatment of the field theory of

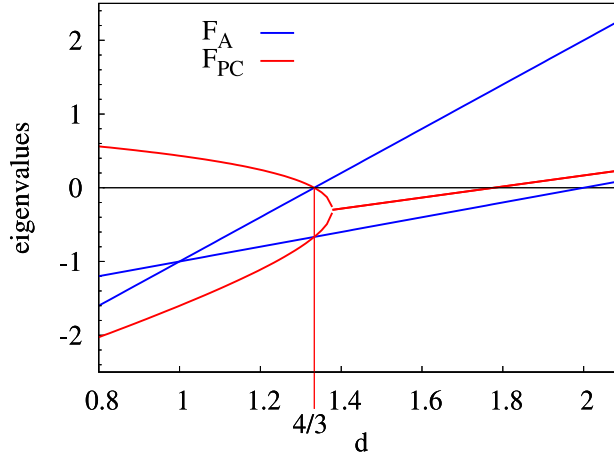
even-BARW, which writes [12]:

$$S[\phi, \hat{\phi}] = \int_{x,t} \hat{\phi}(\partial_t - D\nabla^2)\phi - \lambda(1 - \hat{\phi}^2)\phi^2 + \sigma(1 - \hat{\phi}^2)\phi\hat{\phi}. \quad (15)$$

Their analysis showed the appearance of a second upper critical dimension $d_c \simeq 4/3$ below which the branching rate σ becomes irrelevant, allowing for the annihilation fixed point F_A to become stable and possibly root an absorbing state in $d = 1$. Furthermore, in a direct calculation in $d = 1$, Cardy and Täuber [12] managed to identify a combination of σ and λ that admits a fixed point at one-loop order. However, it yielded poorly determined exponents and the extension of this calculation to higher orders appeared problematic [12].

We therefore re-analysed the even-BARW model by means of NPRG methods [3]. We once again exploited the generic flow equations derived in section 3, specified for even-BARW theory. The action (15) is Z_2 symmetric. The effective potential U_k should hence only depend on the quadratic invariants ψ^2 , $\hat{\psi}^2$ and $\hat{\psi}\psi$, structure which appears to be preserved by the flow equations $\partial_s X_k$'s. We here consider the LPA, which suffices to obtain a non-trivial picture of the physics of the model. The effective potential is Taylor expanded around the stationary solution ($\hat{\psi} = 1, \psi = 0$) and truncated at a given power n of the fields.

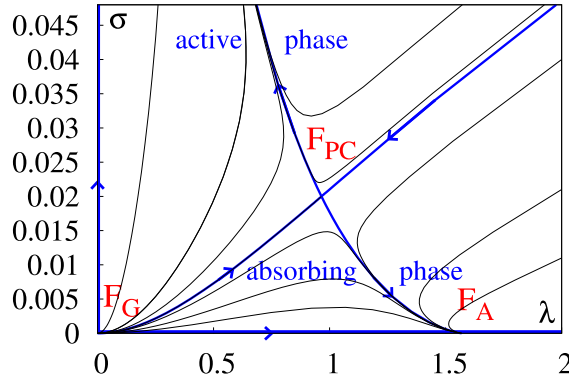
Figure 5. Variation with d of the eigenvalues of the fixed points in the lowest-order LPA, from [3]. Blue lines: pure annihilation fixed point F_A . Red lines: non-perturbative fixed point F_{PC} whose eigenvalues are both negative for $\frac{4}{3} < d < 1.3784\dots$ and complex-conjugated at larger d (only the real part is plotted).



Even from the lowest order $n = 2$ (corresponding to $U_k = -\lambda_k(1 - \hat{\psi}^2)\psi^2 + \sigma_k(1 - \hat{\psi}^2)\psi\hat{\psi}$), the flow equations for the coupling constants λ_k and σ_k exhibit, in addition to the Gaussian fixed point $F_G = \{\sigma = 0, \lambda = 0\}$ and to the pure annihilation fixed point $F_A = \{\sigma = 0, \lambda_A \neq 0\}$, a non-perturbative solution $F_{PC} = \{\sigma_{PC} \neq 0, \lambda_{PC} \neq 0\}$ [3]. The latter governs an absorbing transition in dimension $d < 4/3$. It becomes unphysical (with a negative σ_{PC}) and thus plays no role for $d > 4/3$. Indeed, the

stability of F_{PC} and F_A can be read off from figure 5 which displays their eigenvalues. For $d > 4/3$, the annihilation fixed point is the only physical one and it is once unstable (in the σ direction), implying that the system is always active. When d decreases, σ_{PC} approaches zero until F_{PC} crosses F_A in $d = 4/3$, where they exchange stability. Below $d = 4/3$, F_A is hence stable, describing the absorbing phase and F_{PC} acquires an unstable direction, thus driving a phase transition [3]. The corresponding flow diagram is depicted for $d = 1$ in figure 6.

Figure 6. Flow diagram of the lowest-order LPA in $d = 1$ (arrows represent the RG trajectories as s is decreased towards the “infra-red”, macroscopic limit $s \rightarrow -\infty$), from [3].



The critical exponent ν can be determined in the vicinity of F_{PC} and it converges as the order n of the field truncation is increased to $\nu = 2.0 \pm 0.1$, which is already in fair agreement with Monte Carlo simulations yielding $\nu = 1.85 \pm 0.1$. One would then need to go to the next order in the derivative expansion and include non-trivial kinetic renormalisations analogous to Z_k and D_k to get a determination of the other exponents. We stress that F_{PC} does not become Gaussian in any dimension and is thus genuinely non-perturbative. This explains why it cannot be reached *via* any perturbative expansion.

6. Conclusion and prospects

Through this review we have advocated the use of a non-perturbative field-theoretic method, the NPRG, to investigate non-equilibrium systems. We have highlighted the valuable results that this method has allowed to provide for various reaction-diffusion processes. Most importantly, we have unveiled that non-perturbative effects turn out to play a crucial role in the physics of these models. Indeed, we have first computed the phase diagram of odd-BARW and evidenced the existence of a non-perturbative threshold for the absorbing phase to emerge above two dimensions. This very threshold explains the failure of perturbative treatments which incorrectly lead to the conclusion that the transition disappears for $d > 2$. Furthermore, we have studied the NPRG flow equations of the even-BARW model and found a genuine non-perturbative fixed point — *i.e* non-Gaussian in any dimension — which is responsible for the PC absorbing phase transition in low dimensions and becomes unphysical above $d = 4/3$. It hence provides a theoretical back-up for the transition observed in numerical simulations,

reproducing in particular the numerical results for the ν exponent in $d = 1$.

The NPRG appears as an efficient tool to tackle non-equilibrium systems, which opens many prospects. First, it would allow to investigate critical properties of other reaction-diffusion processes. For instance, the universality class of the so-called pair contact process with diffusion model has been the subject of a long-lasting debate [27]. The influence of quenched disorder in DP models is also of great interest since the latter is suspected to hinder experimental realisations of this universality class [6]. Beyond the scope of reaction-diffusion processes, this technique has been applied to the study of the roughening transition in surface growth, generically modelled by the notorious Kardar-Parisi-Zhang equation [28]. It has allowed to obtain non-trivial results [29] and it would be of utmost interest to push further the investigation of growth phenomena. The NPRG methods could finally allow to give some insights into glassy dynamics.

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