Notes and Examples from Tensor Analysis for Continuum Mechanics

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This is a set of notes on basic vector and tensor algebra and analysis.

1 What is tensor analysis?

The conventional language of continuum mechanics is vectors and tensors. This language concerns first vectors. A vector in two or three dimensions is an arrow, *not a list of numbers*. Picture an arrow pointing at the sun, based at the center of Stonehenge, whose length is the intensity of light at noon on January 1, 2000. Its physical significance is clear, irrespective of how we describe it mathematically. If you ask four people to describe this arrow in mathematical terms (without communicating with each other), one person might give a list of three numbers (a, b, c). Someone else might write different numbers (d, e, f). Each will likely have chosen a different basis. Inspired by the shape of Stonehenge, maybe someone will have used a cylindrical polar coordinate system, with a list (r, θ, z) denote standard polar coordinates of the tip of the arrow with respect to that person's choice of basis. Someone else might simply say $\mathbf{v}$. The purpose of tensor analysis is to give precise rules that relate (a, b, c) and its basis to (d, e, f) and its basis, accounting for the given fact that *the two descriptions denote the same arrow*.

In continuum mechanics one often deals with vector fields. To describe them quantitatively one needs a basis erected at each point on a given region Ω. Of course, each of these bases could be the same (up to their choice of origin): if the basis is orthonormal, this case is called the *natural basis of a rectangular Cartesian coordinate system*. There are ways of constructing "natural" bases associated with other coordinate systems, e.g., well-defined bases that change from point to point. Tensor analysis deals in an automatic way with this case too, giving transformation laws for lists (a, b, c) which now depend on each point in Ω and a choice of basis at that point.

As a simple example, consider the vector field that gives the wind velocity at each point on a weather map. The arrows on the weather map probably approximately satisfy $\operatorname{div} \mathbf{v} = 0$. In components $(v_1(x_1, x_2), v_2(x_1, x_2))$ relative to the natural basis of a rectangular Cartesian system, $\operatorname{div} \mathbf{v} = 0$ is the statement $\partial v_1/\partial x_1 + \partial v_2/\partial x_2 = 0$. The same vector field (same arrows on the map!) expressed in the natural (orthonormal) basis of a polar coordinate system, $(v_r(r, \theta), v_\theta(r, \theta))$ then automatically satisfies "the equation $\operatorname{div} \mathbf{v} = 0$ in polar coordinates", namely $\partial(r v_r)/\partial r + \partial v_\theta/\partial \theta = 0$.

The same ideas apply to linear transformations. The analog of an "arrow" for a linear transformation consists of two pictures: a unit *cube* centered at the origin and a *parallelepiped* centered at the origin, together with a *rule* that says which corner of the cube goes to which corner of the parallelepiped. In short, a linear transformation is a cube-parallelepiped-rule. The vectors giving the positions of the corners of the cube could be given by components in one basis, and the components of the positions of the corners of the parallelepiped could be given in another (or the same) basis. Then the linear transformation would be described by a matrix. Tensor analysis gives the relationship between two matrices describing the same linear transformation when different choices of bases are made. Continuum mechanics has many tensors, and, like vector fields, they can depend on position ("tensor fields"). Typical examples are the deformation gradient, the velocity gradient tensor and the stress tensor.

The bottom line? All these components and bases and covariant components and Christoffel symbols at the beginning of a continuum mechanics course can be pretty daunting. Do they matter? Not much. The equations already must have the property of invariance under a change of basis field illustrated above by the example $\operatorname{div} \mathbf{v} = 0$. A vector is an arrow and a tensor is a cube-parallelpiped-rule. You can always just use one particular, say, rectangular Cartesian coordinate system and its natural basis field. In that particular system vectors are represented by lists and tensors by matrices. A simpler approach is to really think of a vector as an arrow and to write $\mathbf{v}$. Vectors are combined with each other and with scalars according to the rules of a vector space. If $\mathbf{v}(\mathbf{x}),\ \mathbf{x} \in \Omega \subset \mathbb{R}^n$, is a vector field, its gradient $\nabla \mathbf{v}$ is a tensor field. Every calculation can be done just using the abstract rules for derivatives and vector spaces. We will give those rules below and in the lectures.

2 Scalars, vectors and tensors

A scalar is a number, a vector is an arrow, and a tensor is a linear transformation. Two numbers can be added and multiplied and these operations are governed by the usual rules for combining real numbers. Similarly, two vectors can be added. Geometrically this corresponds to the placement of the base of the second arrow at the tip of the first one, the sum being a new arrow that goes from the base of the first to the tip of the second, i.e., the usual rule for vector addition. Similarly, a vector can be multiplied by a scalar to make a new vector that points in the same direction as the first but whose length is the length of the given arrow multiplied by the scalar. This could lead to trouble if we try to multiply a vector by the scalar 0, so it is convenient to also include the vector $\mathbf{0}$, the arrow "of zero length" which when added to any vector gives itself. The rules for vector addition in this way have been formalized as the axioms for a vector space (look them up).

As for notation, people use the real number symbol $\mathbb{R}$ to denote all scalars, and, with a few exceptions, Greek letters in this course will denote scalars, $\alpha \in \mathbb{R}$. The set of all collections of n real numbers like $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is denoted $\mathbb{R}^n$. It is a vector space with the arrow identified as the directed line passing from the origin $(0, 0, \dots, 0)$ to the point $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and the vectors used in this class will always lie in $\mathbb{R}^n$ and will be denoted by lower case Latin letters, $\mathbf{v} \in \mathbb{R}^n$. This will be true for all kinds of vectors – position vectors, forces, moments, velocities – whatever their interpretation (Some folks with a well developed organizational streak like to make up different names for the vector spaces containing these different kinds of vectors). Of course, $\mathbb{R}^1$ can be identified with the set of scalars $\mathbb{R}$. In a finite dimensional vector space (like, e.g., $\mathbb{R}^n$) there exists a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ consisting of n linearly independent vectors such that every vector in the space can be written as a linear combination of these, i.e., if $\mathbf{v} \in \mathbb{R}^n$ then there are scalars $\alpha_1, \dots, \alpha_n$ such that

$$\mathbf{v} = \alpha_1 \mathbf{e}_1 + \cdots + \alpha_n \mathbf{e}_n. \tag{1}$$

The scalars $\alpha_1, \dots, \alpha_n$ are called the *components* of $\mathbf{v}$ with respect to the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. In a finite dimensional vector space one can always choose an orthonormal basis, and in $\mathbb{R}^n$ one such basis is the *standard basis*

$$\begin{aligned} \hat{\mathbf{e}}_1 &= (1, 0, 0, \dots, 0), \\ \hat{\mathbf{e}}_2 &= (0, 1, 0, \dots, 0), \end{aligned}$$

$$\vdots$$

$$\hat{\mathbf{e}}_n \;=\; (0, 0, 0, \ldots, 1). \tag{2}$$

Often in mechanics one writes vectors as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ or, briefly, α_i. Unless a particular basis has been given, it is understood that these are components with respect to the standard basis. The bottom line is that calculations that primarily embody geometrical concepts are usually simplest using the symbol $\mathbf{v}$ while analytical calculations like the study of the differential equations of continuum mechanics are often best done with components. We'll use both descriptions in class.

Good references for finite dimensional vector spaces are the books by P. Halmos (Finite Dimensional Vector Spaces, Springer-Verlag), J. K. Knowles (Linear Vector Spaces and Cartesian Tensors, Oxford) and J. G. Simmonds (A Brief on Tensor Analysis). If this is the first time you have seen these concepts, you should get one of these books and do some of the relevant exercises.

The whole point about tensor analysis is the following. One person sets up a coordinate system, which is to say, he assigns a vector space $\mathbb{R}^n$ and a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for it, and he gets components α_i of a certain vector $\mathbf{v}$. Another person comes along and uses a different basis and gets different components β_i *of the same vector* $\mathbf{v}$. Tensor analysis gives the formula relating α_i and β_i that ensures that they describe the same arrow $\mathbf{v}$. This formula is called a "transformation law", but it has nothing to do with any law of physics: it is essentially a triviality that guarantees that everyone is talking about the same arrow.

A tensor is a linear transformation between vector spaces. Consider a tensor $\mathbf{A}$ that maps $\mathbb{R}^n$ to $\mathbb{R}^m$, $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$. That is, for every vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{A}$ delivers a definite vector (denoted $\mathbf{A}\mathbf{v}$) in $\mathbb{R}^m$ and it is linear:

$$\mathbf{A}(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 \mathbf{A}\mathbf{v}_1 + \alpha_2 \mathbf{A}\mathbf{v}_2, \quad \text{for all } \alpha_1, \alpha_2 \in \mathbb{R}, \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n. \tag{3}$$

If we give a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$ and another basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ for $\mathbb{R}^m$ then we can write the components of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and we can also write the components of $\mathbf{A}\mathbf{v}$ in the basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$. Call these two sets of components $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_m)$, respectively. Imagine that the two bases are fixed. Then it is clear that specification of the tensor $\mathbf{A}$ is equivalent to specifying an $m \times n$ matrix γ_{ij} that maps $(\alpha_1, \ldots, \alpha_n)$ to $(\beta_1, \ldots, \beta_m)$. That is,

$$\beta_i = \gamma_{ij} \alpha_j. \tag{4}$$

$\triangleright$**Explain briefly why specification of γ_{ij} specifies A.** In (4) we have used the summation convention, in which a summation is implied over each repeated index: (4) is in detail $\beta_i = \sum_{j=1}^{n} \gamma_{ij} \alpha_j, \ i = 1, \ldots, m$. If $n = m$ and we use the same basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for the domain and the range of $\mathbf{A}$ and, e.g., get components γ_{ij}, we say that γ_{ij} the components of $\mathbf{A}$ in the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

Below, we will use the Einstein summation convention and write $\beta_i = \sum_{j=1}^{n} \gamma_{ij} \alpha_j, \ i = 1, \ldots, m$ as simply $\beta_i = \gamma_{ij} \alpha_j$. We always sum over repeated indices, and the range of the summation will be understood from what is given earlier.

If someone else comes along and uses their own bases, they will get a different matrix $\hat{\gamma}_{ij}$ representing $\mathbf{A}$, so again we have a problem in tensor analysis: find the relation between $\hat{\gamma}_{ij}$ and γ_{ij} that guarantees that we are talking about the same tensor. (Geometrically, by the "same tensor", we mean the same cube-parallelepiped rule. Or, equivalently, two tensors are the same if, when they act on any given arrow, they produce the same arrow). This leads to "transformation laws" for tensors.

In $\mathbb{R}^n$ we can measure lengths of vectors and angles between vectors, and we have not yet used this important property of $\mathbb{R}^n$. That is, $\mathbb{R}^n$ is an *inner product space* having a "dot product". Inner product spaces are vector spaces with additional axioms governing the dot product, abstracted from their obvious counterparts in $\mathbb{R}^n$ (look these up). We write $\mathbf{a}\cdot\mathbf{b}$ to denote the dot product of the two vectors $\mathbf{a}$ and $\mathbf{b}$. In components in the standard basis, $\mathbf{a}\cdot\mathbf{b} = a_i b_i$. The length of a vector $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

Using the dot product we can develop formulas that relate a tensor to its components in a certain basis. Given two vectors $\mathbf{a}\in\mathbb{R}^m$ and $\mathbf{b}\in\mathbb{R}^n$ we define a tensor $\mathbf{a}\otimes\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^m$ by the rule

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{v} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a} \ \text{ for all } \mathbf{v}\in\mathbb{R}^n. \tag{5}$$

$\triangleright$**Show $\mathbf{a}\otimes\mathbf{b}$ defined in this way is a tensor, i.e., a linear transformation.** The tensor $\mathbf{a}\otimes\mathbf{b}$ is called the *tensor product* of $\mathbf{a}$ and $\mathbf{b}$. Given a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, we can define its so-called *reciprocal basis* $\{\mathbf{e}^1,\dots,\mathbf{e}^n\}$ from the formulas

$$\mathbf{e}_i\cdot\mathbf{e}^j = \delta_i^j. \tag{6}$$

Here, δ_i^j is the Kronecker delta, defined by $\delta_1^1 = \delta_2^2 = \cdots = \delta_n^n = 1, \delta_i^j = 0$ for $i \neq j$. The notational feature that distinguishes the reciprocal basis is that the indices are written up. This notation will be continued below. $\triangleright$**Draw a nonorthogonal basis consisting of two linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbb{R}^2$ and then draw its reciprocal basis $\mathbf{e}^1, \mathbf{e}^2$. Use sufficient accuracy to indicate that you understand** (6). Given the basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ formula (6) in fact uniquely defines the reciprocal basis. To see this fact, express each $\mathbf{e}^j$ in the basis $\{\mathbf{e}_1,\dots\mathbf{e}_n\}$ with certain coefficients and then notice that (6) comprises linear equations for these coefficients; the linear independence of $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ implies that these equations are uniquely solvable.

Let nm numbers $A^i{}_j,\ i = 1,\dots,m,\ j = 1,\dots,n$ be given, and let $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ be a basis for $\mathbb{R}^n$ and $\{\mathbf{f}_1,\dots,\mathbf{f}_m\}$ be a basis for $\mathbb{R}^m$. Define a tensor $\mathbf{A}$ by[1],

$$\mathbf{A} = A^i{}_j \mathbf{f}_i\otimes\mathbf{e}^j \tag{7}$$

Note that the reciprocal basis $\mathbf{e}^1,\dots,\mathbf{e}^n$ is used in this definition. $\triangleright$ **Assume $m = n$. Show, by choosing the numbers $A^i{}_j$ appropriately, that given any vectors $\mathbf{a}_1,\dots\mathbf{a}_m$ in $\mathbb{R}^m$, the action of A on a basis for $\mathbb{R}^n$ gives $\mathbf{a}_1,\dots,\mathbf{a}_m$. (As you can see from (6) and (7), a good basis to choose is $\mathbf{e}_1,\dots,\mathbf{e}_n$).** $A^i{}_j$ are called mixed components of the tensor $\mathbf{A}$ with respect to the bases $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ and $\{\mathbf{f}_1,\dots,\mathbf{f}_m\}$.

The identity tensor or simply *identity* is the tensor $\mathbf{I}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying $\mathbf{I}\,\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^n$. Another equivalent way of defining reciprocal bases is

$$\mathbf{e}_i\otimes\mathbf{e}^i = \mathbf{I} \quad \text{(sum over } i\text{)} \tag{8}$$

$\triangleright$**Show that (6) $\iff$ (8).**

The reader may notice that I suddenly started writing indices up and down. With reciprocal bases denoted with indices up, it becomes useful to extend that notation in a systematic way to components. In this notation repeated indices with components on different levels are summed. (Minor point: we also are using non-Greek letters with indices, like $A^i{}_j$, for components. This should not cause confusion because the indices indicate that these are scalar components). There are also some different kinds of "components" that have various names that aren't worth remembering. These are defined by,

$$\mathbf{A} = A_{ij}\mathbf{f}^i\otimes\mathbf{e}^j = A^{ij}\mathbf{f}_i\otimes\mathbf{e}_j = A_i{}^j\mathbf{f}^i\otimes\mathbf{e}_j. \tag{9}$$

[1]Again, we use the summation convention: $A^i{}_j\mathbf{f}_i\otimes\mathbf{e}^j$ is shorthand for $\sum_{i=1}^m\sum_{j=1}^n A^i{}_j\mathbf{f}_i\otimes\mathbf{e}^j$

Notice the way the indices are written up or down; from now on we will adhere to this convention.

How do I know that $\mathbf{A}$ can be written in the forms given in (9)? Let's look at the first representation $\mathbf{A} = A_{ij}\mathbf{f}^{\,i} \otimes \mathbf{e}^j$. $\mathbf{A}$ is a linear transformation and $\mathbf{e}_1, \dots \mathbf{e}_n$ is a basis for $\mathbb{R}^n$. So, by linearity, $\mathbf{A}$ is determined by its action on a basis. But

$$\mathbf{A}\mathbf{e}_k = (A_{ij}\mathbf{f}^{\,i} \otimes \mathbf{e}^j)\mathbf{e}_k = A_{ij}\mathbf{f}^{\,i}\delta^j_k = A_{ik}\mathbf{f}^{\,i}. \tag{10}$$

Fix k for a moment. We know based on the above discussion of reciprocal vectors that $\mathbf{f}^1, \dots, \mathbf{f}^m$ is a basis for $\mathbb{R}^m$, so, by choosing the coefficients A_{ik} appropriately, I can make $A_{ik}\mathbf{f}^{\,i}$ any vector in $\mathbb{R}^m$. Hence, I can make $\mathbf{A}$ map a basis into any set of m vectors in $\mathbb{R}^m$. Thus, $\mathbf{A}$ is determined because we have specified its action on a basis. Since $\mathbf{A}$ is a linear transformation, we have then specified its action on any vector in $\mathbb{R}^n$.

The representations (7) or (9) show that a tensor can also be regarded as a vector belonging to a finite dimensional vector space. That is, $A^i{}_j$ are the components of $\mathbf{A}$ in the basis $\{\mathbf{f}_i \otimes \mathbf{e}^j\}$. $\triangleright$**Check that this basis is linearly independent.** This is a basis for a vector space that can be regarded as $\mathbb{R}^{n\times m}$. Tensor products of bases are bases for vector spaces of tensors. If $n = m$, the space of tensors on $\mathbb{R}^n$ has a lot of additional structure. For example, one can multiply two tensors: $\mathbf{ABv} = \mathbf{A}(\mathbf{Bv})$. Also, this is an inner product space with inner product $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{AB}^T)$. (The symbol tr is defined just below.)

Suppose $A^i{}_j$ are components of a tensor $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ in a certain basis: $\mathbf{A} = A^i{}_j\mathbf{e}_i \otimes \mathbf{e}^j$. Consider two possible properties of the components $A^i{}_j$: (1) $A^1{}_1 = 1$; (2) $A^i{}_i = 1$ (re: repeated index, sum over i). These two properties have a very different status. Property (1) really has no physical or geometric significance. If I change to a new basis (that is, I find the components of $\mathbf{A}$ in a different basis) then I can change $A^1{}_1$ to a different number. In fact, if $\mathbf{A}$ is not proportional to the identity ($\mathbf{A} \neq \alpha\mathbf{I}$ for some $\alpha \in \mathbb{R}$) and $A^1{}_1 \neq 0$, I can change $A^1{}_1$ to any other nonzero number. So the 11 component of a tensor having a value of, say, 7 has no fundamental significance. However, the statement $A^i{}_i = A^1{}_1 + A^2{}_2 + \dots A^n{}_n = 7$ is very different: if it is true in one basis then it is true in any basis ($\triangleright$**prove this**). One can say that $A^i{}_i$ is really *a property of the tensor* $\mathbf{A}$. In this case the common value of $A^i{}_i$ is called the trace of $\mathbf{A}$, denoted tr$\mathbf{A}$. Obviously, this is an important topic for tensor analysis: find all, say, scalars that one calculate from the components of $\mathbf{A}$ that are independent of the choice of basis, and therefore are properties of $\mathbf{A}$ itself. The determinant of $\mathbf{A}$ is another one. In mechanics $A^1{}_1$ could not enter a basic law or have any fundamental significance. But of course the solution of some particular continuum mechanics problem done in a special basis could result in, say, $A^1{}_1 = 3$. Statements about the traces or determinants of various tensors (like stress or strain) can and do enter fundamental laws of mechanics.

An alternative way to define trace is as done in class. That is, tr is a linear mapping from tensors to scalars with the property that $\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

Using dot products we can also define symmetric and skew matrices. Let us suppose for simplicity that $n = m$ and consider a tensor $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$. The transpose of $\mathbf{A}$, denoted $\mathbf{A}^T$, is the unique linear transformation defined by

$$\mathbf{y} \cdot \mathbf{A}^T\mathbf{x} = \mathbf{x} \cdot \mathbf{A}\mathbf{y}, \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{11}$$

($\triangleright$**Show that this defines $\mathbf{A}^T$ uniquely**). By definition a symmetric tensor satisfies $\mathbf{A}^T = \mathbf{A}$ while a skew tensor satisfies $\mathbf{A}^T = -\mathbf{A}$.

Note that (11) defines $\mathbf{A}^T$ without reference to any basis, but, if one looks at the components of $\mathbf{A}$ in a particular basis, this has implications for the symmetry of the matrix. Note that $(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}$, and therefore, in view of (7), the symmetry of $\mathbf{A}$ implies the symmetry $A^i{}_j = A^j{}_i$ of its components only for special bases.

In 3 dimensions ($n = m = 3$), a skew tensor $\mathbf{W}$ has an associated *axial vector* $\mathbf{w} \in \mathbb{R}^3$ such that

$$\mathbf{W}\mathbf{v} = \mathbf{w} \times \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbb{R}^3. \tag{12}$$

Any tensor $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ can be decomposed into skew and symmetric parts:

$$\mathbf{A} = \mathbf{E} + \mathbf{W}, \quad \mathbf{E} = \mathbf{E}^T, \mathbf{W} = -\mathbf{W}^T. \tag{13}$$

In fact, $\mathbf{E} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and $\mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ $\triangleright$**Show that this decomposition is unique: if $\mathbf{A} = \mathbf{E}' + \mathbf{W}'$ where $\mathbf{E} = \mathbf{E}^T$ and $\mathbf{W} = -\mathbf{W}^T$ then $\mathbf{E} = \mathbf{E}'$ and $\mathbf{W} = \mathbf{W}'$.**

3 Vector and tensor fields and their derivatives

Often in mechanics we have vector or tensor *fields*. These are mappings that assign to each point in a domain $\Omega \subset \mathbb{R}^n$ a vector or tensor. Examples are velocity fields or stress fields. Consider a vector field $\mathbf{v} : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$. For each fixed $\mathbf{x}_1 \in \Omega$ we have a vector $\mathbf{v}(\mathbf{x}_1)$. We can do all the operations described above to this vector: add it to other vectors, express in in different bases, etc. If we go to a different point $\mathbf{x}_2 \in \Omega$ we get a different vector $\mathbf{v}(\mathbf{x}_2)$. If we were intent on making life difficult, we could express $\mathbf{v}(\mathbf{x}_2)$ in a different basis than we used for $\mathbf{v}(\mathbf{x}_1)$. If we were particularly troublesome, we could even have a different basis for every choice of $\mathbf{x}$ in Ω. That would be a *basis field* $\{\mathbf{e}_1(\mathbf{x}), \dots, \mathbf{e}_n(\mathbf{x})\}, \mathbf{x} \in \Omega$. In fact, it can actually be simplifying to introduce basis fields. For example, in problems with, say, cylindrical symmetry (cylindrical axis $\mathbf{e}$), one can introduce right handed orthonormal bases with the property that one of the vectors points in the direction $\mathbf{e}$ and another points in the radial direction. This might be helpful, for example, in a torsion problem in elasticity. For the analysis of tornadoes it is helpful to write the Navier-Stokes equations of fluid mechanics in the natural basis of a spherical coordinate system (defined below), with the origin at the apex of the tornado. Then a suitable inverse assumption reduces these equations to relatively simple ordinary differential equations which can be integrated to give quite a clear picture of the dynamics of tornadoes.

The use of basis fields complicates the calculation of derivatives. This is easily appreciated. Even if the components of a vector or tensor are constants, the vector or tensor may be varying with position if the basis (in which the components are expressed) is changing with position. Similarly, a vector that is constant – the same arrow at every $\mathbf{x} \in \Omega$ – has zero gradient. However, it will not have constant components in general, if the basis field is not constant. That's why, for example, the expression for the gradient in the natural basis of a cylindrical coordinate system looks quite different from the gradient in rectangular Cartesian coordinates. Components of a vector or tensor in the standard basis field are called *rectangular Cartesian components*. I denote this as RCC in class.

In class I gave definitions of the gradient of a vector field or a tensor field. These definitions follow a similar pattern. In the simplest case the gradient of a *scalar field* $\alpha : \Omega \to \mathbb{R}^n$ is defined by the formula,

$$\nabla\alpha(\mathbf{x}) \cdot \mathbf{v} = \lim_{\varepsilon \to 0} \frac{\alpha(\mathbf{x} + \varepsilon\mathbf{v}) - \alpha(\mathbf{x})}{\varepsilon} \quad \text{for all } \mathbf{v} \in \mathbb{R}^3. \tag{14}$$

The right hand side of (14) is a scalar-valued function of $\mathbf{v}$. If it is a linear function of $\mathbf{v}$ then there is a vector (called $\nabla\alpha$) such that (14) is true. This vector is called the gradient of α at $\mathbf{x}$. $\triangleright$**Prove the following: if the scalar-valued mapping $\rho : \mathbb{R}^n \to \mathbb{R}$ is linear, i.e. $\rho(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1\rho(\mathbf{v}_1) + \alpha_2\rho(\mathbf{v}_2)$ holds for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, then there is an $\mathbf{a} \in \mathbb{R}^n$ such that $\rho(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.** Similar formulas hold for vectors and tensors: the gradient of a vector is a tensor, the gradient of a tensor is a "higher order tensor", that is, a linear transformation on tensors.

If $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $\beta : \mathbb{R}^n \to \mathbb{R}$ are smooth scalar fields and $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields, then

$$\begin{aligned} \nabla(\alpha\beta) &= \beta\nabla\alpha + \alpha\nabla\beta, \\ \nabla(\alpha\mathbf{v}) &= \alpha\nabla\mathbf{v} + \mathbf{v} \otimes \nabla\alpha, \\ \nabla(\mathbf{u} \cdot \mathbf{v}) &= \nabla\mathbf{u}^T\mathbf{v} + \nabla\mathbf{v}^T\mathbf{u}. \end{aligned} \tag{15}$$

These are all derivable from the definitions like (14) in the same way the product rule is derived in calculus. One quick way to derive them is to note that (since *you* get to choose the basis field) you can choose the standard basis field at each $\mathbf{x} \in \Omega$. This basis field is constant, that is, independent of $\mathbf{x}$, and therefore the basis factors out of all formulas like (14). If in addition we use the components of $\mathbf{x}$ in the standard basis, $\mathbf{x} = x_1\hat{\mathbf{e}}_1 + \cdots + x_n\hat{\mathbf{e}}_n$, then gradients just become partial derivatives, once again, because the basis is constant. The usual notation for partial derivatives in this case is $\frac{\partial(\ )}{\partial x_i} = (\ )_{,i}$. In this constant rectangular Cartesian basis, the formulas (15) become, respectively,

$$\begin{aligned} (\alpha\,\beta)_{,i} &= \beta\,\alpha_{,i} + \alpha\,\beta_{,i}\,, \\ (\alpha\,v_i)_{,j} &= \alpha\,v_{i,j} + v_i\,\alpha_{,j}\,, \\ (u_i v_i)_{,j} &= u_{i,j}\,v_i + v_{i,j}\,u_i. \end{aligned} \tag{16}$$

Another useful operator is the Laplacian, defined for scalar fields by

$$\triangle\alpha = \mathrm{tr}\nabla\nabla\alpha = \mathrm{div}\nabla\alpha. \tag{17}$$

The Laplacian of a vector field can be defined with reference to a constant vector $\mathbf{a} \in \mathbb{R}^n$. We define,

$$(\triangle\mathbf{u}) \cdot \mathbf{a} = \triangle(\mathbf{u} \cdot \mathbf{a}) \quad \text{for all } \mathbf{a} \in \mathbb{R}^n. \tag{18}$$

(This clearly defines $\triangle\mathbf{u}$.) In rectangular Cartesian components, this is $(\triangle u)_i = u_{i,kk}$.

The definitions of the divergence of a vector or tensor field are given below:

$$\begin{aligned} \mathrm{div}\,\mathbf{u} &= \mathrm{tr}\nabla\mathbf{u}, \\ \mathbf{a} \cdot \mathrm{div}\mathbf{L} &= \mathrm{div}(\mathbf{L}^T\mathbf{a}) \quad \text{for all } \mathbf{a} \in \mathbb{R}^m. \end{aligned} \tag{19}$$

Once again, we used the device of contracting with a constant vector to help define the divergence of a tensor; this avoided a lot of excess notation. (▷**Using these definitions write the divergences of a vector and tensor in rectangular Cartesian components**). From the latter and (18) we get

$$\mathrm{div}(\nabla\mathbf{u}) = \triangle\mathbf{u}. \tag{20}$$

We have

$$\begin{aligned} \mathrm{div}(\alpha\mathbf{u}) &= \nabla\alpha \cdot \mathbf{u} + \alpha\,\mathrm{div}\mathbf{u}, \\ \mathrm{div}(\mathbf{L}\mathbf{u}) &= (\mathrm{div}\,\mathbf{L}^T) \cdot \mathbf{u} + \mathrm{tr}(\mathbf{L}\nabla\mathbf{u}). \end{aligned} \tag{21}$$

Divergence does not quite commute with gradient:

$$\mathrm{div}(\nabla\mathbf{u}^T) = \nabla\mathrm{div}\mathbf{u}. \tag{22}$$

If we combine (20) and (22), we get expressions for the skew and symmetric parts of the gradient of a vector field:

$$\mathrm{div}(\frac{1}{2}(\nabla\mathbf{u}\pm(\nabla\mathbf{u})^T))=\frac{1}{2}(\triangle\mathbf{u}\pm\nabla\mathrm{div}\mathbf{u}). \tag{23}$$

We will often also need various forms of the chain rule. For example, if we have

$$\sigma(\mathbf{x})=\alpha(\mathbf{u}(\mathbf{x})), \tag{24}$$

then

$$\nabla\sigma=\nabla\mathbf{u}^T\,\nabla\alpha. \tag{25}$$

Similarly, we can have

$$\mathbf{g}(\mathbf{x})=\mathbf{w}(\alpha(\mathbf{x})),\quad \mathbf{h}(\tau)=\mathbf{v}(\mathbf{w}(\tau)), \tag{26}$$

where $\mathbf{w}$ is a vector-valued function of scalars. then,

$$\nabla\mathbf{g}=\frac{d\mathbf{w}}{d\alpha}\otimes\nabla\alpha,\quad \frac{d\mathbf{h}}{d\tau}=\nabla\mathbf{v}\frac{d\mathbf{w}}{d\tau}. \tag{27}$$

All these formulas follow directly from the definitions. The following is a collection of formulas in $\mathbb{R}^3$. φ is a scalar, $\mathbf{v}$ is a vector and $\mathbf{M}$ is a tensor from $\mathbb{R}^3$ to $\mathbb{R}^3$, all assumed smooth on an open set Ω.

$$\begin{aligned}
\nabla(\varphi\mathbf{v}) &= \varphi\nabla\mathbf{v}+\mathbf{v}\otimes\nabla\varphi,\\
\mathrm{div}(\varphi\mathbf{v}) &= \varphi\,\mathrm{div}\mathbf{v}+\mathbf{v}\cdot\nabla\varphi,\\
\nabla(\mathbf{v}\cdot\mathbf{u}) &= \nabla\mathbf{v}^T\,\mathbf{u}+\nabla\mathbf{u}^T\,\mathbf{v},\\
\mathrm{div}(\mathbf{u}\otimes\mathbf{v}) &= \mathbf{u}\,\mathrm{div}\,\mathbf{v}+(\nabla\mathbf{u})\mathbf{v},\\
\mathrm{div}(\mathbf{M}^T\mathbf{u}) &= \mathbf{u}\cdot\mathrm{div}\,\mathbf{M}+\mathbf{M}\cdot\nabla\mathbf{u},\\
\mathrm{div}(\varphi\mathbf{M}) &= \mathbf{M}\nabla\varphi+\varphi\,\mathrm{div}\mathbf{M},\\
\mathrm{div}(\nabla\mathbf{u}^T) &= \nabla\mathrm{div}\mathbf{u},\\
\triangle\varphi &= \mathrm{div}\nabla\varphi,\\
\mathrm{div}(\mathbf{u}\times\mathbf{v}) &= \mathbf{v}\cdot\mathrm{curl}\,\mathbf{u}-\mathbf{u}\cdot\mathrm{curl}\mathbf{v},\\
\mathrm{curl}\,\nabla\varphi &= 0,\\
\mathrm{div}\ \mathrm{curl}\,\mathbf{u} &= 0,\\
\mathrm{curl}\ \mathrm{curl}\,\mathbf{u} &= \nabla\mathrm{div}\mathbf{u}-\triangle\mathbf{u},\\
\mathrm{curl}\,\nabla\mathbf{u} &= 0,\\
\nabla\mathbf{u}+\nabla\mathbf{u}^T &= 0,\ \Longrightarrow\nabla(\nabla\mathbf{u})=0,\\
\mathrm{curl}\,\mathbf{u} &= 0\ \ \Longrightarrow\mathbf{u}=\nabla\varphi\ \ (\Omega\text{ simply connected}),\\
\mathrm{div}\,\mathbf{u} &= 0\ \text{ and }\ \mathrm{curl}\,\mathbf{u}=0\Longrightarrow\triangle\mathbf{u}=0;\\
\text{given }\mathbf{u}\text{ there exist }\varphi,\mathbf{v}\text{ such that }\mathbf{u} &= \nabla\varphi+\mathrm{curl}\,\mathbf{v},\quad \mathrm{div}\,\mathbf{v}=0.
\end{aligned} \tag{28}$$

4 Christoffel components and covariant derivatives

This will not be particularly important for this course, but it is good to know about it. From time to time it does become useful, particularly if it is anticipated that a solution of a problem will have a symmetry, like cylindrical, spherical or even helical symmetry. This section also explains where those strange formulas in the appendices of textbooks, like some equations of elasticity or fluid dynamics in cylindrical coordinates, come from.

Coordinate systems themselves generate basis fields. Recall that we are working on a region $\Omega \subset \mathbb{R}^n$ and points in Ω are labelled by $\mathbf{x}$. A coordinate system is an invertible function which maps Ω into $\mathbb{R}^n$. For example, the following defines a coordinate system:

$$\begin{aligned} x^1 &= \varphi^1(\mathbf{x}), \\ x^2 &= \varphi^2(\mathbf{x}), \\ &\vdots \\ x^n &= \varphi^n(\mathbf{x}), \end{aligned} \tag{29}$$

where $(\varphi^1(\mathbf{x}), \ldots, \varphi^n(\mathbf{x}))$ is an invertible mapping on Ω. The numbers $(x^1, x^2, \ldots, x^n)$ are called the coordinates of $\mathbf{x}$. We will denote the inverse of the coordinate system by

$$\mathbf{x} = \mathbf{g}(x^1, \ldots, x^n) \tag{30}$$

For example, with $n = 3$, Ω taken to be all of $\mathbb{R}^3$, and $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ an orthonormal basis, we have, say, $x^1 = r = |\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3|$, $x^2 = \theta = \arctan(\mathbf{x} \cdot \hat{\mathbf{e}}_2 / \mathbf{x} \cdot \hat{\mathbf{e}}_1)$, $x^3 = z = \mathbf{x} \cdot \hat{\mathbf{e}}_3$ as the definition of the *cylindrical coordinate system* (compare with (29)). The inverse of this coordinate system is

$$\mathbf{x} = \mathbf{g}(r, \theta, z) = r\cos\theta\ \hat{\mathbf{e}}_1 + r\sin\theta\ \hat{\mathbf{e}}_2 + z\ \hat{\mathbf{e}}_3, \tag{31}$$

(compare with (30)). This defines cylindrical coordinates in three dimensions.

The *natural basis* of a coordinate system is the basis field defined from the mapping (30) by

$$\mathbf{e}_i = \frac{\partial \mathbf{g}}{\partial x^i}, \quad i = 1, \ldots, n. \tag{32}$$

▷**Calculate the natural basis of the cylindrical coordinate system and draw a picture of the basis. Note that it is an orthogonal but not an orthonormal basis**. This expression for $\mathbf{e}_i$ gives it as a function of $(x^1, \ldots, x^n)$. It can also be expressed as a function of $\mathbf{x} \in \Omega$ by using the coordinate system (29).

The natural basis of a coordinate system is a basis field, so it provides a basis at every point $\mathbf{x} \in \Omega$. We can therefore write vector and tensor fields in terms of it. If $\mathbf{v}(\mathbf{x})$ is a vector field on Ω, we can write

$$\mathbf{v}(\mathbf{x}) = v^i(\mathbf{x})\ \mathbf{e}_i(\varphi^1(\mathbf{x}), \ldots \varphi^n(\mathbf{x})). \tag{33}$$

Here, I've put in all the arguments of functions. From (33) one can see that the gradient of $\mathbf{v}$ is not going to be simply the sum of the gradients of its components, because both v^i and $\mathbf{e}_i$ vary from point to point.

Extending earlier notation, if the natural basis field is constant then the coordinate system is *Cartesian*. If it is orthonormal and constant, then it is *rectangular Cartesian*. Cylindrical and spherical coordinate systems are not Cartesian, but their natural bases are each orthogonal (but not orthonormal).

Using (32), reciprocal natural basis and metric components g_{ij} and g^{ij} are defined by

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \quad g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \quad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{34}$$

It is easily seen that $\mathbf{e}^i = \nabla\varphi^i$. As matrices, the metric components are positive-definite and they are inverses of each other:

$$g_{ij}\, g^{jk} = \delta^k_i. \tag{35}$$

The gradients of various vector fields are important tensor fields in mechanics. They are by definition independent of the choice of a basis, but it is useful to be able to calculate them in the natural basis of a coordinate system. By direct calculation of the gradient of (33), we have

$$\nabla \mathbf{v}(\mathbf{x}) = \mathbf{e}_i \otimes \nabla v^i + v^j \nabla \mathbf{e}_j. \tag{36}$$

We have two problems, to calculate ∇v^i and to calculate $\nabla \mathbf{e}_j$, and to express the answers in the basis $\{\mathbf{e}_i\}$ or its reciprocal. The gradient of the natural basis is traditionally given another name,

$$\Gamma_i = \nabla \mathbf{e}_i. \tag{37}$$

The n tensors $\Gamma_1, \dots \Gamma_n$ have components in the natural basis $\{\mathbf{e}_i\}$, and we can write

$$\Gamma_i = \Gamma_i{}^j{}_k \mathbf{e}_j \otimes \mathbf{e}^k. \tag{38}$$

The $\Gamma_i{}^j{}_k$ are called Christoffel components or Christoffel symbols. The origin of this latter terminology is traditional tensor analysis. Traditional expositions make a big deal about how the Christoffel symbols are not tensors; "tensors" in these treatments are defined as quantities with indices having certain kinds of transformation laws. Here, this is clear: the index i in (37), (38) labels which tensor and has nothing to do with components.

Notice that for each j, $\nabla \mathbf{e}^j$ (but not necessarily $\nabla \mathbf{e}_j$) is a symmetric tensor. That is,

$$\nabla \mathbf{e}^j = \nabla\nabla\varphi^j. \tag{39}$$

We can write using (37) and (38)

$$\Gamma_i{}^j{}_k = \mathbf{e}^j \cdot \Gamma_i \, \mathbf{e}_k = \mathbf{e}^j \cdot (\nabla \mathbf{e}_i \, \mathbf{e}_k), \tag{40}$$

and use the product rule $(15)_3$ to get

$$\Gamma_i{}^j{}_k = \mathbf{e}_k \cdot \nabla(\mathbf{e}^j \cdot \mathbf{e}_i) - \mathbf{e}_k \cdot (\nabla \mathbf{e}^{j\,T} \mathbf{e}_i) = -\mathbf{e}_k \cdot (\nabla \mathbf{e}^{j\,T} \mathbf{e}_i). \tag{41}$$

Since $\nabla \mathbf{e}^j$ is symmetric, we have

$$\Gamma_i{}^j{}_k = \Gamma_k{}^j{}_i. \tag{42}$$

In three dimensions this reduces the calculation of the 27 Christoffel components to 18, and for simple coordinate systems some of these 18 will vanish.

The easiest way to calculate the Christoffel components is to first calculate the metric components of the coordinate system and then to use the formula,

$$\Gamma_i{}^j{}_k = \frac{1}{2} g^{jm} \left(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m}\right), \tag{43}$$

which can be proved from the definitions of Γ_i and the metric components.

Now we return to the calculation of ∇v^i of (36). It is useful to substitute the inverse of the coordinate system and define $\tilde{v}^i(x^1, \dots, x^n) = v^i(\mathbf{g}(x^1, \dots, x^n))$. Then we take the partial derivative with respect to x^j:

$$\frac{\partial \tilde{v}^i}{\partial x^j} = \nabla v^i \cdot \frac{\partial \mathbf{g}}{\partial x^j} = \nabla v^i \cdot \mathbf{e}_j. \tag{44}$$

Typically, people will remove the $\tilde{}$, as it is clear from the differentiation that v^i has been re-expressed as a function of the coordinates, and we will also do this. Combining (36), (38) and (44) we get

$$\nabla \mathbf{v} = \left(\frac{\partial v^i}{\partial x^j} + v^k \, \Gamma_k{}^i{}_j\right) \mathbf{e}_i \otimes \mathbf{e}^j. \tag{45}$$

The components of $\nabla\mathbf{v}$ are the quantities in parentheses of (45). These components are usually indicated with a semicolon as below,

$$v^i_{;j} = \frac{\partial v^i}{\partial x^j} + v^k\,\Gamma_{k\,j}^{\,i}, \tag{46}$$

and $v^i_{;j}$ are called the *covariant derivatives* of $\mathbf{v}$.

Other results can be derived in a similar way. As a simple application we see immediately from (45) that

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\nabla\mathbf{v} = v^i_{;i} \tag{47}$$

The divergence of a vector field in cylindrical coordinates listed in the appendices of textbooks is not quite this expression; those books use physical components, as explained below. In elasticity, the equilibrium equations in the absence of body forces are $\operatorname{div}\sigma = 0$, where the stress σ is a symmetric tensor on $\mathbb{R}^3$. This can easily be found in any coordinate system by using the device used above; note that

$$\mathbf{a}\cdot\operatorname{div}\sigma = \operatorname{div}(\sigma^T\mathbf{a}) \quad \text{for all } \mathbf{a}\in\mathbb{R}^3. \tag{48}$$

The right hand side is then the divergence of a vector field and we can use the formula (47). At the last step, we put everything on one side of the equation and use that if $\mathbf{a}\cdot\mathbf{f} = 0$ for all $\mathbf{a}\in\mathbb{R}^3$ then $\mathbf{f} = 0$. In this way one gets these equilibrium equations as

$$\frac{\partial\sigma^{ij}}{\partial x^i} + \sigma^{kj}\Gamma_{k\,i}^{\,i} = 0. \tag{49}$$

There are other forms of this using other kinds of components.

There are other sets of components obtained by using other choices of combinations of the basis and its reciprocal. They are written,

$$v^{i;j}, \quad v_i^{\,;j}, \quad v_{i;j}. \tag{50}$$

They can be obtained by doing the whole calculation over again or by the procedure of "raising and lowering indices using metric components" (This is, in general, an easy way of changing one set of components to another in the same basis.) In a Cartesian coordinate system the Christoffel components vanish, so the components of a gradient reduce to partial derivatives.

5 Rectangular Cartesian components

The coordinate system most often used in mechanics is the rectangular Cartesian coordinate system. It is trivially defined, using the standard basis, by

$$\begin{aligned}
\varphi^1(x^1\hat{\mathbf{e}}_1 + \cdots + x^n\hat{\mathbf{e}}_n) &= x^1,\\
\varphi^2(x^1\hat{\mathbf{e}}_1 + \cdots + x^n\hat{\mathbf{e}}_n) &= x^2,\\
&\vdots\\
\varphi^n(x^1\hat{\mathbf{e}}_1 + \cdots + x^n\hat{\mathbf{e}}_n) &= x^n,
\end{aligned} \tag{51}$$

and the inverse mapping is $\mathbf{x} = x^i\hat{\mathbf{e}}_i$. Hence the natural basis and the reciprocal natural basis are both $\{\hat{\mathbf{e}}_1,\ldots,\hat{\mathbf{e}}_n\}$, the metric components are $\delta^{ij} = \delta_{ij}$.

Since the natural basis coincides with the reciprocal basis there is no difference between the various kinds of components: for the rectangular Cartesian components of any tensor field,

$$A_{ij} = A^{ij} = A_i^{\,j} = A^i_{\ j}. \tag{52}$$

Thus, it is common practice to write all indices on the same (lower) level. In rectangular Cartesian coordinates covariant derivatives are simply partial derivatives – usually a comma replaces the semicolon. I will follow these practices and write

$$\nabla \mathbf{v} = v_{i,j}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{53}$$

for example. I'll also follow common practice of dropping the basis vectors when using rectangular Cartesian components. It is then understood that every free index is associated to a basis vector, e.g., the term

$$v_{i,j}\, u_j \tag{54}$$

is really the vector

$$(v_{i,j}\, u_j)\hat{\mathbf{e}}_i, \tag{55}$$

which in the fully explicit form is

$$(v_{i,j}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)(u_k \hat{\mathbf{e}}_k), \tag{56}$$

and therefore corresponds to $\nabla \mathbf{v}\, \mathbf{u}$.

Since $\mathbb{R}^n$, and in fact every finite dimensional vector space with an inner product, has an orthonormal basis, we will always have rectangular Cartesian components at our disposal. Often it is easiest to use them to derive results involving vectors and tensors as we've indicated above. Sometimes people become sufficiently familiar with the more abstract forms that they use them for everyday calculations.

6 Physical components for orthogonal natural bases

The components of a vector or tensor in the natural basis of a coordinate system do not always have a direct physical meaning. For example, the component v_θ of the velocity of a particle at $\mathbf{x}$ in a cylindrical coordinate system does not represent "the speed of the particle at $\mathbf{x}$ in the direction $\mathbf{e}^\theta$". The reason is that $\mathbf{e}^\theta$ is not a unit vector.

Worse than that, $\mathbf{e}^\theta$ has the dimension $[\text{length}]^{-1}$ so that v_θ has the dimension $[\text{length}]^2/[\text{time}]$, which is not even the dimension of a velocity. This is not a good situation, because one of reasons for changing to a particular coordinate system is that specializing physical assumptions assume a simple form. The dimensional problems could be fixed by always requiring that the equations defining the coordinate system be dimensionless, but there would still be the problem with the non-normality.

To avoid these problems, we introduce still another kind of components called *physical components*. Here, we describe them for the common choice of an orthogonal coordinate system. They are defined as components relative to a basis field that is parallel to the natural basis field but which has unit length and therefore is orthonormal. Specifically, physical components in a coordinate system with orthogonal natural basis $\{\mathbf{e}_i\}$ are components in the basis

$$\bar{\mathbf{e}}_i = \frac{1}{|\mathbf{e}_i|}\, \mathbf{e}_i \quad \text{no sum}, \quad i = 1, \ldots, n. \tag{57}$$

Since the basis $\{\bar{\mathbf{e}}_i\}$ is orthonormal, there is no distinction between the various kinds (covariant, etc.) of physical components. We can simply say *the* physical components of a vector or tensor. Often, people used the notation

$$\begin{aligned} \mathbf{A} &= A\langle ij\rangle\, \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j, \\ \mathbf{v} &= v\langle i\rangle\, \bar{\mathbf{e}}_i \end{aligned} \tag{58}$$

for physical components.

The most direct way to get the physical components is first to get the components with respect to the natural basis or the reciprocal natural basis and then to adjust the result accounting for (57), e.g., the physical component $v\langle i\rangle$ of velocity in the example above is given by

$$v\langle i\rangle = |\mathbf{e}_i|\, v^i \quad \text{(no sum)}. \tag{59}$$

In most modern textbooks, when authors say that they are giving an expression in cylindrical or spherical components, they mean physical components.