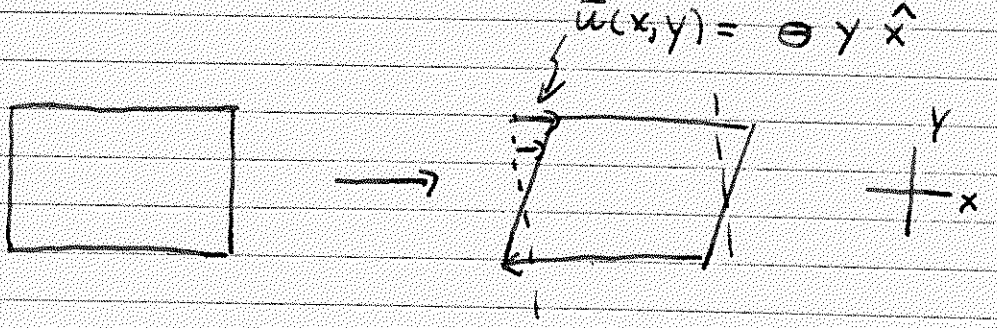
2D ~~Plastic Sheet~~ Elasticity Theory

I) Flat elastic sheet

deform : $\vec{u}(\vec{r})$

example

pure shear



$\vec{u}(\vec{r})$ displacement mass point located at $\vec{r}$. Let $F(\{u\})$ be elastic (free) energy cost.

* Elastic energy cost (F) cannot depend on $\vec{u}$ itself : $dF = 0$ if $\vec{u}$ is constant

* F depends on derivatives of $\vec{u}$

$$\partial_i u_j = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \leftarrow u_{ij}$$

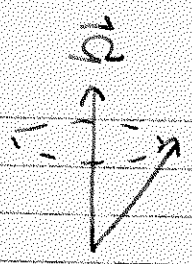
+ SYMMETRIC

$$\frac{1}{2}(\partial_i u_j - \partial_j u_i) \leftarrow A_{ij}$$

ANTI-SYMM

* $dF = 0$ for a pure rotation

$$\vec{u}_R = \frac{1}{2}\vec{\Omega} \times \vec{r} \qquad \vec{\Omega} = \hat{z}$$



$$= \left(-\frac{\Omega}{2}y, \frac{\Omega}{2}x\right)$$

$u_{ij} = 0$ for a pure rotation.

$$A_{ij} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}$$

$\Rightarrow$ F can only depend on u_{ij} "Strain Tensor"

nondimensionless

* F must be invariant under symmetry operations of sheet. Assume rotation, reflection are symmetry operations

Invariants of u_{ij}: $\mathrm{Tr}\,\overleftrightarrow{u}$

3 to second order $\mathrm{Tr}\,\overleftrightarrow{u}^2$

$\sum_{i,j} u_{ij}^2$ or $\mathrm{Det}\,\overleftrightarrow{u}$

* $F = \int d^3r \left\{ \alpha\, \mathrm{Tr}\overleftrightarrow{u} + \beta\, \mathrm{Tr}\overleftrightarrow{u}^2 + \gamma\, u_{ij}^2 \right\}$

Demand $\vec{u} = 0$ equilibrium state

$\Rightarrow \alpha = 0$

$$F = \frac{1}{2} \int_A d^2s \left(2\mu \, u_{ij}^2 + \lambda \, u_{ii}^2 \right)$$

μ: "shear modulus"

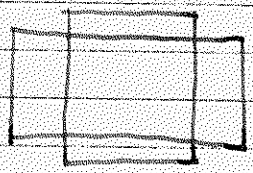
$\lambda + \mu$: "Area modulus"

1) Pure shear

$$u_{ij} = \frac{\theta}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathrm{Tr}\, \vec{u} = 0$$

$$F = \frac{1}{2} \times 2\mu \times \left(2 \left(\frac{\theta}{2} \right)^2 \right) A = \frac{1}{2} \mu \theta^2 A$$

2) Elongational Shear



$$\vec{u} = \frac{dL}{2L} (x, -y)$$

$$u_{ij} = \frac{dL}{2L} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathrm{Tr}\, \vec{u} = 0$$

Compute F

3) Hydrostatic Compression

$$u_{ij} = \frac{dA}{2A} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathrm{Tr}\, u = dA/A$$

$$F = \frac{1}{2} A \left[2\mu \times 2 + \lambda \times 4 \right] \left(\frac{dA}{2A} \right)^2 = \frac{1}{2} (\mu + \lambda) \frac{dA^2}{A}$$

(area modulus)

Stress Tensor

W : work done on system to cause deformation

$dW \propto (d\,u_{ij} \times A)$ like $dW = F\,dx$

(du_{ij}: change in u_{ij}; A: area)

proportionality constant must be a tensor: σ_{ij}

$$\frac{dW}{A} = \sigma_{ij}\, du_{ij}$$

"stress tensor" units: force/area → Pa

Thermodynamics (entropy S, temperature T)

$$dF = dW - S\,dT$$

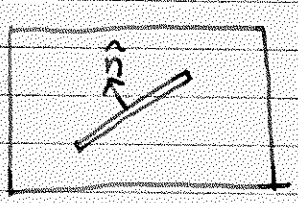
$$dT = 0 \qquad dW = dF = \frac{1}{2}\int d^3r \left\{ 2\mu\, u_{ij}\, du_{ij} + \lambda\, u_{\delta\delta}\, du_{\delta\gamma} \right\}$$

$$\Rightarrow \boxed{\sigma_{ij} = 2\mu\, u_{ij} + \lambda\, u_{ll}\, \delta_{ij}}$$

Formal:

Give proof that: functional derivative $\boxed{\frac{\partial F}{\partial \epsilon_{ij}} = \sigma_{ij}}$.

Show that if you cut an elastic sheet that is under stress along a slit, normal $\hat{n}$



Then $\vec{f} = \overleftrightarrow{\sigma} \cdot \hat{n}$ is the force / unit length on slit.

Equilibrium Equations

Minimize F: $\delta F = 0$

since $\delta F = \sigma_{ij}\, \delta \epsilon_{ij} \Rightarrow \sigma_{ij} \overset{?}{=} 0$ NO

$$\delta F = \int d^2 r\ \sigma_{ij}\, \delta \epsilon_{ij}$$

$$= \int d^2 r\ \sigma_{ij} \left\{ \tfrac{1}{2} \partial_i \delta u_j + \tfrac{1}{2} \partial_j \delta u_i \right\}$$

* partial integration. Assume $\sigma_{ij} = 0$ on boundary

$$= \int d^2 r \left(\partial_i \sigma_{ij} \right) \delta u_j$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \overleftrightarrow{\sigma} = 0} \qquad \left(\partial_i \sigma_{ij} = 0 \right)$$

for $\sigma_{ij} = 2\mu\, \epsilon_{ij} + \lambda\, \epsilon_{\ell\ell}\, \delta_{ij}$

$\partial_i \sigma_{ij} = 0$

$$\mu \nabla^2 \vec{u} + (\lambda + \mu)\, \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) = 0$$

Equilibrium Eq.

Airy Stress Function

- For a function $\chi(\vec{r})$ define

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \qquad \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$$

Check that $\partial_i \sigma_{ij} = 0$ for any χ.

This does not mean that any χ is permissible! Express strain tensor in terms of stress tensor. Show yourself that

$$u_{ij} = \left(\frac{1+\sigma}{Y}\right) \sigma_{ij} - \frac{\sigma}{Y}\, \sigma_{\ell\ell}\, \delta_{ij}$$

$K = \dfrac{4\mu(\mu+\lambda)}{2\mu+\lambda}$: Young's Modulus

$\sigma = \frac{\lambda}{2\mu+\lambda}$: Poisson's Ratio

Check yourself that

$$u_{ij} = \left(\frac{1+\sigma}{Y}\right) \underbrace{\epsilon_{im}\epsilon_{jn}\partial_m\partial_n\chi}_{\sigma_{ij}} - \frac{\sigma}{K}\underbrace{\nabla^2\chi}_{\sigma_{kk}}\delta_{ij}$$

where Operator $\epsilon_{ik} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Operate with $\epsilon_{ik}\epsilon_{jl}\partial_k\partial_l$ on strain tensor

$$\epsilon_{ik}\epsilon_{jl}\partial_k\partial_l u_{ij} = \frac{1}{Y}\nabla^4\chi$$

because $(\epsilon_{ik}\epsilon_{jl}\partial_k\partial_l)^2 = \nabla^4$

Write as

$$\frac{1}{Y}\nabla^4\chi = \underbrace{\epsilon_{ik}\epsilon_{jl}\partial_k\partial_l u_{ij}}_{\text{"incompatibility"}}$$

A) If $\vec{u}(\vec{r})$ is a well-behaved, single-valued function

combine

$$I = \epsilon_{ik}\,\epsilon_{jl}\,\partial_k\,\partial_l\,\partial_i\,u_j$$

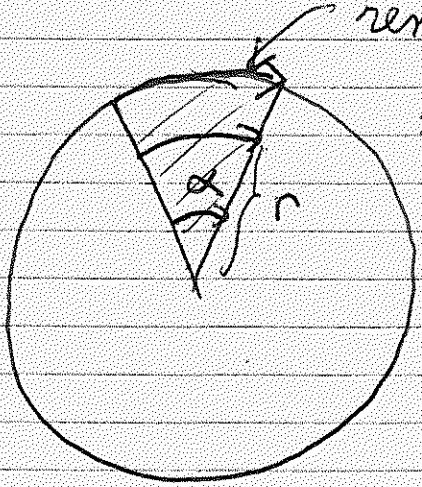
ϵ_{ik} antisymmetric, $\partial_k\,\partial_i$ symmetric

$$\epsilon_{ik}\,\partial_k\,\partial_i = 0$$

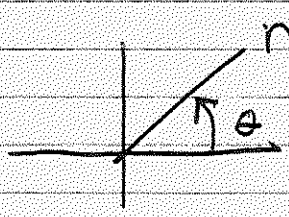
$$I = 0 \Rightarrow \boxed{\nabla^4 \chi = 0}$$

"Biharmonic Equation"

B) "Disclination"



$\Rightarrow$ "disclination"



Example displacement field for a disclination

$$\vec{u}_D = \frac{\alpha}{2\pi}\,(r\theta)\,\hat{\theta}$$

true
(not necessarily solution of equilibrium eq.!)

$r\theta$ = arc length

$\vec{u}_p(\theta=0) = 0$

$\vec{u}_p(\theta=2\pi) = \alpha r \hat{\theta}$ } discontinuous at $\theta = 0 = 2\pi$ "cut" along radial direction

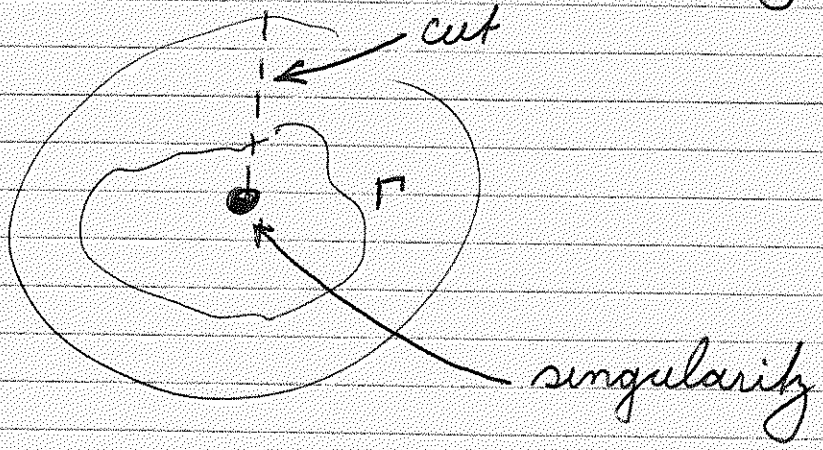
Define "bond-angle" field

$$\boxed{\theta_B = \frac{1}{2} \epsilon_{ij} \partial_i u_j}$$

$$= \frac{1}{2} (\vec{\nabla} \times \vec{u})_z$$

For example, show that $\theta_B(\vec{r}) = \frac{S}{2\pi} \theta$ for $\vec{u}_D(\vec{r})$

Let Γ be a path surrounding dislocation



$$\oint_\Gamma d\theta_B = S$$ (show true for $\vec{u}_D$ problems)

$$S = \oint d\theta_B = \oint \vec{\nabla}\theta_B \cdot d\vec{s} \qquad \text{use } \theta_B = \tfrac{1}{2}\epsilon_{ij}\nabla_i u_j$$

$$= \frac{1}{2}\oint (\epsilon_{ij}\nabla_i \nabla_k u_j)\, dx_k$$

now use Stokes Theorem $\oint \vec{A}\cdot d\vec{x} = \int (\vec{\nabla}\times\vec{A})\cdot d^2\vec{s}$

$$= \frac{1}{2}\int d^2r \left\{ \epsilon_{kl}\nabla_l \epsilon_{ij}\nabla_i \nabla_k u_j \right\}$$

$$= \frac{1}{2}\int d^2r\, I$$

$\Rightarrow$ I must have a δ function singularity at the disclination site

$$\boxed{\frac{1}{Y}\nabla^4\chi = s\,\delta(\vec{r})}$$

Show that this equation is solved by

$$\chi(\vec{r}) = \frac{Y}{8\pi} s \left(r^2 \ln r + \underbrace{A r^2}_{\text{constant}} \right)$$

Impose boundary condition $\sigma_{rr} = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\varphi^2}$

vanishes at $r = R$

$$\Rightarrow A = -\frac{1}{2} - \ln R$$

$$\chi = \frac{K_0 S}{8\pi} r^2 \left[\ln r/R - \frac{1}{2} \right]$$

Compute $\vec{\vec{\sigma}}$ & $\vec{\vec{\epsilon}}$

$$F_D = \frac{1}{2} \int a^2 r \; \vec{\vec{\sigma}} : \vec{\vec{\epsilon}}$$

$$\boxed{F_D = \frac{V_0 S^2}{32\pi} R^2}$$