Physics of membranes III

- Fluctuations of fluid membranes
- Membrane persistence length

S. A. Safran, Statistical thermodynamics of surfaces, interfaces, membranes (Westview Press)
S. A. Safran, Safran SA. (**1999**) Curvature Elasticity of Thin Films. *Advances in Physics*, **48**:395-448.

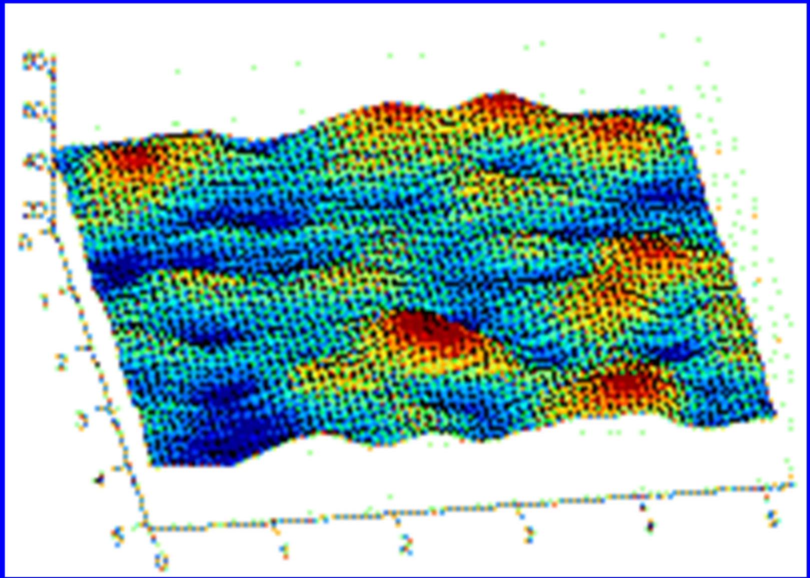
Curvature free energy

$$f_c = 2k\,(H - c_0)^2 + \bar{k}\;K$$

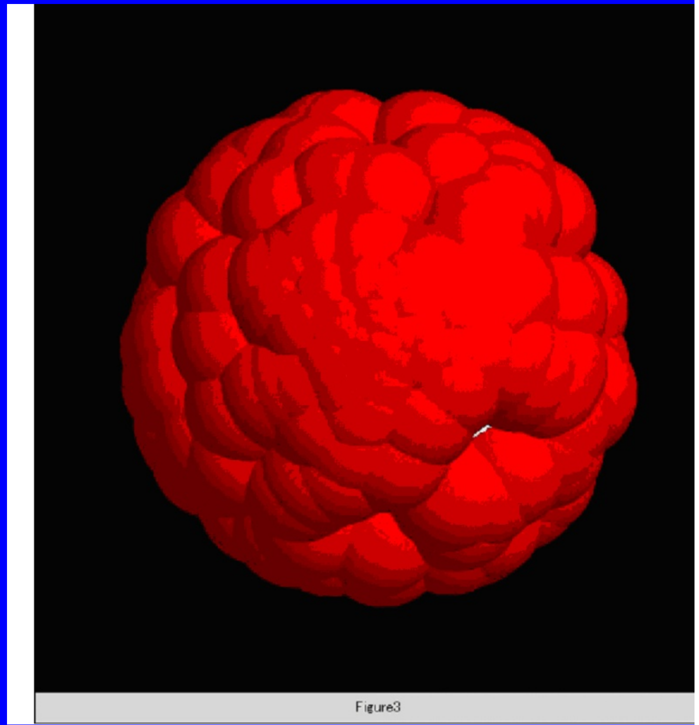
$$f_c = \frac{1}{2}k\,(\kappa_1 + \kappa_2 - 2c_0)^2 + \bar{k}\;\kappa_1\kappa_2$$

- Lowest energy state: flat if $c_0 = 0$
- Entropy: thermal undulations

Fluctuating membrane

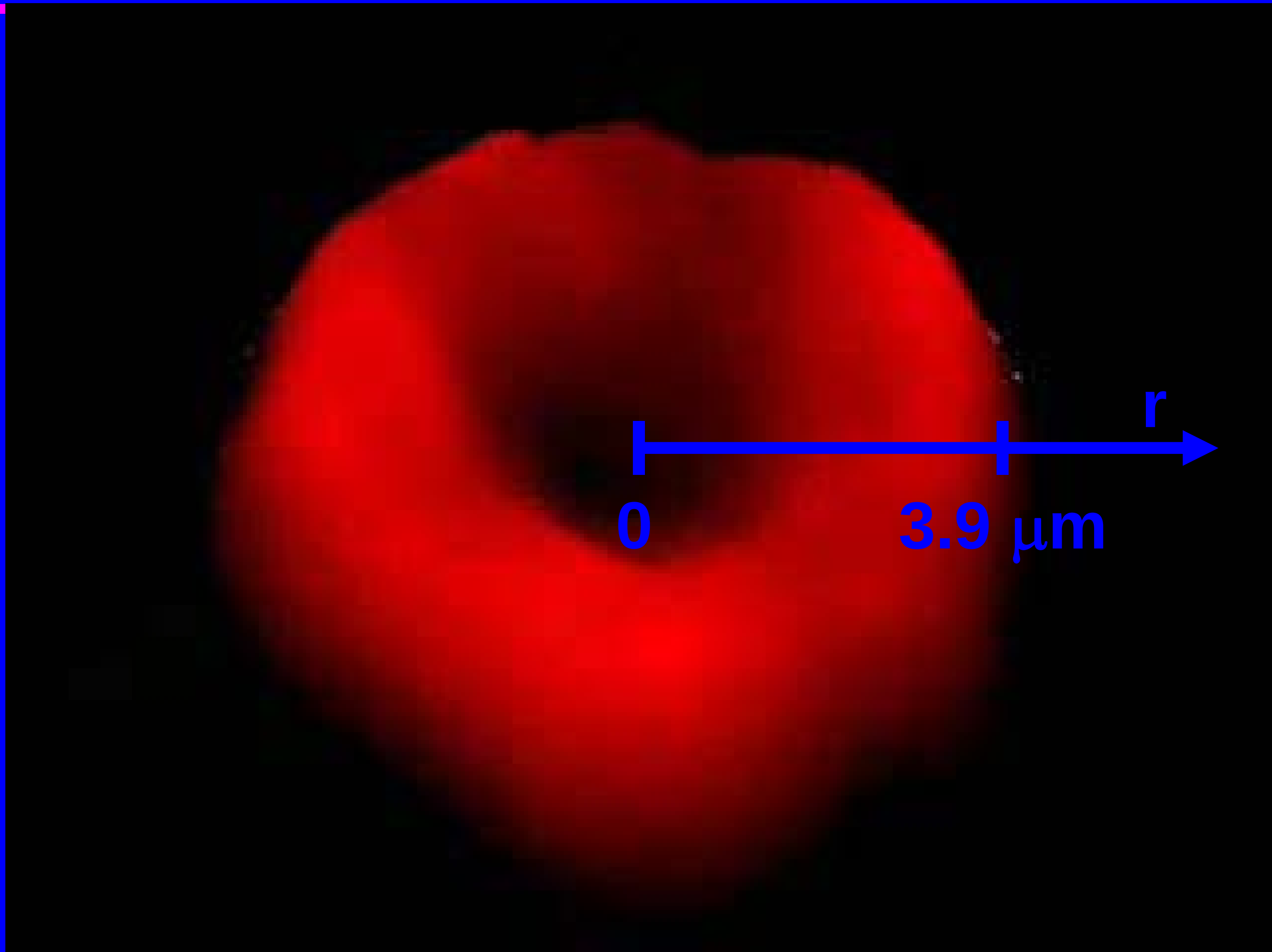


http://www.berkeley.edu/news/media/releases/2002/10/24_laser.html



http://www.rgi.tut.fi/ijbem/volume6/number2/003.htm

Experiments: fluctuation amplitudes



Surface Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int d\vec{r} \int d\vec{r}\,' h(\vec{r})G(\vec{r}-\vec{r}\,')h(\vec{r}\,')$$

$$h(\vec{r}) = \frac{1}{\sqrt{L^d}} \sum_{\vec{q}} h(\vec{q}\,)\, e^{-i\vec{q}\cdot\vec{r}}$$

$$h(\vec{q}) = \frac{1}{\sqrt{L^d}} \int d\vec{r}\, h(\vec{r}\,)\, e^{i\vec{q}\cdot\vec{r}}$$

$$G(\vec{q}) = \int d\vec{r}\, G(\vec{r}\,)\, e^{i\vec{q}\cdot\vec{r}}$$

$$\sum_{\vec{q}} = \left(\frac{L}{2\pi}\right)^d \int d\vec{q}$$

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{q}} h(\vec{q}\,)G(\vec{q}\,)h(-\vec{q})$$

Gaussian Fluctuations

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{q}} h(\vec{q}\,)G(\vec{q}\,)h(-\vec{q})$$

$$\left\langle |h(\vec{k}\,)|^2 \right\rangle = \prod_{\vec{q}} \int dh(\vec{q}\,)\, h(\vec{k}\,)\, h(-\vec{k})\, P[h(\vec{q}\,)]$$

$$P[h(\vec{q}\,)] = \frac{e^{-\mathcal{H}/T}}{\prod_{\vec{q}} \int dh(\vec{q}\,)\, e^{-\mathcal{H}/T}}$$

$$\langle |h(\vec{q}\,)|^2 \rangle = \frac{T}{G(\vec{q}\,)}$$

$$\langle h^2(\vec{r}) \rangle = \frac{1}{L^d} \sum_{\vec{q}} \langle |h(\vec{q}\,)|^2 \rangle = \left(\frac{1}{2\pi}\right)^d \int d\vec{q}\, \frac{T}{G(\vec{q}\,)}$$

Membrane fluctuations

$$f_c = \tfrac{1}{2}k\left(h_{xx} + h_{yy}\right)^2$$

$$f_c = \frac{1}{2A}k\sum_{\vec{q}} q^4 |h_q|^2$$

$$\left\langle |h_q|^2 \right\rangle = \frac{T}{kq^4}$$

$$\left\langle h(\vec{r})^2 \right\rangle = \frac{1}{A}\sum_{\vec{q}} \left\langle |h_q|^2 \right\rangle \sim \frac{T}{k}L^2$$

$$g_n(\vec{r}) = \left\langle (\hat{n}(\vec{r}) - \hat{n}(0))^2 \right\rangle$$

$$\hat{n} \approx \hat{z} - h_x\hat{x} - h_y\hat{y}$$

$$g_n(\vec{r}) = \frac{2}{A}\sum_{\vec{q}} q^2 \left\langle |h_{\vec{q}}|^2 \right\rangle (1 - \cos\vec{q}\cdot\vec{r})$$

Persistence length

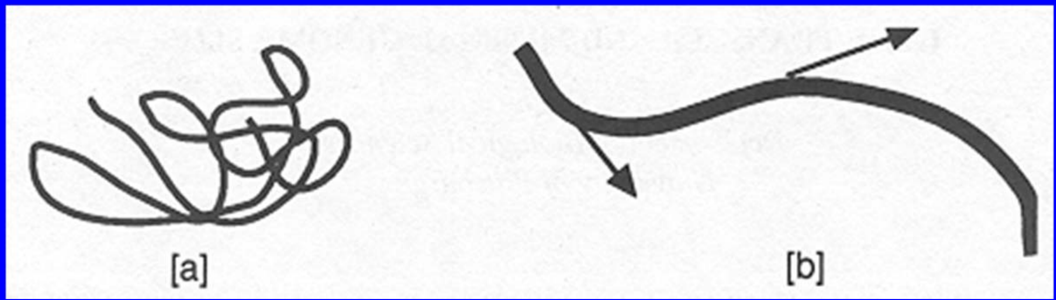
$$g_n(\vec{r}) = \frac{2}{A}\sum_{\vec{q}} q^2 \left\langle |h_{\vec{q}}|^2 \right\rangle (1 - \cos\vec{q}\cdot\vec{r})$$

$$\int_0^{2\pi} \cos(qr\cos\theta)d\theta = 2\pi J_0(qr)$$

$$g_n(\vec{r}) \approx \frac{\alpha T}{4\pi k}\log\frac{r}{a}$$

$$\xi_k = a\exp\left[\frac{4\pi k}{\alpha T}\right]$$

Large scales



Small scales

http://www7.nationalacademies.org/ssb/nanopanel1boal.html

Curvature energy renormalization: I

$$H = \frac{(1+h_x^2)h_{yy} + (1+h_y^2)h_{xx} - 2h_x h_y h_{xy}}{2\sqrt{(1+h_x^2+h_y^2)^3}}$$

Mean curvature

$$H = \frac{(1+\eta)h_{yy} + (1+\eta)h_{xx}}{2(1+\eta+\eta)^{3/2}}$$

$$\eta = \langle h_x^2 \rangle = \langle h_y^2 \rangle \text{ and } \langle h_x h_y \rangle = 0.$$

$$\eta = \langle h_x^2 \rangle = \frac{k_B T}{4\pi k}\int \mathrm{d}q \frac{1}{q} = \frac{k_B T}{4\pi k}\log\left(\frac{L}{a}\right)$$

$$H^2 \approx \tfrac{1}{4}(h_{xx} + h_{yy})^2(1 - 4\eta)$$

Curvature energy renormalization: II

$$\eta = \langle h_x^2 \rangle = \frac{k_B T}{4\pi k} \int \mathrm{d}q \frac{1}{q} = \frac{k_B T}{4\pi k} \log\left(\frac{L}{a}\right)$$

$$H^2 \approx \tfrac{1}{4}(h_{xx} + h_{yy})^2 (1 - 4\eta)$$

$$F_b = \frac{k}{2} \int dS \quad H^2$$

$$dS = dx\, dy \sqrt{\left(1 + h_x^2 + h_y^2\right)} \approx dx\, dy\, (1 + \eta)$$

$$F_b = \frac{k}{2} \int dx\, dy\, (1 - 3\eta)\left(h_{xx} + h_{yy}\right)^2$$

$$F_\mathrm{b} \approx \frac{k}{2} \left[1 - 3\, \frac{k_B T}{4\pi k} \log\left(\frac{L}{a}\right)\right] \int \mathrm{d}\mathbf{r}\, (h_{xx} + h_{yy})^2$$

Renormalized bending modulus

Renormalized bending modulus

$$k_r = k\left[1 - 3\frac{k_B T}{4\pi k}\log\left(\frac{L}{a}\right)\right].$$

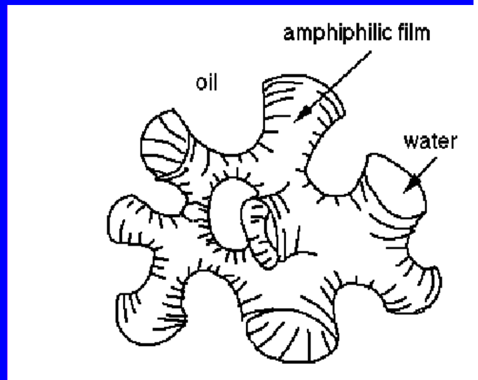
Finite membrane patch

$$k(q) = k\left(1 - \frac{3k_B T}{4\pi k}\log\left(1/(qa)\right)\right)$$

Wavevector dependent

Physics – curvature softening

- Large wavelength fluctuations less energy
- Weak preference for persistence length
- Sponge structures: phase separation
  - Coexistence: nearly pure oil, water
  - Sponge near persistence length



- Gaussian curvature effects
  - Palmer/Morse, J. Chem. Phys. 1996