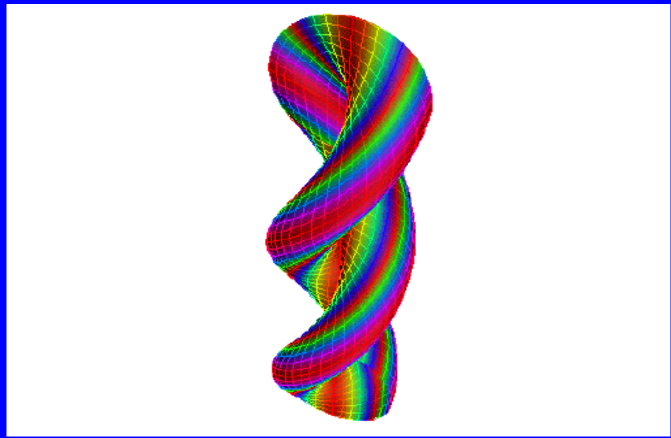
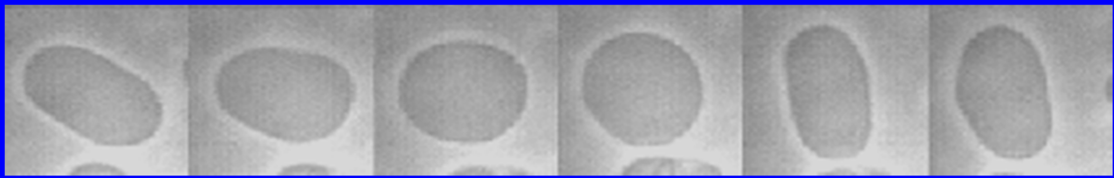


Physics of membranes I

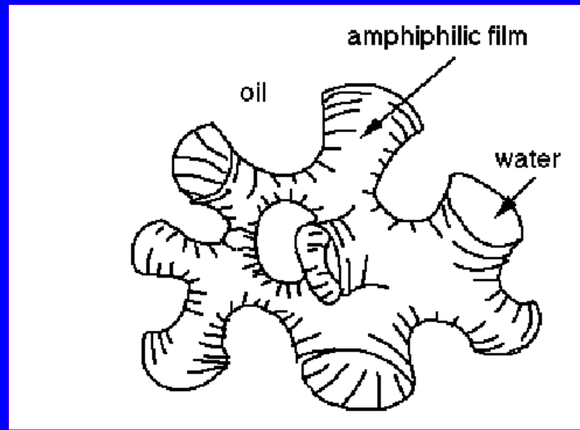
- Fluctuating Lines and Surfaces
- Lines: length, curvature
- Surfaces: parameterization, area, curvature
- Physical, biological and graphic examples

S. A. Safran, Statistical thermodynamics of surfaces, interfaces, membranes (Westview Press)
S. A. Safran, Safran SA. (**1999**) Curvature Elasticity of Thin Films. *Advances in Physics*, **48**:395-448.

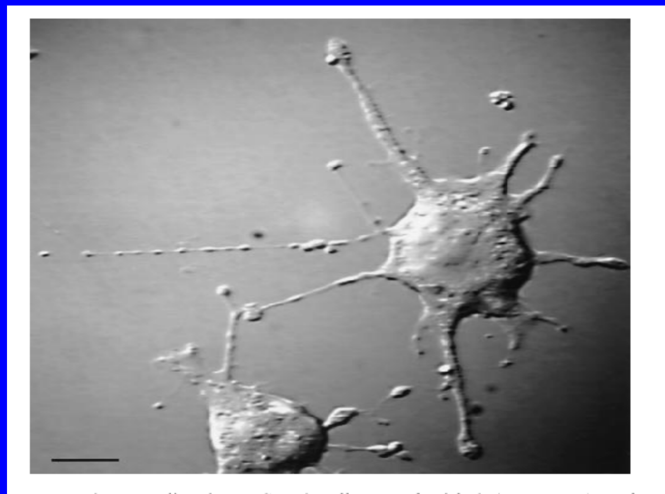
Surface fluctuations



vesicle fluctuations
(mpi-golm)



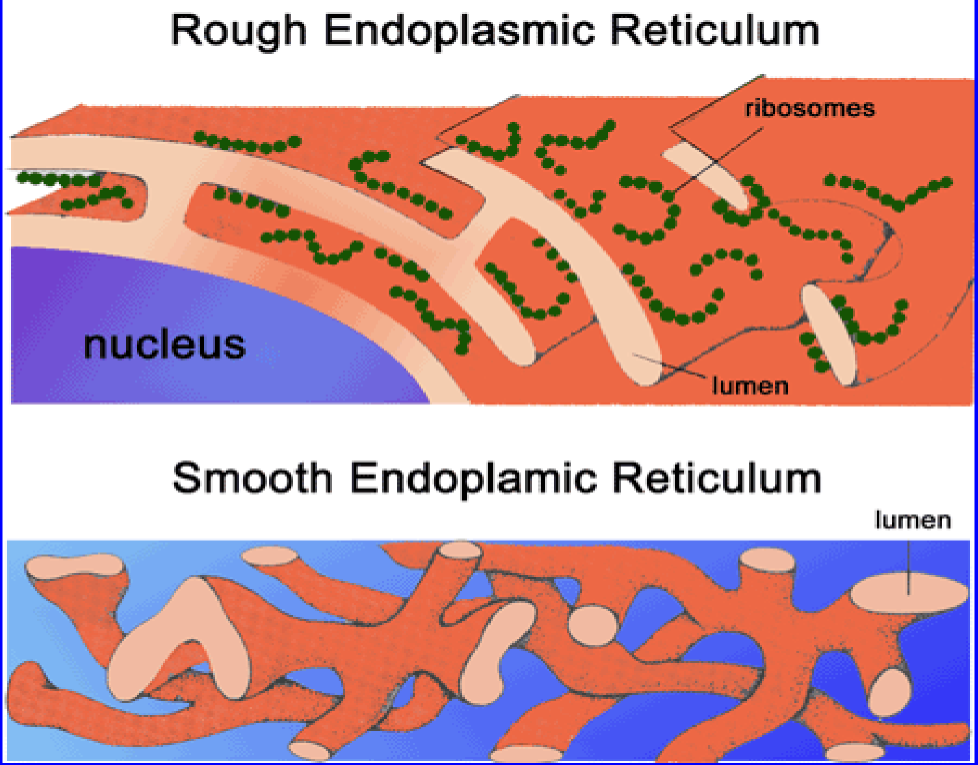
http://www.humanillnesses.com/original/Se-Sy/Sickle-cell-Anemia.html



Bar-ziv et al. PNAS, 1999

Endoplasmic reticulum shape

- Single sheet membrane
- Proteins and lipids produced here

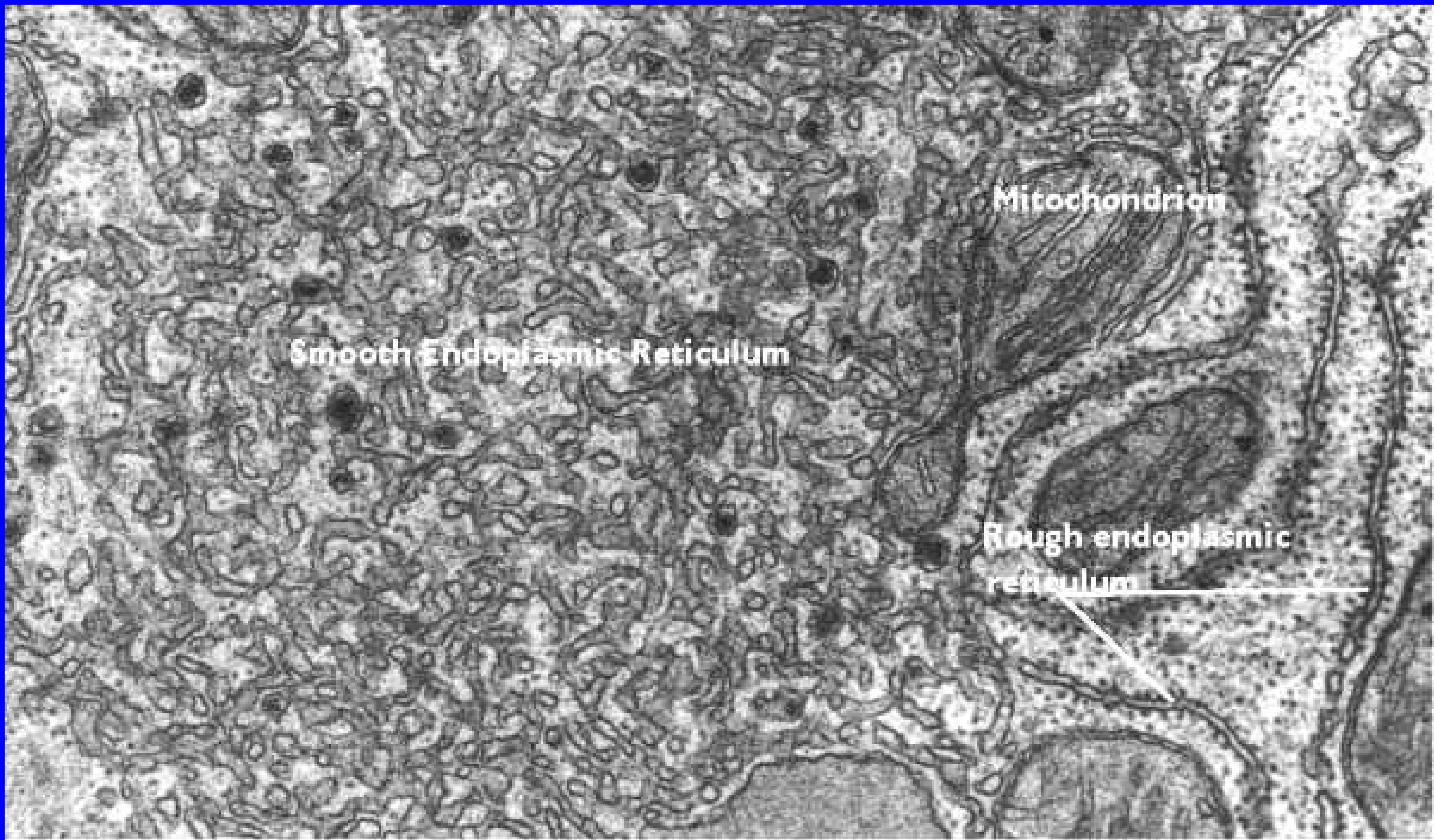


Proteins produced here

Vesicles + proteins budded off from here

http://library.thinkquest.org/C004535/endoplasmic_reticulum.html

Endoplasmic reticulum in cells



Liver Cell
ER

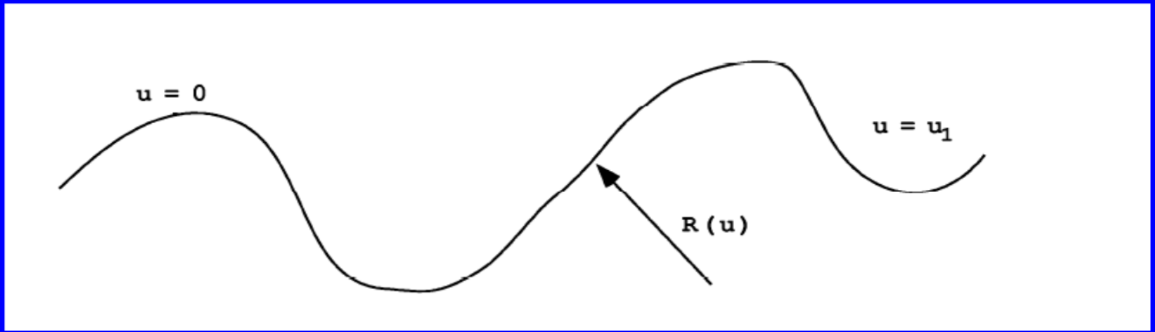
cellbio.utmb.edu/cellbio/ser.htm

Space Curves

$$\vec{r} = \vec{R}(u)$$

$$\Delta s = |\vec{R}(u_1) - \vec{R}(u_2)|$$

$$ds = |\frac{\partial \vec{R}}{\partial u}|\, du$$



Tangent vector:
Unit vector

$$\hat{t} = \frac{d\vec{R}}{ds} = \frac{d\vec{R}/du}{ds/du}$$

Curvature vector::
Change of tangent

$$\frac{d\hat{t}}{ds} = \kappa \hat{n}_c = \frac{d^2\vec{R}}{ds^2}$$

Curvature perpendicular to tangent

- Tangent is unit vector:

$$\vec{t}^{\,2}(s) = 1 = \frac{\partial \vec{R}(s)/\partial s}{\left|\partial \vec{R}(s)/\partial s\right|} \bullet \frac{\partial \vec{R}(s)/\partial s}{\left|\partial \vec{R}(s)/\partial s\right|}$$

- Curvature vector:

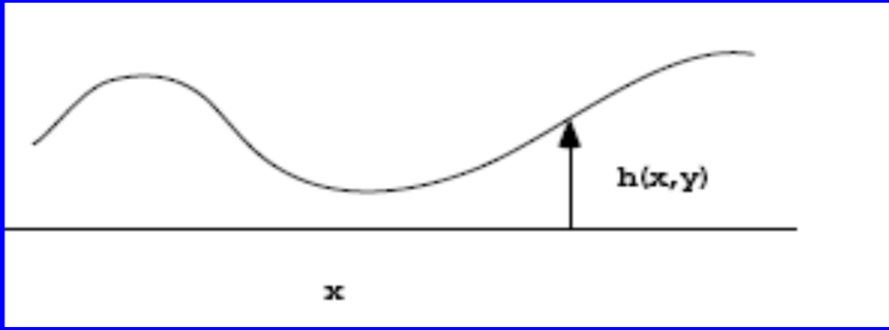
$$\frac{d\hat{t}}{ds} = \kappa \hat{n}_c = \frac{d^2 \vec{R}}{ds^2}$$

- Derivative of t^2 w.r.t. s is zero:

$$\frac{\partial \vec{t}(s)}{\partial s} \bullet \vec{t}(s) + \vec{t}(s) \bullet \frac{\partial \vec{t}(s)}{\partial s} = 0$$

- Tangent is perpendicular to curvature vector

Surfaces: Monge gauge



$$F(x, y, z) = z - h(x, y) = 0$$

On the surface: $$dF = d\vec{r} \cdot \nabla F = 0$$ Gradient perpendicular to surface

Normal vector: $$\hat{n} = \frac{\nabla F}{|\nabla F|}$$ $$\hat{n} = \frac{\hat{z} - h_x \hat{x} - h_y \hat{y}}{\sqrt{1 + h_x^2 + h_y^2}}$$

Surface geometry: area

Locus of points
where F=0
are set of points r_s

$$\int dA = \int d^3 r \sum_s \delta(\vec{r} - \vec{r}_s)$$

Convert the
delta function

$$\int dA = \int d^3\vec{r}\ \delta(F)\ |\nabla F|$$

Monge gauge:
Integrate over z

$$F(x, y, z) = z - h(x, y)$$

$$dA = dx\ dy\sqrt{1 + h_x^2 + h_y^2}$$

Curvature tensor

Change of normal as move along surface

$$d\hat{n} = d\vec{r} \cdot \mathbf{Q}$$

$$\hat{n} = \frac{\nabla F}{|\nabla F|}$$

$$F_i = \partial F / \partial r_i$$

$$\Upsilon = |\nabla F|$$

$$Q_{ij} = \frac{1}{\Upsilon} \left[F_{ij} - \frac{F_i \Upsilon_j}{\Upsilon} \right]$$

$$\Upsilon = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

Invariants of curvature tensor

- Description of shape independent of coordinate system
- Determinant and one eigenvalue of Q are zero
- Remaining 2 eigenvalues are principal curvatures
- Trace is twice the mean curvature, H (sum of curvatures)
- Sum of minors of Q is Gaussian curvature, K

  (product of principal curvatures)
- Proof that trace, minors are invariants:

  Safran, *Statistical Thermodynamics....* Cptr. 1

Invariants: eigenvalues (Mean curvature), minors (Gaussian curvature

$$H = \frac{1}{2\Upsilon^3}\left[F_{xx}(F_y^2 + F_z^2) - 2F_xF_yF_{xy} + \mathrm{Perm}\right]$$

$$K = \frac{1}{\Upsilon^4}\left[F_{xx}F_{yy}F_z^2 - F_{xy}^2F_z^2 + 2F_{xz}F_x(F_yF_{yz} - F_zF_{yy}) + \mathrm{Perm}\right]$$

$$H = \frac{(1 + h_x^2)h_{yy} + (1 + h_y^2)h_{xx} - 2h_xh_yh_{xy}}{2\sqrt{(1 + h_x^2 + h_y^2)^3}}$$

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}$$

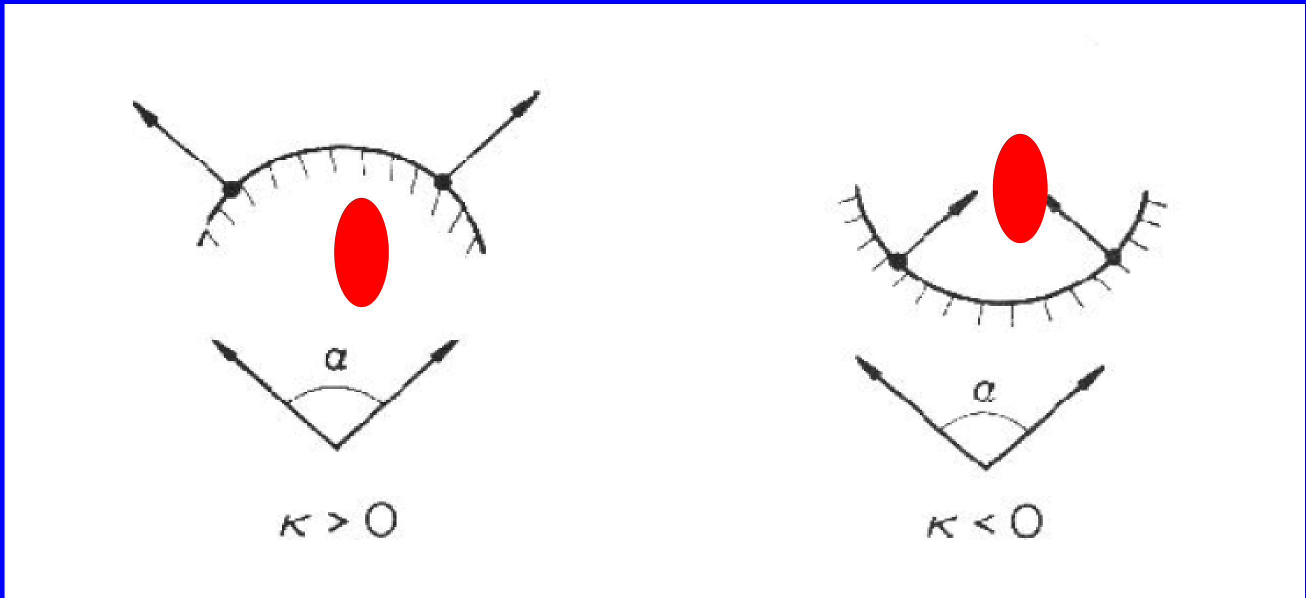
Sign of curvature

Gaussian curvature is however in fact an *intrinsic* property of the surface, meaning it does not depend on the particular embedding of the surface; intuitively, this means that ants living on the surface could determine the Gaussian curvature. For example, an ant living on a sphere could measure the sum of the interior angles of a triangle and determine that it was greater than 180 degrees, implying that the space it inhabited had positive curvature. On the other hand, an ant living on a cylinder would not detect any such departure from Euclidean geometry, in particular the ant could not detect that the two surfaces have different mean curvatures (see below) which is a purely extrinsic type of curvature.

http://en.wikipedia.org/wiki/Curvature

Sphere: equal positive curvatures
Cylinder: one zero, one positive
Plane: both zero
Saddle: one positive one negative

Sign of curvature



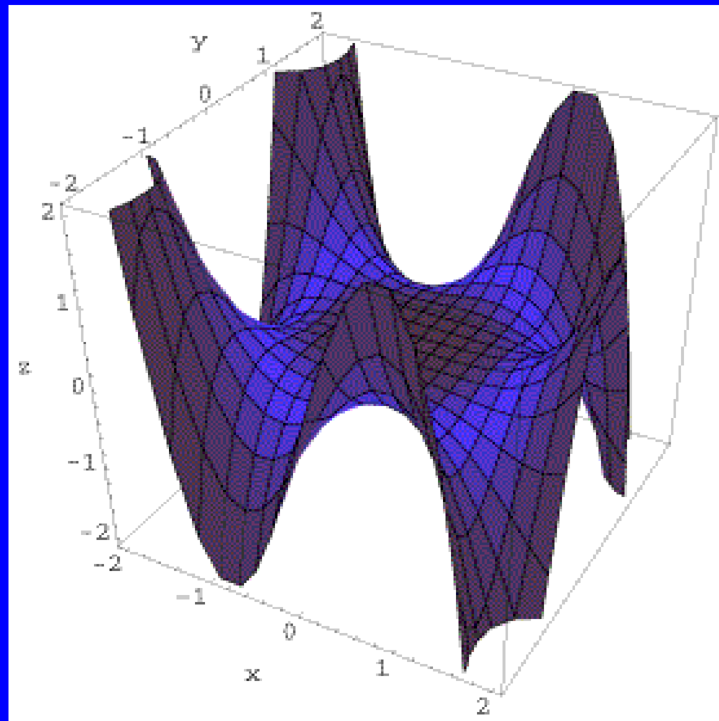
For convex surfaces the surface normals point away from the center of curvature

In the concave case the surface normals point towards the center of curvature

http://homepages.inf.ed.ac.uk/rbf/BOOKS/FSTO/node28.html

Surface shapes and curvature

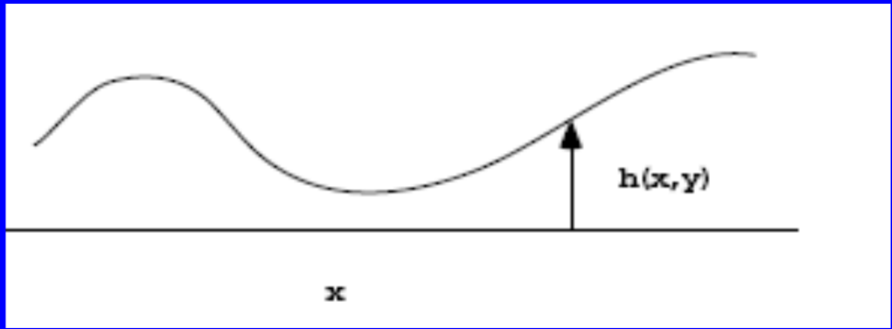
Sphere: equal positive curvatures
Cylinder: one zero, one positive
Plane: both zero
Saddle: one positive one negative



Saddle surface colored by Gaussian curvature

http://www.math.hmc.edu/~gu/
curves_and_surfaces/surfaces/genmonkey.html

Nearly flat Surface



$$A = \int dx\ dy \left(1 + \frac{1}{2}({h_x}^2 + {h_y}^2)\right)$$

Area

$$H \approx \frac{1}{2}(h_{xx} + h_{yy})$$

Mean curvature

$$K \approx h_{xx}h_{yy} - h_{xy}^2$$

Gaussian curvature

Minimal area surfaces

$$A = \int dx\, dy \sqrt{1 + h_x^2 + h_y^2}$$

Find shape that minimizes area

Euler Lagrange minimization of:

$$I = \int f[\psi(\vec{r}), \nabla\psi(\vec{r})]$$

$$\frac{\delta I}{\delta\psi(\vec{r})} = \frac{\partial f}{\partial\psi(\vec{r})} - \nabla\cdot\frac{\partial f}{\partial\nabla\psi(\vec{r})} = 0$$

Minimal surfaces: zero mean curvature

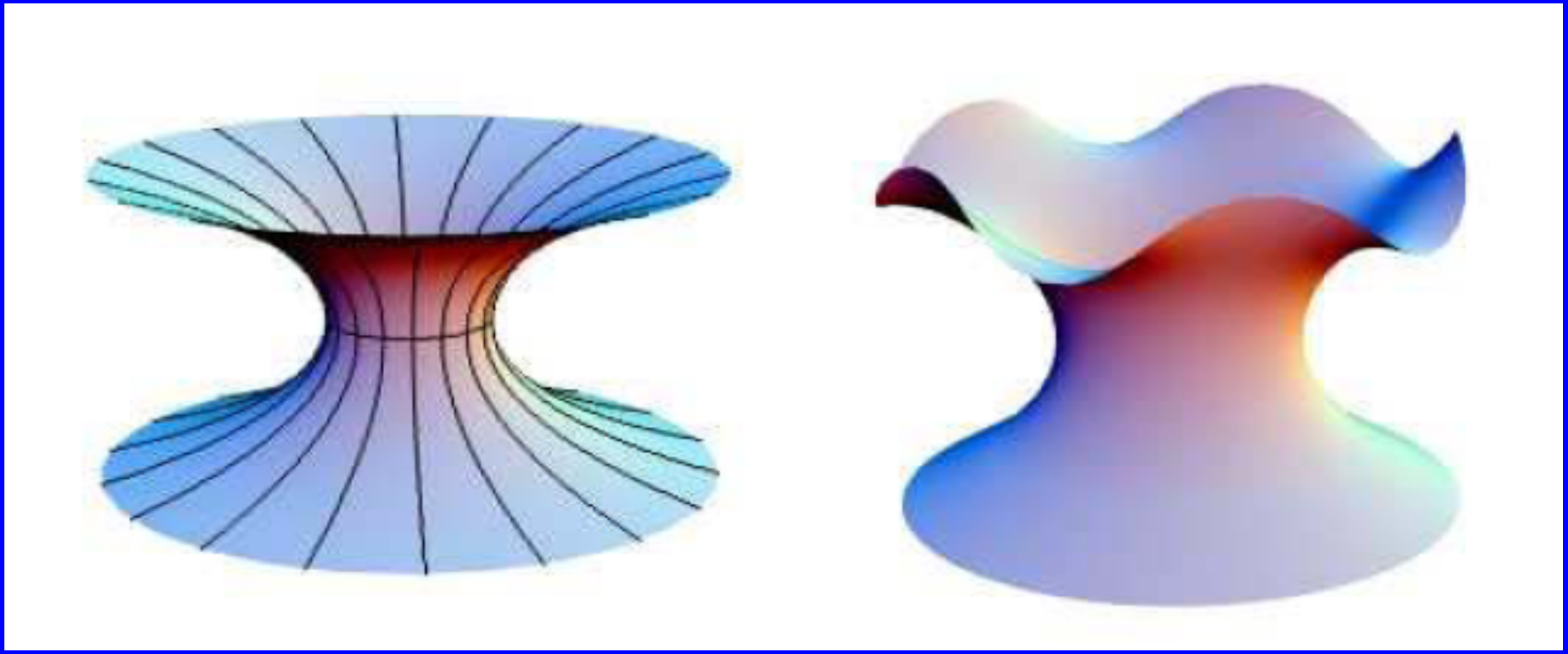
$$\frac{\partial}{\partial x}\frac{h_x}{\sqrt{1+h_x^2+h_y^2}}+\frac{\partial}{\partial y}\frac{h_y}{\sqrt{1+h_x^2+h_y^2}}=0$$

Gaussian curvature is negative: saddle shaped

$$\frac{(1+h_x^2)h_{yy}+(1+h_y^2)h_{xx}-2h_xh_yh_{xy}}{\sqrt{(1+h_x^2+h_y^2)^3}}=0$$

Physical examples: Soap bubbles in frame – minimize tension energy (area)

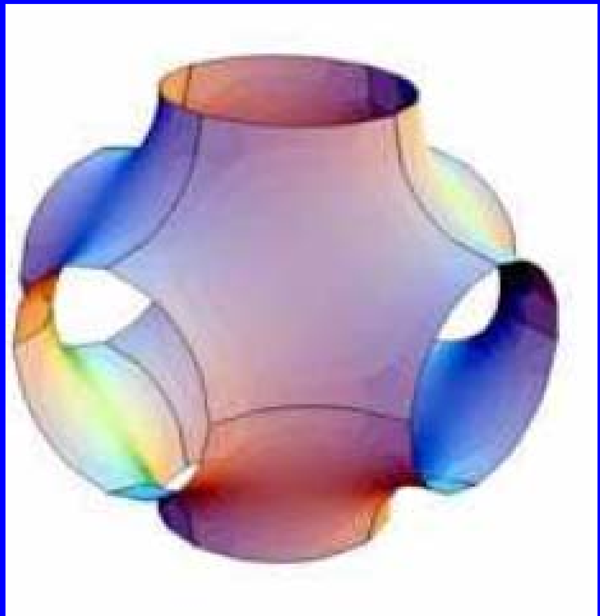
Minimal Surfaces

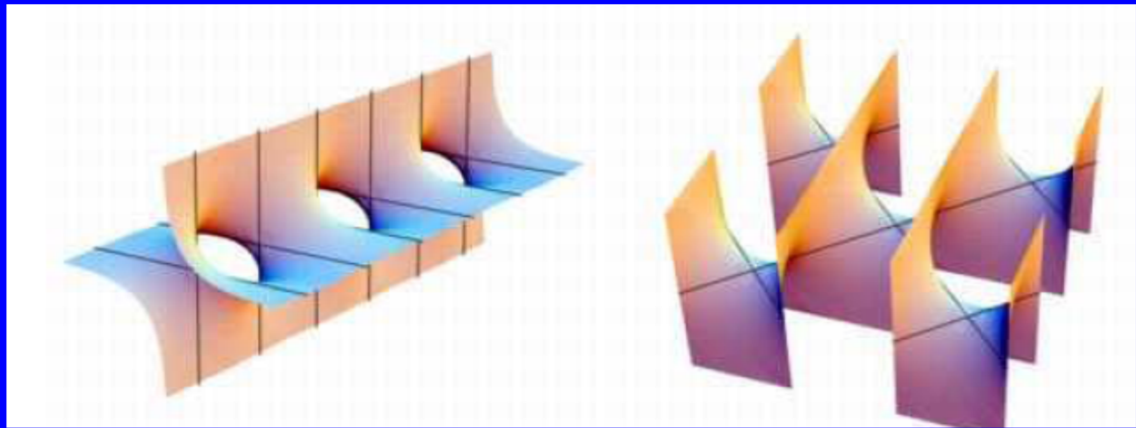


Catenoid

http://www.math.uni-bonn.de/people/weber/research/minimal/index.html

Periodic Minimal Surfaces

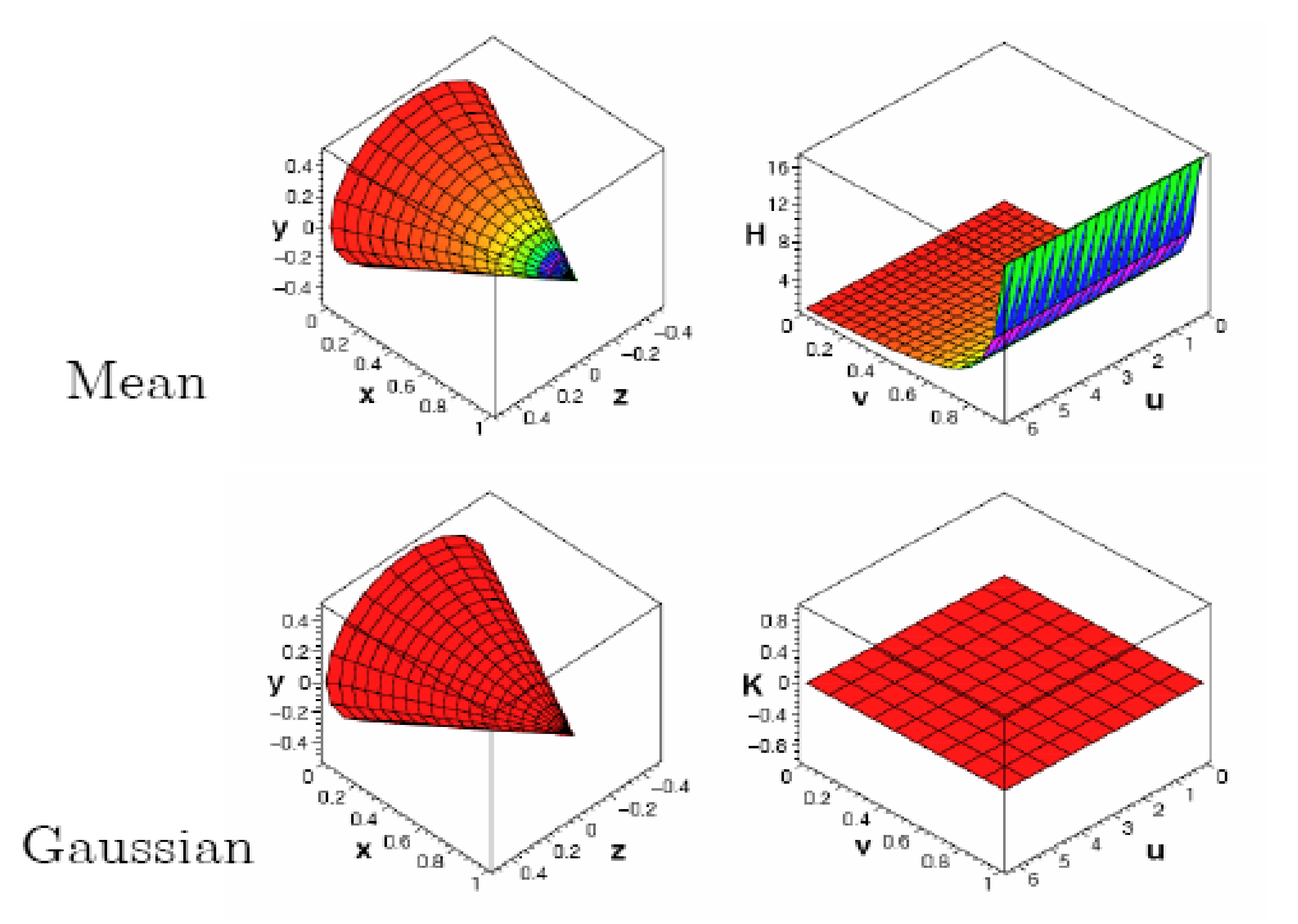




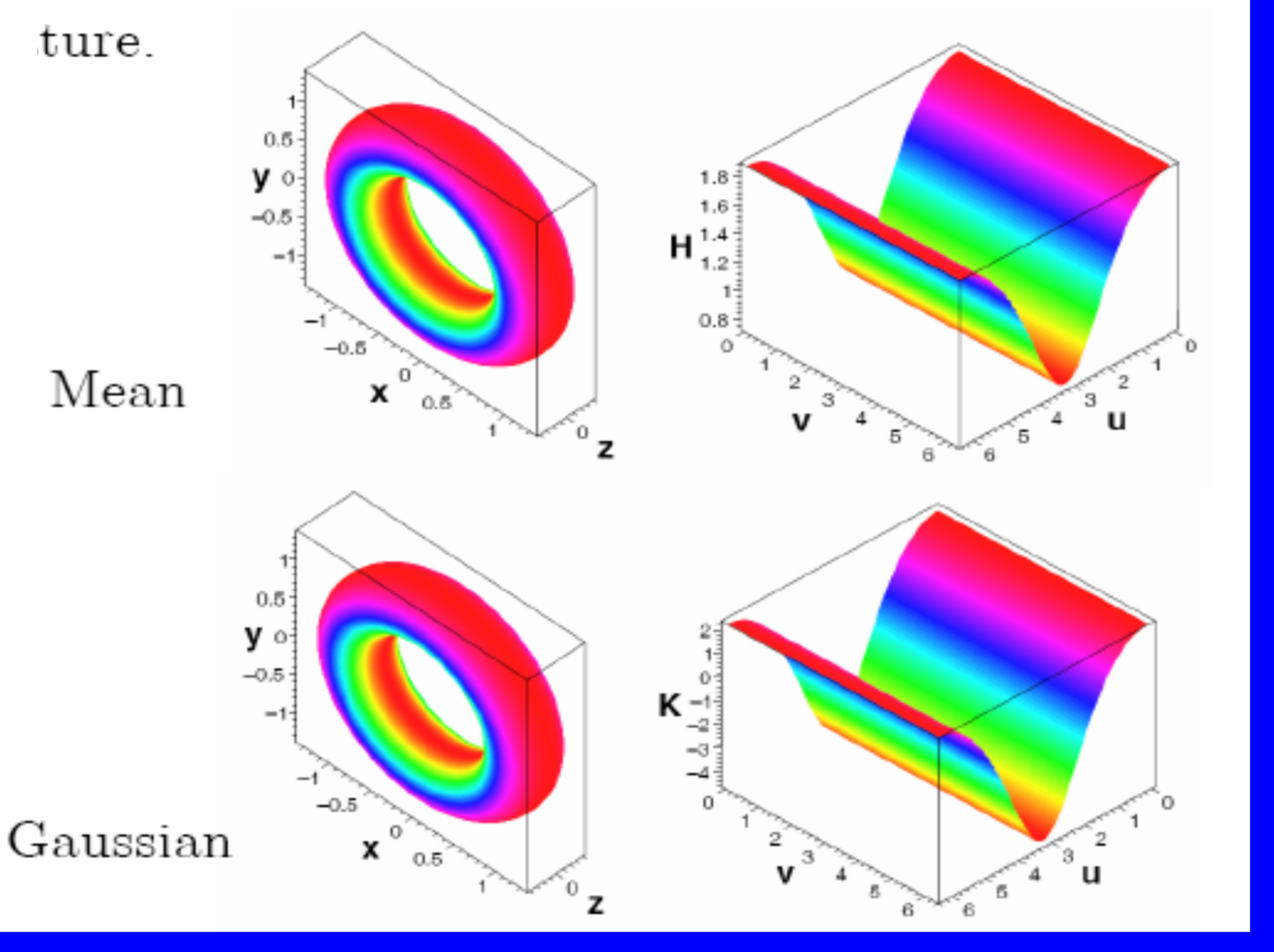
Schwarz

Supplementary material

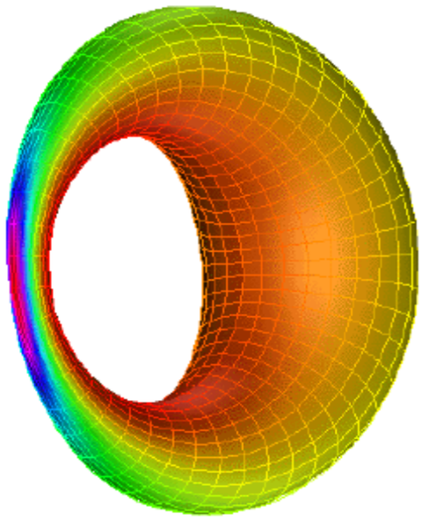
Cone



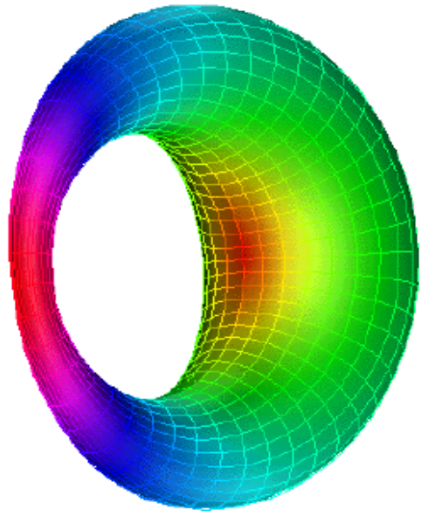
Torus



Curvature and shape I



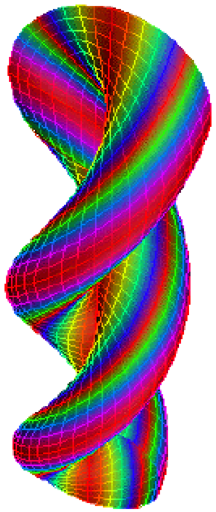
Ring cyclide
colored by Gaussian curvature



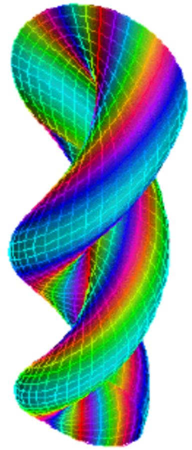
Ring cyclide
colored by mean curvature

http://www.math.umd.edu/research/bianchi/Gifsurfs/surfs.html

Curvature and shape II



Corkscrew surface
colored by Gaussian curvature



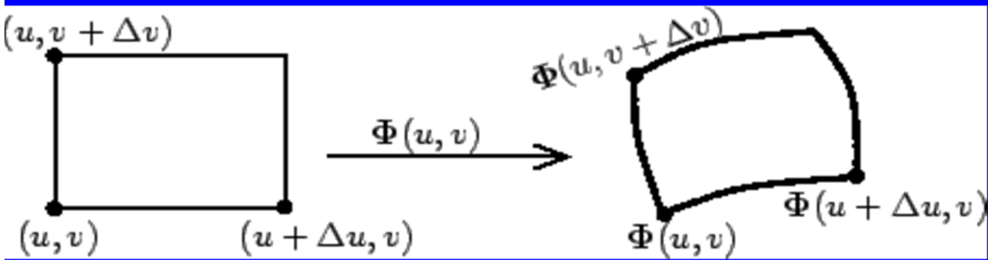
Corkscrew surface
colored by mean curvature

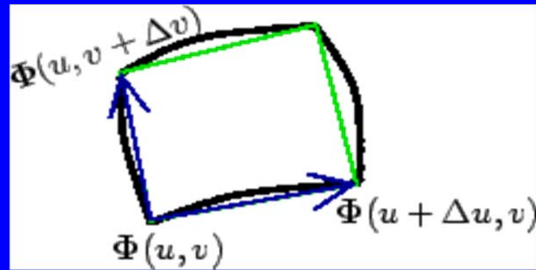
http://www.math.umd.edu/research/bianchi/Gifsurfs/surfs.html

Curvature vector

- The curvature vector is directed away from the curve on its concave side toward the z-axis
- This means that the curvature vector points in the direction that the curve is turning
- The curvature vector is ***orthogonal*** to the unit tangent vector
- The **scalar curvature,** k(s), is the magnitude of the curvature vector, **k**(s): i.e., |**k**(s)| =k(s)
- The scalar curvature is equivalent to the spatial rate of change of the orientation of the unit tangent vector with arc length along the curve
  - *The curvature is greater where the orientation of the unit tangent vector changes more rapidly with position along the curve*
  - A point on the curve where the curvature is zero is called the **inflection point**

Surface area geometry





$$\vec{r_0} = (x, y, h(x, y))$$

$$\vec{r_1} = (x+dx, y, h(x, y) + h_x(x, y)dx)$$

$$\vec{r_2} = (x, y+dy, h(x, y) + h_y(x, y)dy)$$

$$dA = |(\vec{r_1} - \vec{r_0}) \times (\vec{r_2} - \vec{r_0})| = dx\, dy \sqrt{1 + h_x(x, y)^2 + h_y(x, y)^2}$$

/http://www.math.umn.edu/~nykamp/m2374/readings/surfareacalc

http://www.math.oregonstate.edu/home/programs/undergrad/
CalculusQuestStudyGuides/vcalc/surface/surface.html