Spinors lectures

1. Introduction
    a. Bose gases with spin:
        i. Alkali atoms:
            1. hyperfine structure, Breit Rabi
            2. magnetic vs.optically trapped states
            3. hyperfine relaxation
            4. Start a table:

OK. Magnetization conserved	OK. Mag. not conserved	Not OK
Na F=1		
Rb F=1, etc.		

        ii. High spin atoms, half filled shells…
        iii. Other atoms: metastable noble gases, alkali earth
    b. Magnetic order and Bose statistics
        i. BEC's pick out low energy states
        ii. Yamada's Bose ferromagnetism
    c. Symmetry breaking
        i. argue why this is relevant
        ii. go through argument
2. High-spin states
    a. Bloch sphere
    b. Majorana representation
        i. spin nodes
        ii. orthogonal states
    c. Some examples
    d. Relation to spin moments: have to recreate derivation?
3. Rotational symmetry and collisions
4. External fields
    a. Linear Zeeman shift
    b. quadratic shifts, static and microwave derived
5. Ground states and detection methods
    a. Mean-field
        i. F=1. Show Stenger diagram and energy of extremal states. Show experimental examples and use them to demonstrate methods of probing. Sodium and rubidium
        ii. F=2. Describe experiments.
    b. Bigelow non-mean field state. Describe how this is fragmented? Give a calculation.
6. Spin mixing
    a. Two particle spin mixing
    b. Coherent spin mixing oscillations. Chapman results.
    c. Spin mixing instability

    d. Quantum quenches, connection to Kibble Zurek
    e. Spin squeezing
7. Topological defects
    a. Spin vortices:
        i. Mermin Ho vortices
        ii. Berkeley experiment
    b. Polar-state skyrmions
        i. Seoul experiment
    c. Polar-state half-vortices
        i. 2D physics
8. Dipolar stuff
    a. Predicted spin textures
    b. Berkeley experiment that "demonstrated" dipolar interactions
9. Last words
    a. Equilibration
    b. Magnetometry, limit to coherence?
    c.

Spinor Bose gases

"text book" for those lectures: arxiv: 1205.1888

Lectures outline: 6×45 minutes

1. Introduction + general concepts
2. High spin states - spin moments
3. Interactions with rotational symmetry
4. Effect of external fields
5. *Ground states + experimental methods
6. *Spin dynamics
7. Topological defects
8. Magnetic dipole interactions
9. Leftovers + future

Longer

1. Introduction

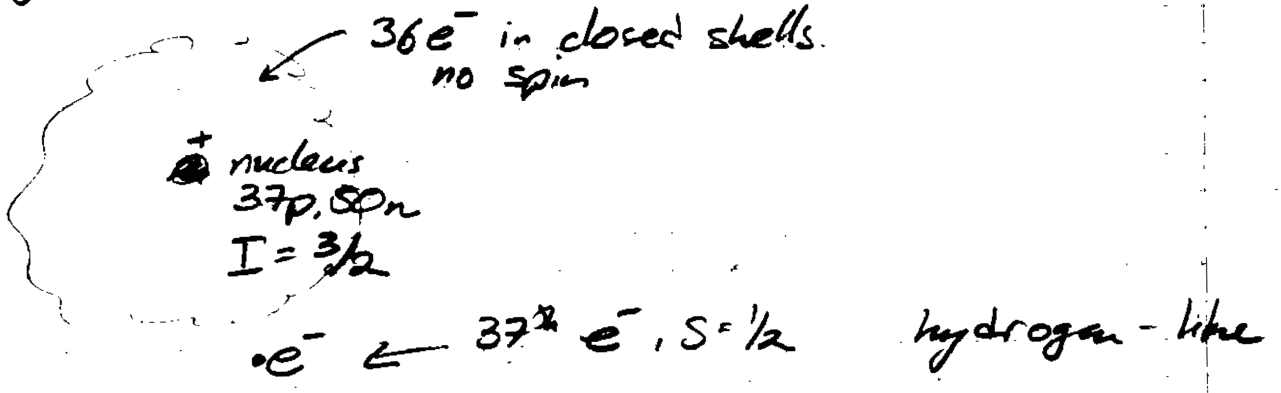
Bose gases w/internal-state degree of freedom

→ quantum fluids w/possibility of multicomponent order parameter

Out of very many possibilities, here are some examples.

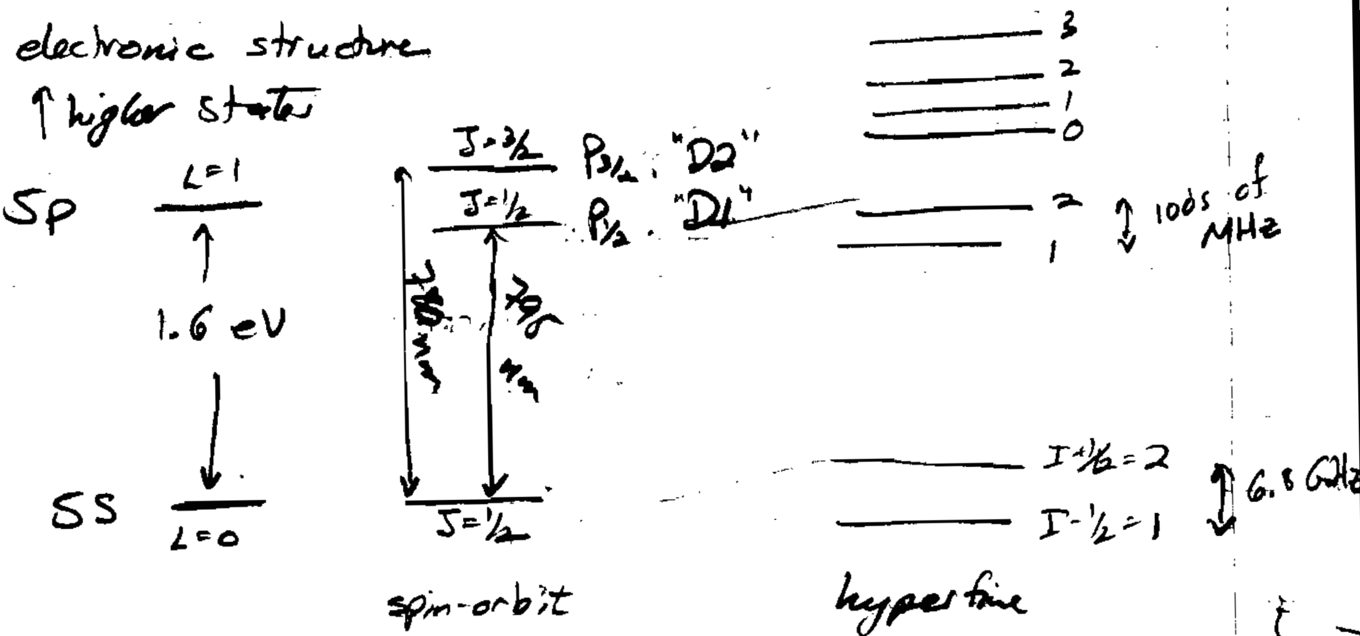
a. Alkali atoms, hyperfine spin

Eg. ^{87}Rb



electronic structure

↑ higher states



$$\mathcal{H}_{HF} = a h \frac{\vec{I}\cdot\vec{J}}{\hbar^2} - \vec{\mu}\cdot\vec{B}$$

$\mu_n \ll \mu_B$

$$\vec{\mu} = -g_J \mu_B \frac{\vec{J}}{\hbar} + g_I \mu_B \frac{\vec{I}}{\hbar}$$

↑ given by Landé formula

for $\vec{J} = \vec{L} + \vec{S}$, $g_J = g_L \left(\frac{J(J+1) - S(S+1) + L(L+1)}{2J(J+1)} \right) + g_S \left(\frac{\quad + \quad -}{\quad} \right.$

For $\vec{B}=0$, $\mathcal{H}_{HF}$ is rotationally invariant. Solved by state of total ang. momentum

$$\vec{F} = \vec{J} + \vec{I}$$

Now add small $\vec{B}$: $F \simeq$ good quantum number

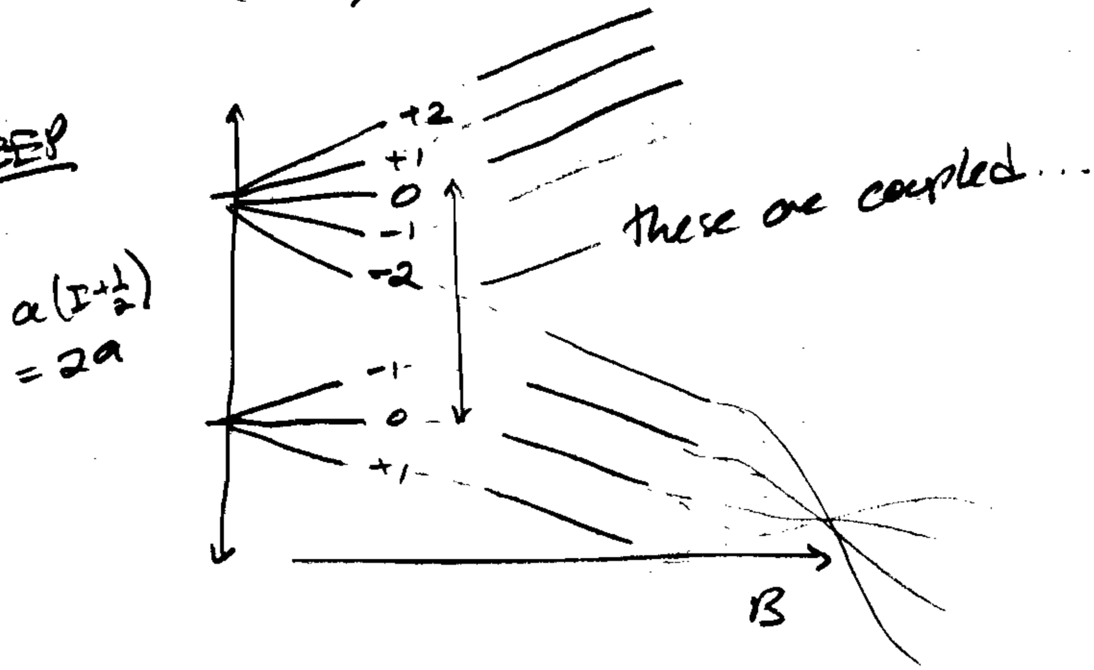
$$\langle F, m_{F_1} | \vec{\mu} | F, m_{F_2} \rangle \propto \langle \dots | \vec{F} | \dots \rangle$$

$$= -g_F \mu_B \frac{\vec{F}}{\hbar}$$

$$g_F \simeq \overset{\simeq 2}{g_J} \; \frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)}$$

$$= \frac{\pm 1}{(I+\frac{1}{2})} \quad \text{for } F = I \pm 1/2.$$

KEEP



+ Magnetic vs. optical traps:

In a mag trap, we imagine the atom moves so slowly that it remains in a fixed projection of spin along field axis

$$V(\vec{r}) \simeq -g_F \mu_B m_F |B|$$

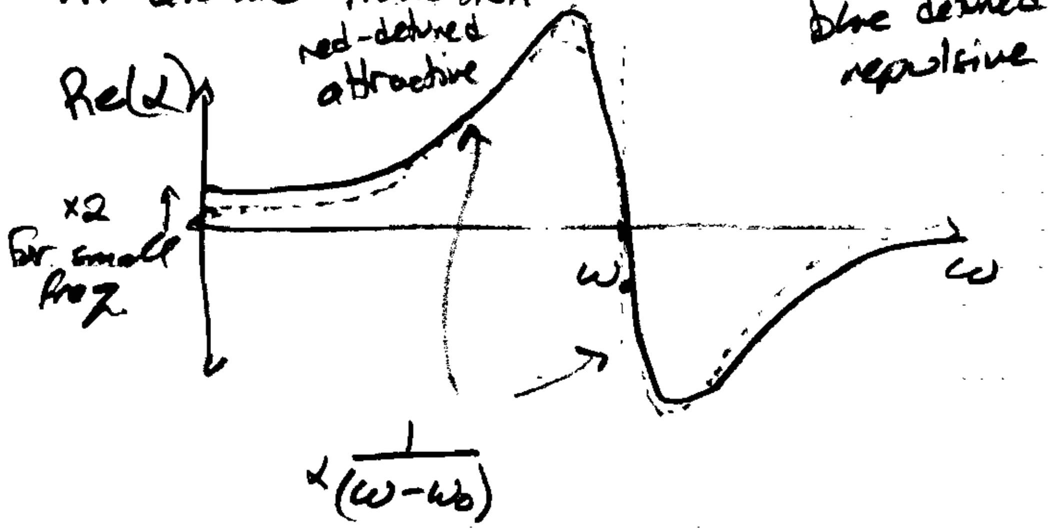
+ $|B|$ cannot have maximum in current-free region where $\nabla \cdot B = \nabla \times B = 0$

So can only trap weak-field seeking : mark diagram
→ Pseudo-spin states (more later)

<u>Optical traps</u>:

$$V = -\underset{\substack{\uparrow \\ =0 \text{ in absence} \\ \text{of } \vec{E}, \text{ by parity}}}{\vec{d}} \cdot \vec{E} \qquad = -\frac{1}{2}\langle \underset{\substack{\uparrow \\ \alpha(\omega) = \text{atomic polarizability}}}{\alpha} \vec{E}^* \cdot \vec{E} \rangle$$

At atomic transition



$$V = \sum_{\text{transitions}} \frac{\hbar\Omega_i^2}{4} \left(\frac{1}{\omega-\omega_{0,i}} + \frac{1}{\omega+\omega_{0,i}} \right)$$

Some key properties.

- Can hold all F, m_F states

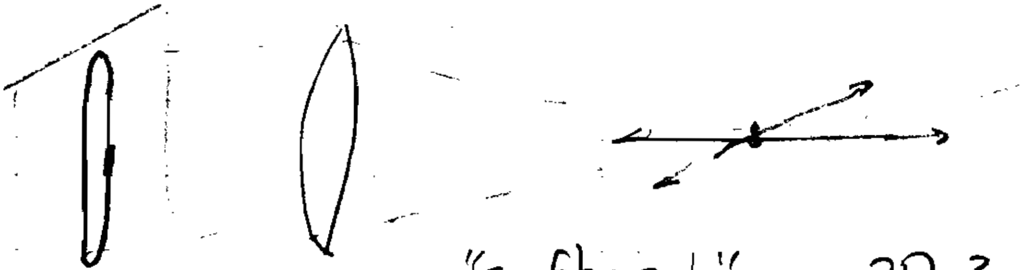
Candidates for interesting q. fluids:

$\left\{ \begin{array}{l} F = I - 1/2, \\ F = I + 1/2 \end{array} \right.$ "spinor" gases

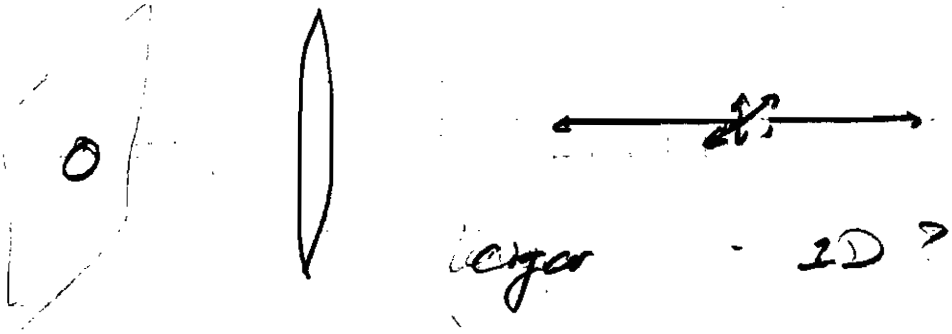
mixtures of these ? Not yet explored

[other properties, pp ~~[illegible]~~ 1.4–1.6 later]

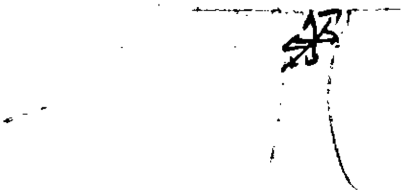
- Variable geometry / dimension



"surfboard" - 2D ?

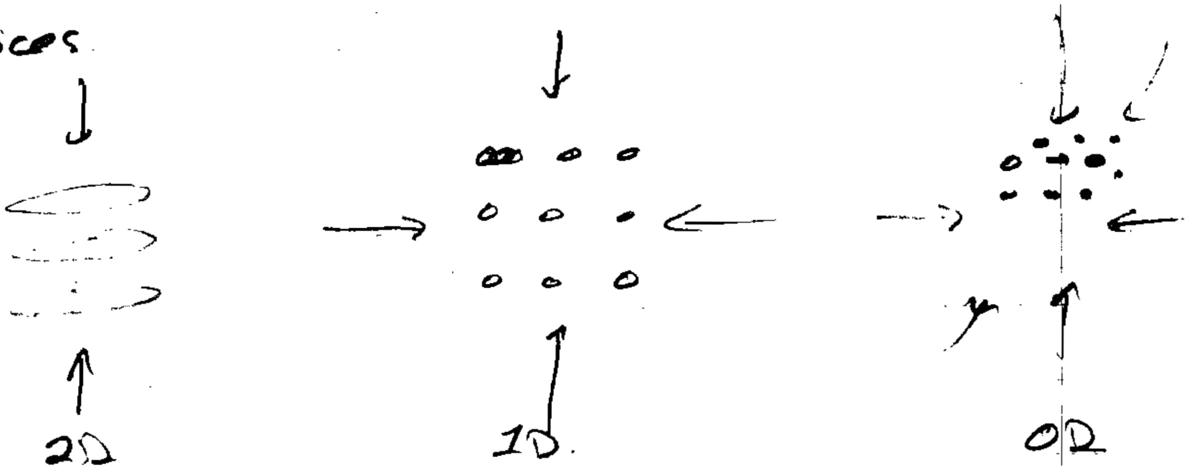


cigar - 1D ?



"crossed dipole" ~ 0D?

or lattices



Criteria for dimensionality:

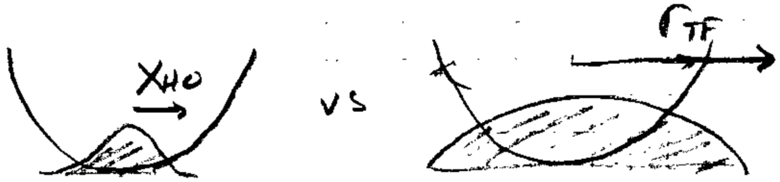
Energy	Length	Implication
$\hbar\omega \gg kT$	$x_{HO} = \sqrt{\frac{\hbar}{2m\omega}}$ $\lambda_{dB} = \sqrt{\frac{h^2}{2\pi mkT}}$ $x_{HO} \ll \lambda_{dB}$	Thermodynamics is strictly in restricted dim. Achieved sometimes in lattices

for a degenerate Bose gas...

$\mu_{int} = \frac{4\pi\hbar^2}{m}\bar{a}\, n$, $\xi_n = \sqrt{\frac{\hbar^2}{2m\mu}} = \frac{1}{\sqrt{8\pi\bar{a}n}}$

$r_{TF} = \sqrt{\frac{2\mu}{m\omega^2}}$

$\hbar\omega \gg \mu_{int}$	$x_{HO} \gg r_{TF}$ $x_{HO} \ll \xi_n$	Density dynamics of BEC frozen out - Quasi-cond + th gas in 2D - Kosterlitz-Thouless in 2D



spin dependent interactions

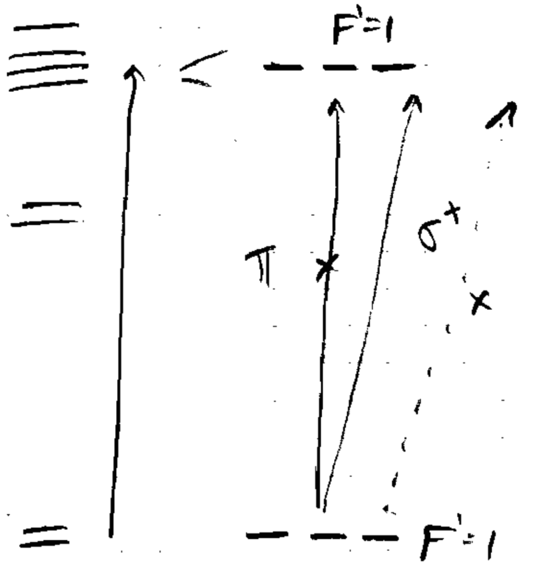
$\mu_{int} = \frac{4\pi\hbar^2}{m}\Delta a\ n$, $\xi_s = \frac{1}{\sqrt{8\pi\Delta a n}}$

$x_{HO}, r_{TF} \ll \xi_s$	Spin dynamics of BEC frozen out - low D spinor physics

- Trapping pot. may be spin dependent 1.6
  - Polz
  - Detuning

Key: light couples strongly to orbital motion of electron

→ More here space

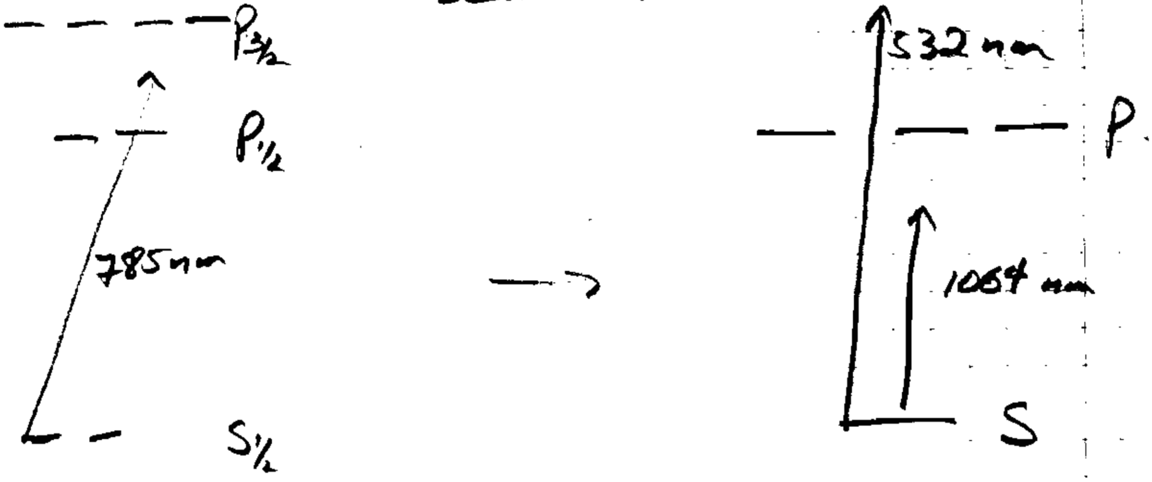


near detuning

2nd order energy

$$\sim \vec{E}^{*}\cdot\left(\vec{d}^{\dagger}\vec{d}\right)\cdot\vec{E}$$

scalar + vector + rank 2 tensor.



farther detuning: ignore nucleus in excited state

ignore tensor part

→ "Fictitious B-field".

even farther

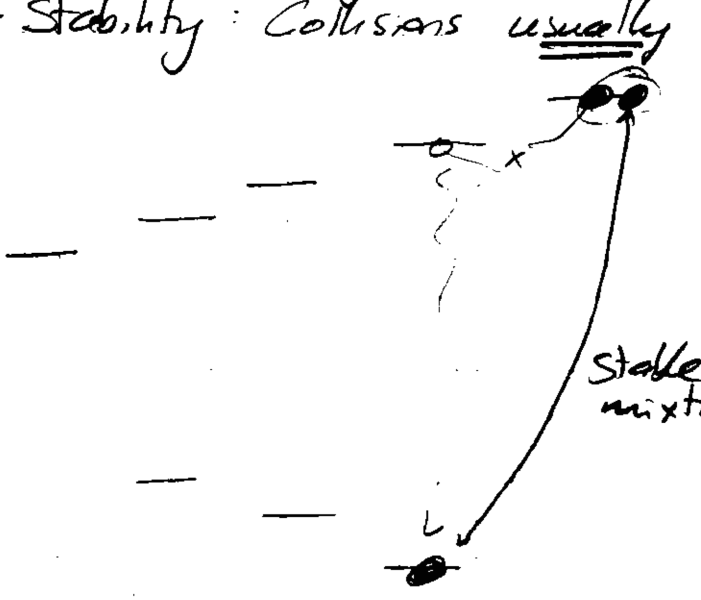
ignore electron spin

ignore vector part

scalar potential

Back to alkali gases

+ Stability: Collisions *usually* conserve m_F



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lower enough room

most lit on spinor Bose gas

OK m_F conserved	OK m_F not cons.	Not OK
^{23}Na F=1 (antiferro)	^{133}Cs F=3 (?)	^{23}Na F=2
^{87}Rb F=1 (ferro).	^{52}Cr J=3	^{133}Cs F=4
^{87}Rb F=2 (af or cyclic)	Dy J=8	^{85}Rb (?)
^{87}Rb pseudospin	Er J=6	
$\lvert F=1, m=0\rangle + \lvert F=2, m=0\rangle$	Th F=4 (?)	
$\lvert F=1, m=\pm1\rangle - \lvert F=2, m=\mp1\rangle$		

^{7}Li? ^{39,41}K? Fr?

High spin atoms

Many in recent years

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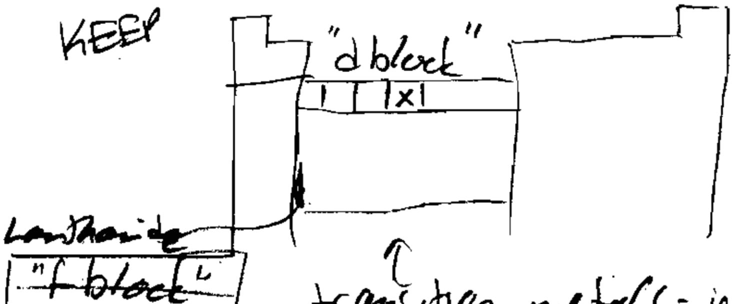


transition metals - incomplete d shells

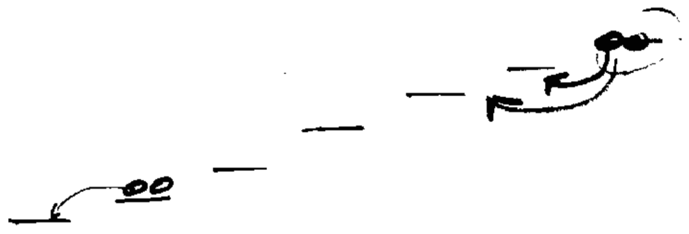
Chromium:

[Ar] $4s^1\ 3d^5$ 6 electron

$S = J = 3$, $\mu = 6\mu_B$

^{52}Cr - no nuclear spi

Cr: strong dipolar relaxation



so cannot be mag trapped

strong dipolar interaction.

$V_{dd} \propto \vec{\mu} \cdot \vec{\mu}$ → $36 \mu_B^2$ for Cr

$\frac{1}{4} \mu_B^2$ for $F=1$ Na, Rb

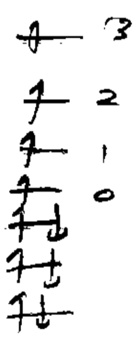
"rare earths"

Lanthandos [illegible]

Dy $[Xe]\, 6s^2\, 4f^{10}$

$S=2,\ L=6,\ J=8$

μ 4 6 $\underline{\underline{10 \mu_B}}$



"most magnetic atom"

both Bosonic ($I=0$) + fermionic

Er $[Xe]\, 6s^2\, 4f^{12}$

$S=1,\ L=5,\ J=6$

$7 \mu_B$

(named after Ytterby, Sweden along with Ytterbium, Yttrium Terbium !)

Tm $[Xe]\, 6s^2\, 4f^{13}$

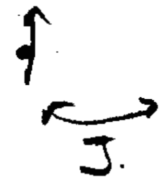
$S=\frac{1}{2},\ L=3,\ J=\frac{7}{2}$

$I=\frac{1}{2} \rightarrow F=4$. g.s

Add to table

b. Magnetic order + Bose statistics.

Usual paradigm for magnetic order



$kT \lesssim J$ → order.

Bose gases order at very *high* temperature

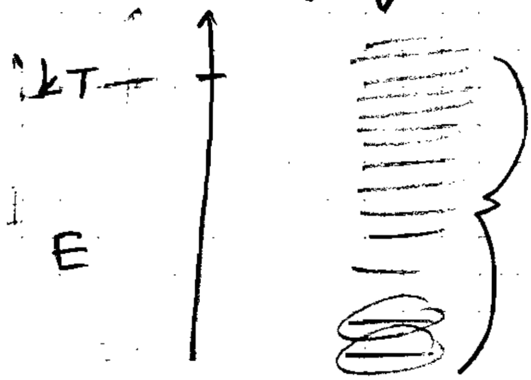
we'll find

$$E_{spin} \sim \frac{4\pi\hbar^2}{m}(\Delta a)\cdot n \sim k_B \times nK \ll k_B T_c \sim k_B \times 100nK$$

why?

Start w/observation that *scalar* BEC "knows" its ground state.

Argument roughly as follows:



M(T) states w/in reach

when $N \sim M$, q. statistics become significant

What happens? Consider just g.s. + 1st excited state

Does BEC occur? ≟ Is g.s. the only one macroscopically occupied?

$$N_1 \sim \frac{k_B T_c}{\Delta E} \sim N^{2/D}.$$

So, in 3D, large N, BEC can distinguish even tiny energies.

$$\text{in Box } E = \underbrace{\frac{\hbar^2}{2m}\times\left(\frac{2\pi}{L}\right)}_{\sim \Delta E}{}^{D}\cdot\left(n_1^2+n_2^2+\ldots n_D^2\right)$$

$$M(T) \sim n_T^D \sim \left(\frac{kT}{\Delta E}\right)^{D/2} \sim N$$

→ "Bose-Einstein ferromagnetism"

Yamada, Prog. Th. Physics 67, 443 (1982)

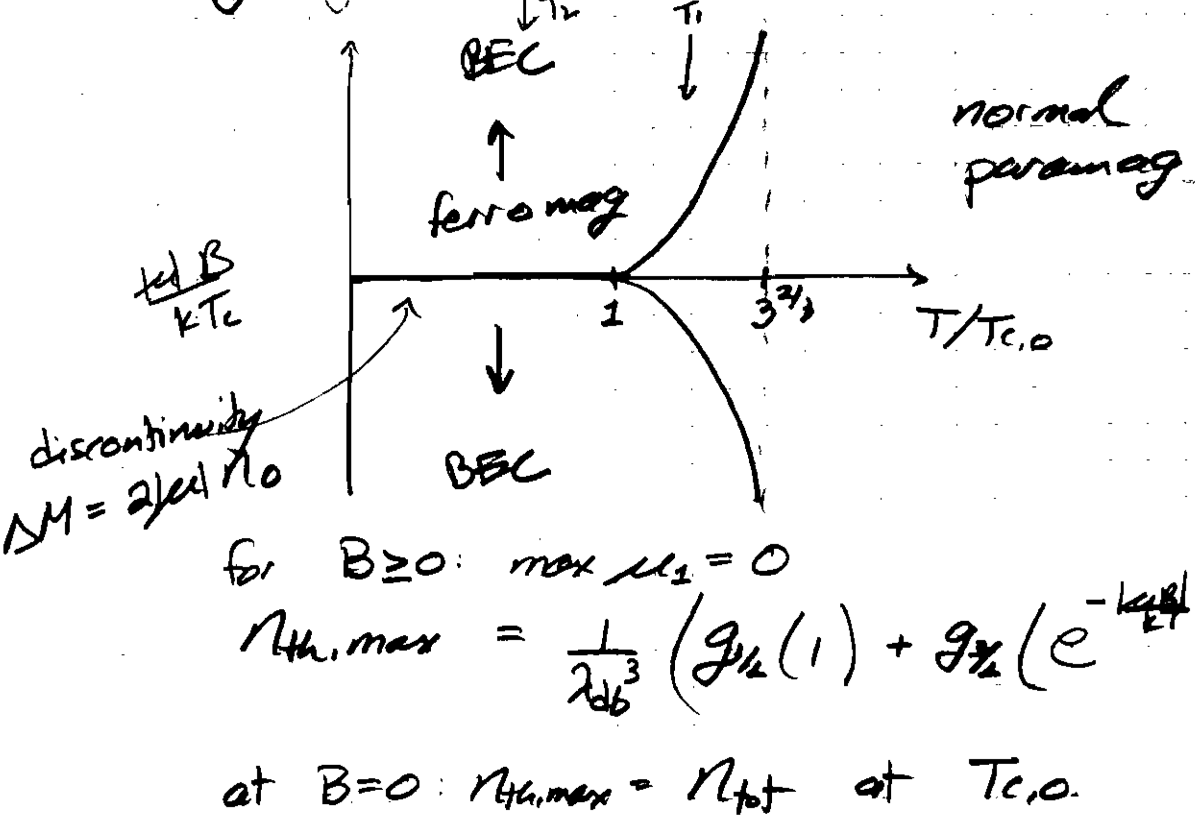
consider spin 1 BEC, no interactions, in magnetic field.

$$\mu_{m_F} = \mu_0 \mp \mu B m_F$$

$$n_{thermal, m_F} = \lambda_{dB}^{-3} \times g_{3/2}\left(e^{-\frac{\mu_{m_F}}{kT}}\right)$$

$$\uparrow \quad \sum_i \frac{x^i}{i^{3/2}}$$

say $\mu < 0$ (electron spin)



for $B \geq 0$: max $\mu_1 = 0$

$$n_{th,max} = \frac{1}{\lambda_{db}^3}\left(g_{3/2}(1) + g_{3/2}\left(e^{-\frac{|\mu B|}{kT}}\right) + g_{3/2}\left(e^{-\frac{2|\mu B|}{kT}}\right)\right)$$

at $B=0$: $n_{th,max} = n_{tot}$ at $T_{c,0}$.

at $B \gg \left|\frac{kT_c}{\mu}\right|$: $n_{th,max} = n_{tot}$ at $T_{c,0} \times (2F+1)^{2/3}$.

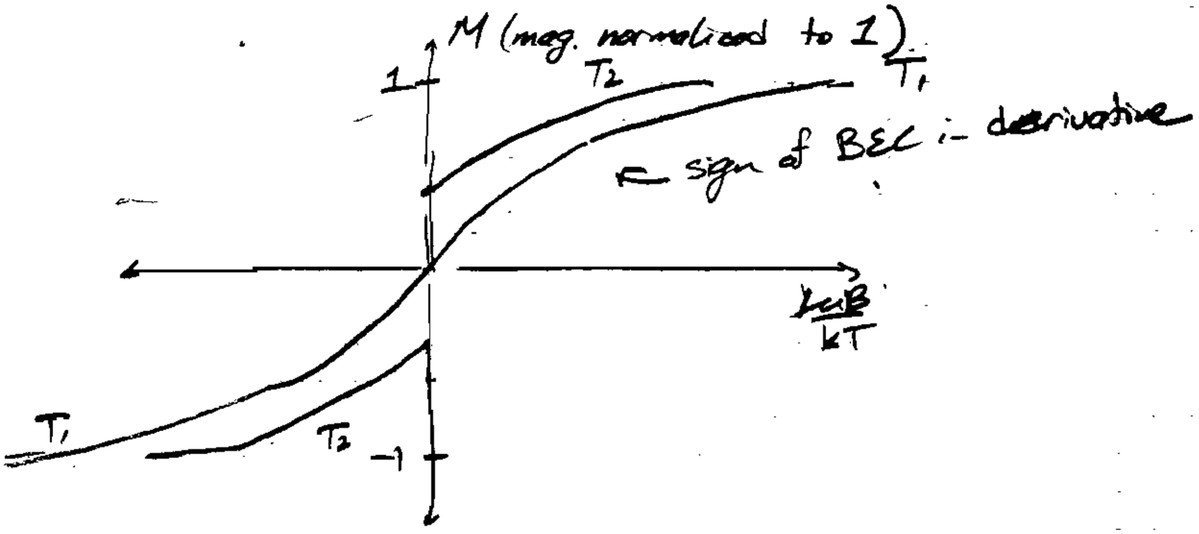
in BEC regime, $B>0$, μ_1 remains pegged at zero.

extra density → $m_F = +1$ state

Still unanswered:

→ What about spin conservation?

Eg. (back to sketch)



T_1 : use $\frac{\mu_B B}{kT}$ as Lagrange multiplier to get M

T_2 : ? ∃ Range of M where prediction fails

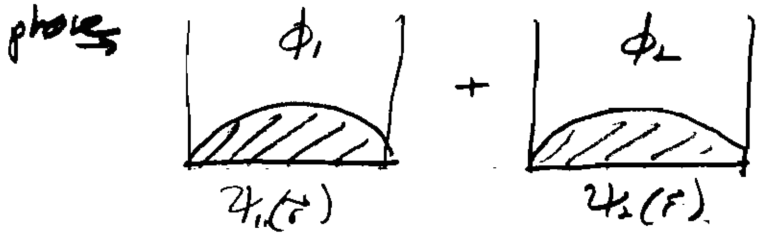
⇒ "Double condensation" Isoshima, Ohmi, Machida
(seen in Paris). JPSJ 69, 3864 (2000)

→ What is magnetic order in this case, or in case of B=0?

→ Interactions?

Simpler case: 2 component BEC ("pseudo spin ½")
Possibilities:

1) 2BEC picture



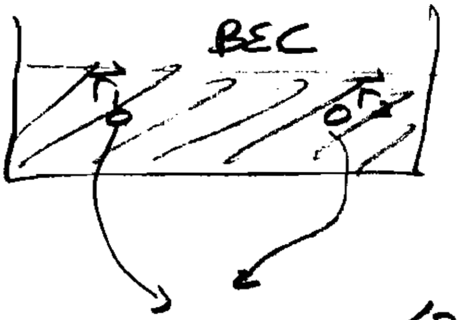
order parameter = spinor wavefunction

$$\vec{\psi} = e^{i\frac{\phi_1+\phi_2}{2}} \begin{pmatrix} e^{i\frac{\Delta\phi}{2}}|\psi_1| \\ e^{-i\frac{\Delta\phi}{2}}|\psi_2| \end{pmatrix}$$

BEC wfn when, say
$\langle n_1 \rangle = n/2$

This spinor wfn. implies long range coherence

Accessible in interference expt

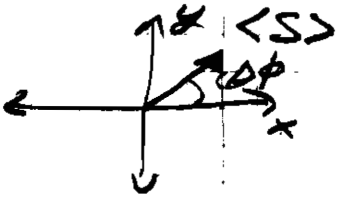


Definite outcome

$$\langle \psi_i^\dagger(\vec r_1)\psi_j(\vec r_2)\rangle = \psi_i^*(\vec r_1)\psi_j(\vec r_2)$$

→ Broken symmetry state

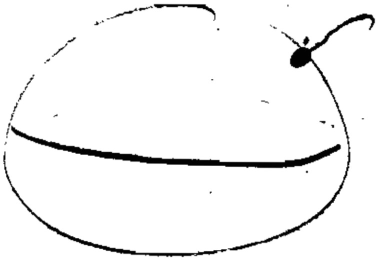
- $\Delta\phi$: orientation of pseudospin



- $\phi_1+\phi_2$: choice of absolute phase [optional].
  (seen in interference w/another BEC)

For $\langle N_1\rangle = \langle N_2\rangle$: choose pseudospin on equator

For unconstrained $N_1 : N_2$?



choose any $\vec\psi$ out of $SU(N)$?
here 2.
can be more...

2) Statistical picture (molecular chaos)

All zero E states equally likely...

$\rho = |N,0\rangle\langle N,0| + |N-1,1\rangle\langle N-1,1| + \ldots$?

"Fragmentation" - more than one macroscopically occupied single particle state.

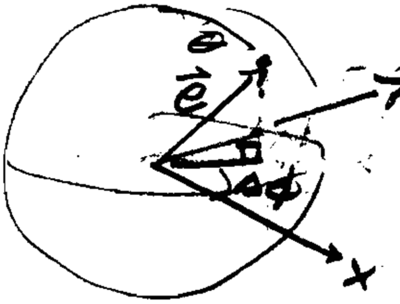
How to tell difference? [Discuss]

2. High-spin states:

Spin ½: Bloch sphere.

All spin ½ q. states point somewhere

$$\vec{\Psi} = e^{i\Phi}\begin{pmatrix} e^{\frac{i\phi}{2}} & \cos\theta/2 \\ e^{-i\phi/2} & \sin\theta/2 \end{pmatrix}$$



$F_{\vec{e}} = \vec{F}\cdot\vec{e}$ drop usually

$$\hat{F}_{\vec{e}}\vec{\Psi} = \frac{\hbar}{2}\vec{\Psi}$$

Also, even for mixed states, all information in vector spin moments

$$\rho = \begin{pmatrix} \frac{1}{2}+S_z & S_x+iS_y \\ S_x-iS_y & \frac{1}{2}-S_z \end{pmatrix}$$

<u>Higher spin states</u>

Derived by Majorana (1932) trying to explain evolution of high spin states

Construction 1: Build spin F state as composite state of 2F spin ½ particles

→ must be fully symmetric under particle exchange

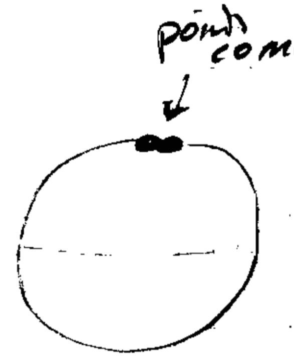
Example $F=1$: $|1,1\rangle = |\uparrow\rangle|\uparrow\rangle$ ← b.c. were just lowering this

$$|1,0\rangle = \frac{|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle}{2}$$

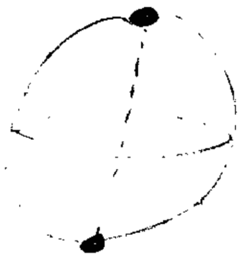
$$|1,-1\rangle = |{\downarrow}{\downarrow}\rangle.$$

Represent state as 2F points on Bloch sphere

$$\{\vec{e}_\Psi\}$$



$|1,+1\rangle$



$|1,0\rangle$

$|1,-1\rangle$

Construction 2:

"Ferromagnetic state" $|\vec{e}_F'\rangle = \hat{R}_{\hat{z}\to\hat{e}} |F,F\rangle$

find all $\vec{e}_\psi'$ s.t. $\langle \vec{e}'_{\psi,F}|\psi\rangle = 0$.

can be expressed as polynomial ~~of~~ order 2F.

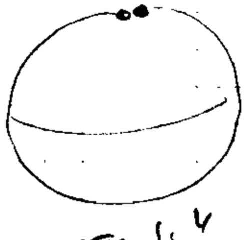
Clearly $\{\vec{e}_\psi'\} = \{-\vec{e}_\psi\}$

$$\langle -\vec{e}_i|\langle\vec{e}_i|\ \langle\vec{e}_i|\sum P\left(|\vec{e}_1\rangle|\vec{e}_2\rangle|\vec{e}_3\rangle.\right) = 0$$

Representation is useful to see geometric group symmetry

<u>CAUTION</u>: Misses "Berry phase" information [Kip]

e.g.

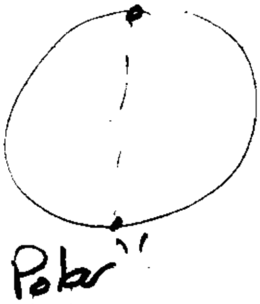


"Ferromagnetic"

looks symmetric for rot. about $\hat{z}$

actually no: picks up phase

"spin-gauge symmetry" - connection b/w rotations of $\vec{M}$ and superfluid velocity



"Polar"

looks symmetric for π rotation about x or y.

Actually no: $e^{-i\pi F_1}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$

→ half vortices: topological binding of phase + nematicity

More physical quantities are the spin/mag. moments

3×3 matrix

$$\rho_{F=1} = \underbrace{\begin{pmatrix} \rho_{11} & \times & \times \\ \times & \rho_{00} & \times \\ \times & \times & \rho_{-1-1} \end{pmatrix}}_{\substack{2 \text{ diagonal} \\ 6 \text{ off diagonal}}} = \underset{(\frac{1}{3}I)}{\text{scalar}} + \underset{\substack{(3) \\ \text{vector spin}}}{\text{vector}} + \underset{\substack{(5) \\ \text{spin} \\ \text{quadrupole}}}{\text{rank 2}}$$

= 8 degrees of freedom

$$\hat{N}_{\mu\nu} = \frac{1}{2}\left(F_\mu F_\nu + F_\nu F_\mu\right) - \frac{1}{3}(F^2)_{\mu\nu}$$

Can relate $\{|\vec{\varepsilon}\rangle\}$ to these moments, but not simple

Even higher order mag moments for F=2, etc.

3. Interactions under rotational symmetry

As in many areas of physics, symmetry helps us simplify a problem + make general (universal) predictions.

Here: what does rotational symmetry imply re: interactions in spinor Bose gas?

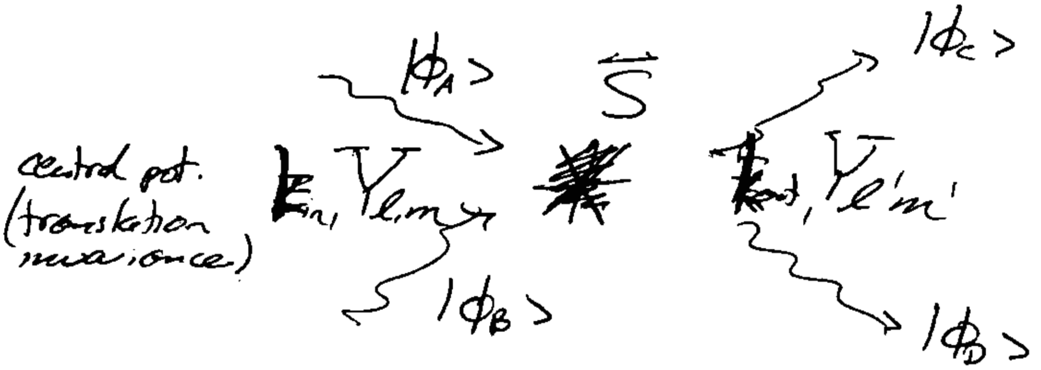
[use notes from MI summer school]. (3.2 and 3.3)

continues...

a) Ultracold (Quantum) collisions

b) Example of an alkali atom ~~slide~~

Implication for interactions. Consider binary collisions

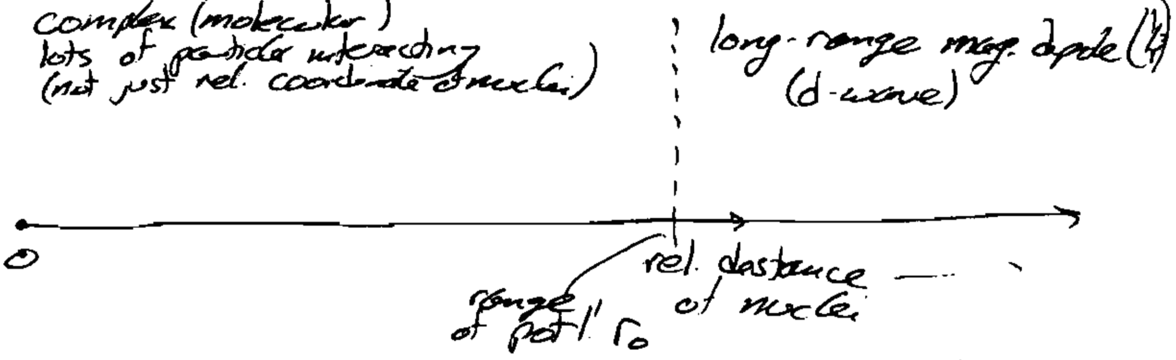


How complicated is $\vec{S}$?

- Classical view of identical particles $A \leftrightarrow B$, $C \leftrightarrow D$
- Time reversal in $\leftrightarrow$ out
  $S \leftrightarrow S^*$
- Energy conservation: diff b/w kin, out made up by internal energies

Further simplifications:

[KEEP]



Common approx 1 - incident low energy ($\lambda_{dB} \ll r_0$)

s-wave only (quantum collision regime)

affected by short range potential

+ treat long range separately.

Focus for now on short range - still pretty open

Spinor gas approx ~~(pot'l at zero field)~~ (note - imperfect)

- interactions are rotationally symmetric

Total ang mom in = " out

Note: Imperfect assumption - in applied B field (see Fesh. res)
- non spherical container

Common approx 2 - Short-range ptl: dipolar, hyperfine interactions are weak

⇒ Ignore "spin-orbit coupling" → e.g. shape resonances in Cs

Orbital ang. mom separately conserved

$F_{tot}(in) = F_{tot}(out)$

Common approx 3 - Collision keeps you w/in ang. mom. manifold

- $\langle F_1', F_2'; F_{tot} \| \substack{\text{short-range}\\\text{stuff}} \| F_1, F_2; F_{tot} \rangle = 0$

~~2&3 are related~~

or
- energetics - stay in low energy hyperfine manifold.

After all these assumptions ...
- s-wave δ fn interactions, characterized by scat. length
- $a_{F_{tot}}$ depends only on total spin F_{tot}

$$F: V_{\substack{short\\range}} = \frac{4\pi\hbar^2}{m}\left[a_0\hat{P}_0 + 0\hat{P}_1 + a_2\hat{P}_2 + a_4\hat{P}_4 \cdots + a_{2F}\hat{P}_{2F}\right]$$

Bose statistics

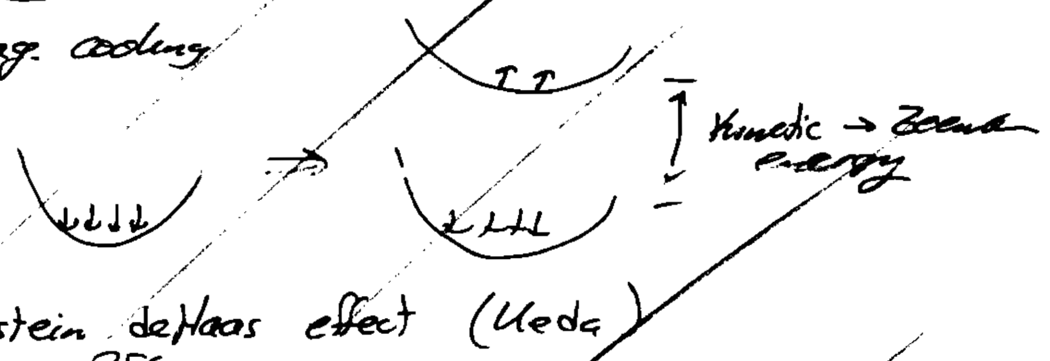
Famous exception: ~~Einstein deHaas effect~~ ~~{Ueda theory~~

Chromium, Pfau, Santos, Ueda

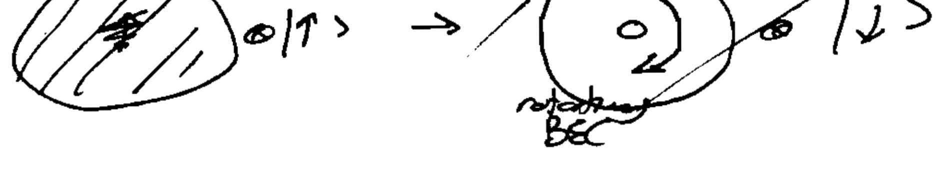
~~52Cr~~ $\mu = 6\mu_B$. $F=3$

Both short + long-range dipolar effects are very large

- Demag. cooling



- Einstein deHaas effect (Ueda)



b) Spin-dependent s-wave interactions in terms of spin operators.

Spin 1: use 2 identities.

① $$\underbrace{\hat{I}_1 \otimes \hat{I}_2}_{\text{identity op in 2 atom tensor space}} = \sum_{F_{pair}} \hat{P}_{F_{pair}} = \hat{\vec{P}}_0 + \hat{\vec{P}}_1 + \hat{\vec{P}}_2$$

For just symmetric states:

$$\left(\hat{I}_1 \otimes \hat{I}_2\right)_s = \sum_{\substack{F_{pail} \\ \text{even}}} \hat{P} = \hat{P}_0 + \hat{P}_2$$

② $$\hat{F}_1 \cdot \hat{F}_2 = \sum_{F_{pair}}' \frac{1}{2}\left(F_{pair}(F_{pair}+1) - 2F(F+1)\right) \hat{\vec{P}}_{F_{pair}}$$

$$\left(\hat{F}_1 \cdot \hat{F}_2\right)_S = P_2 - 2P_0$$

$$\text{So } P_0 = \frac{1}{3}\left(\hat{I} \otimes \hat{I} - \hat{F} \cdot \hat{F}\right)$$

$$P_2 = \frac{1}{3}\left(2I \otimes \hat{I} + \vec{F} \cdot \hat{F}\right)$$

$$\text{So } \underset{\text{range}}{V_{short}} = V_s = \frac{4\pi\hbar^2}{m}\left[\underbrace{\frac{2a_2 + a_0}{3}}_{C_0^{(1)}} + \underset{C_1^{(1)}}{\frac{a_2 - a_0}{3}} \hat{F} \cdot \hat{F}\right] \delta^3(\vec{r})$$

Spin-2: similar approach →

$$\frac{4\pi\hbar^2}{m}\left[\frac{4a_2 + 3a_4}{7} I \otimes I + \frac{(a_4 - a_2)}{7} \vec{F_1} \cdot \vec{F_2} + \frac{(7a_0 - 10a_2 + 3a_4)}{35} \hat{\vec{P}}_0\right]$$

$A^{+}A$ ← "singlet amplit

Note: F=1 spin interaction leads to following terms

$$\frac{c_1^{(1)}}{2} \sum_{\substack{k,l,\\ m,n}} \psi_k^+ \psi_l^+ \psi_m \psi_n \times \left[F_{x,kn} F_{x,lm} + F_{y\ldots} F_{y\ldots} + F_{z\ldots} F_{z\ldots} \right.$$

all these collisions preserve m_F

the sum generates terms like

$$c_1^{(1)} \left(\psi_{+1}^+ \psi_{-1}^+ \psi_0 \psi_0 + \psi_0^+ \psi_0^+ \psi_{+1} \psi_{-1} \right)$$

"spin mixing collisions"

$$|0\rangle + |0\rangle \leftrightarrows |+1\rangle + |-1\rangle$$

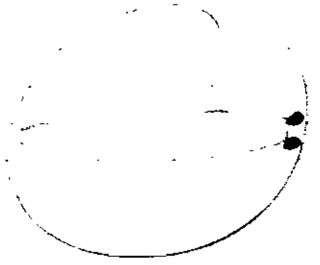
- These occur only for $c_1^{(1)} \neq 0$, $a_2 \neq a_0$
- Relative phase b/w components of spinor wfn is meaningful. In particular

$$\theta = \phi_{+1} + \phi_{-1} - 2\phi_0$$

determines direction of spin mixing.

Also for energetics:

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \quad \text{vs.} \quad \begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ 1/2 \end{pmatrix}$$



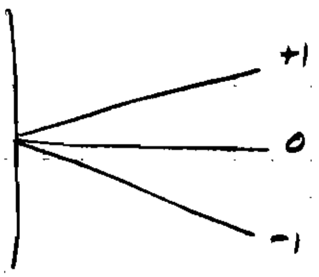
ferromagnetic

polar. ($\hat{y}\hat{z}$ plane)

4. External fields - experimental tools.

a. Magnetic fields

1. Linear Zeeman shift



This is "typically" a HUGE energy!

Eg $B \sim 50$ mG or more

$$\mu B = \frac{700 \text{ kHz}}{\text{G}} \times 50 \text{ mG}$$

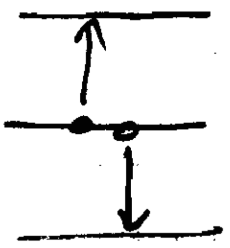
$$= 35{,}000 \text{ Hz}$$

$$\rightarrow 1.75 \,\mu\text{K}.$$

compare to $T_c \sim 100$ nK

$\mu_s \sim 1$ nK

<u>BUT</u> for alkali atoms, at least, $m_F \simeq$ conserved in collisions



no change in Zeeman energy.

NO "mag. reservoir"

Linear Zeeman has no effect!

In contrast, for Cr, etc. it does

Laburthe-Tolra: gas totally polarized for large B

condensate " " for B 7 mG

To accommodate spin conservation, try Lagrange multiplier

say field is uniform:

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_0 \underbrace{- \mu_z B_z}_{\underbrace{\left(+ g_F \mu_B B_z - \lambda \right)}_{p} F_z} - \lambda F_z$$

By symmetry, for $\langle F_z \rangle$ set $p=0$ → Field gone completely!

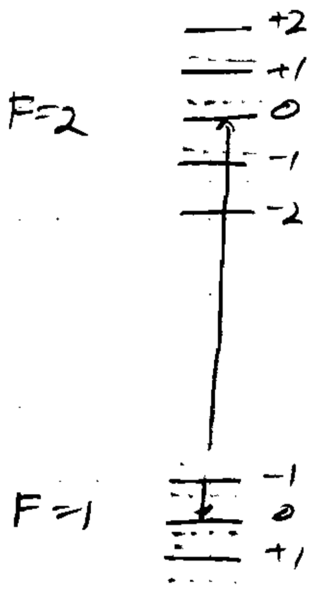
non uniform? If constant orientation:

$p = p(z)$

↑ inhomogeneity of Zeeman shift is still important.

if non-constant orientation → potentially complicated

2. Quadratic Zeeman shift - next order term



$$\langle F=2, m_F \,|\, \underset{\uparrow}{-\vec{\mu}_z} B_z \,|\, F=1, m_F \rangle \neq 0$$

$$\sim -g_s \mu_B S_z$$

Solving simple Q.M. problem, find net

$$H_q \simeq \mp \frac{(g_s \mu_B)^2}{\underset{\uparrow}{(\Delta W)}\,(2I+1)^2} B_z^2 F_z^2 = q F_z^2$$

h.f. splitting

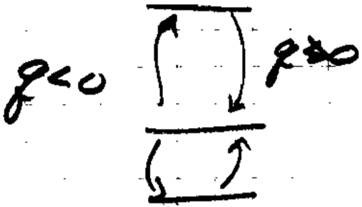
$$\text{Rb} : \quad q = \mp h \times 70\,\mathrm{Hz} \left(\frac{B}{G}\right)^2$$

$$\text{Na} : \quad \times \left(\frac{6.8}{1.7}\right) = \mp h \times 290\,\mathrm{Hz} \left(\frac{B}{G}\right)^2$$

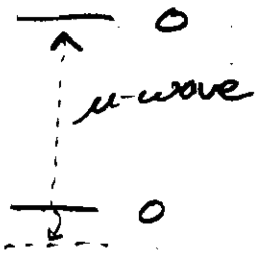
Is relevant.

$q > 0$: favors $m = 0$

$q < 0$: favors $m = \pm 1$.



Can also be obtained by "μ-wave dressing"



AC Zeeman (2nd order)

shifts $m_F = 0$ state up or down

5. Ground states

a. Mean-field + single mode approx.

Ansatz
$$|\vec{\Psi}_{g.s.}\rangle = \Big(\underbrace{|\vec{\psi}(\vec{r})\rangle}_{\text{single particle spinor}}\Big)^N \qquad \text{mean-field}$$

and
$$|\vec{\psi}(\vec{r})\rangle = \underset{\substack{\uparrow \\ \text{all components have same} \\ \text{spatial wfn}}}{\phi(\vec{r})} \times |\vec{\psi}\rangle$$

valid in tight traps.

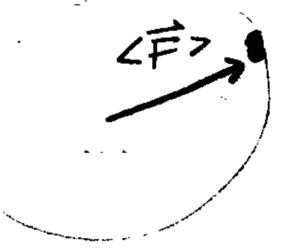
also sheds light on preferred state in extended systems

$$\hat{H} = \hat{V}_s + p\hat{F}_z + q\hat{F}_z^2 \quad \underbrace{\left(+ \hat{H}_{\text{spin-indep}} \right)}_{}\ \text{ignore}$$

spin 1: mean-field energy functional

$$E^{(1)} = \underbrace{\frac{c_1^{(1)} \overset{\uparrow\ \text{density-averaged density}}{n}}{2} \langle \vec{F} \rangle^2}_{\text{interactions}} + p\langle F_z \rangle + q\langle F_z^2 \rangle$$

$c_1^{(1)} < 0$ [^{87}Rb] "ferromagnetic" favors states that maximize $\langle \vec{F} \rangle$



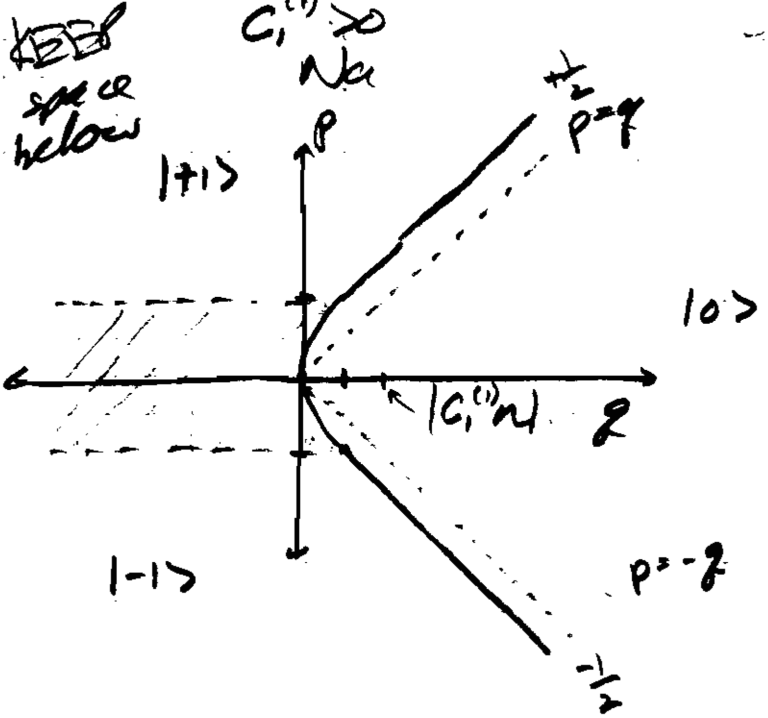
$\hat{R}\,|m_F = +1\rangle$

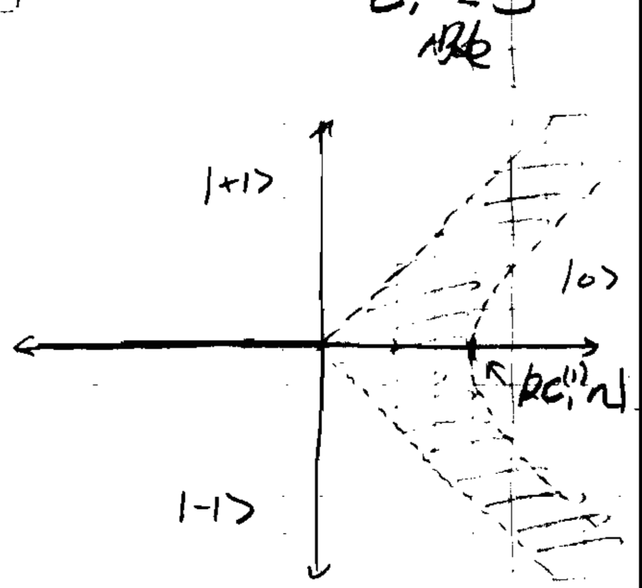
$c_1^{(1)} > 0$ [^{23}Na] "antiferromagnetic"
favors minimizing $\langle \vec{F} \rangle$

polar states
$\hat{R}|m_F=0\rangle$

phase diagrams Stenger et al, Nature 396, 345 (199

KEEP space below





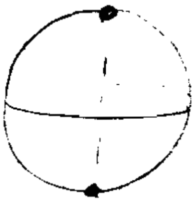
- hard vs. soft boundaries
- explain regions

Experimental evidence: Stenger et al. result

- SG imaging
- size of $m=0$
- immiscibility

$\boxed{\text{PPT}}$

Also examine along $p=0$.
Consider several extremal energy states



$$\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \langle \vec{F} \rangle = 0 \quad \langle F_z^2 \rangle = 0 \quad E^{(1)} = 0$$

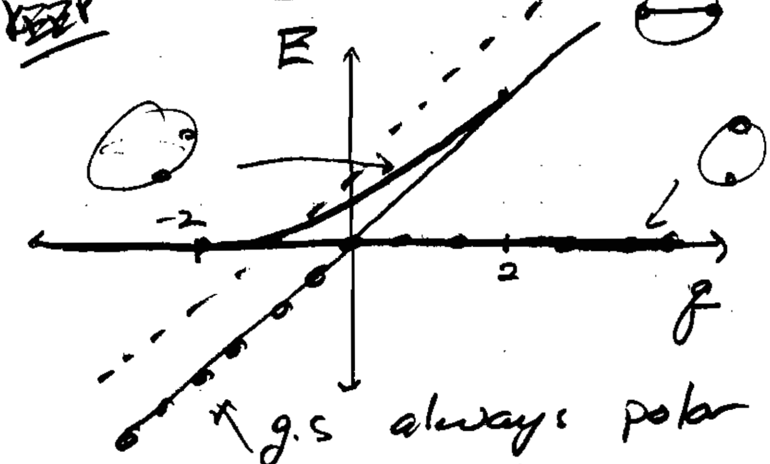
$$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} \quad \langle \vec{F} \rangle = 0 \quad \langle F_z^2 \rangle = 1 \quad E^{(1)} = q$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \langle \vec{F} \rangle = 1 \quad \langle F_z^2 \rangle = 1 \quad E^{(1)} = c_1^{(1)} n + q$$

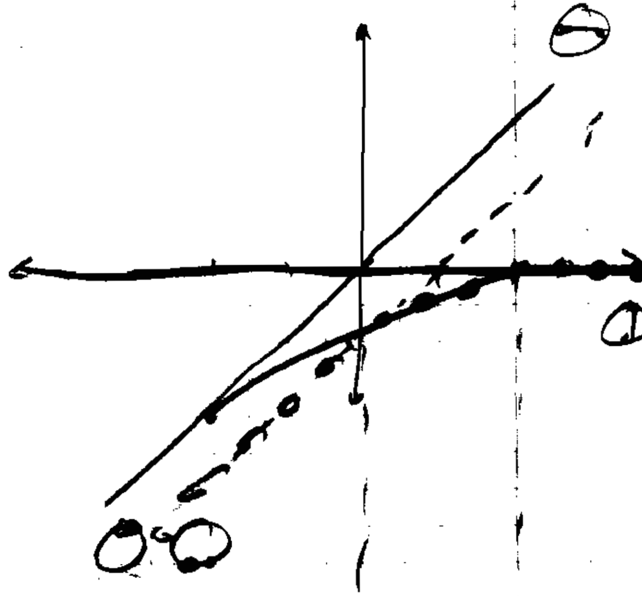
+ another state emerges for $-2 < \frac{q}{|c_1^{(1)} n|} < 2$:

partial transverse magnetization

KEEP



g.s always polar but switches alignment



- Raman Na QPT PPT

Rb evidence

- Chapman data PPT

  F=1 populations correct

- Cohoences

  How to determine coherences, so as to detect e.g. transverse mag. vs. polar state?

  1. Apply pulses to convert nematicity into spin

     → Pulse on a quadratic Zeeman shift

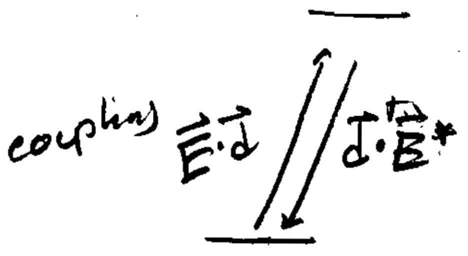
$$\begin{pmatrix}\frac{1}{2}\\ i/\sqrt{2}\\ 1/2\end{pmatrix} \underset{g}{\text{pulse}} \rightarrow \begin{pmatrix}1/2\\ 1/\sqrt{2}\\ 1/2\end{pmatrix} \xrightarrow[\text{with rf}]{\text{rotate spin vector}} \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \text{image}$$

Chapman:

Nature Physics 1, 111 (2005) - Launch coherent spin dynamics, confirms starting point

Nature Physics 8, 305 (2012) - Detect spin squeezing by rotating phase space before imaging.

2. Detect coherence optically



optical susceptibility

$$\chi = \mathrm{Tr}\left(\hat{\rho}\sum_{eg}\frac{1}{\hbar\Delta_{eg}}\vec{d}\,|e\rangle\langle e|\vec{d}\,|g\rangle\langle g|\right)$$

or $\vec{E}\cdot\langle\vec{d}\rangle\langle\vec{d}^{\dagger}\rangle\cdot\vec{E}$

so

So $(b_0^\dagger)^N|0\rangle$ cannot be correct ground state

$\vdots$

$$\frac{4\pi\hbar^2 a_s n}{m} = \mu_{int} \quad \updownarrow \quad \begin{array}{l} |N-2, 1_q, 1_{-q}, \ldots \rangle \\ \\ |N, 0_q, 0_{-q}, \ldots \rangle \end{array} \quad \updownarrow \text{K.E.}$$

K.E limits how much these $\neq 0$ momentum states can mix in

$$\frac{\sum_{q\neq 0}(\text{pop})}{\text{total}} = \text{"quantum depletion"} \propto \sqrt{na^3} \ll 1.$$

But spin mixing collisions need not increase energy ... Do we get massive quantum depletion??

Law, Pu, Bigelow, PRL 81, 5257 (1998).

For F=1. $\mathcal{H}$ is rot. + exchange symm so contains only:

$$\hat{N} \quad , \quad \hat{\vec{F}}_{tot}\cdot\vec{F}_{tot} \qquad \vec{F}_{tot} = \sum_i \hat{\vec{F}}_i$$

$$\hat{\mathcal{H}} = \frac{c_1^{(1)} n}{2}\left(\frac{\hat{\vec{F}}_{tot}\cdot\hat{\vec{F}}_{tot}}{N} - 2\right)$$

ferro: $c_1^{(1)} < 0$, max $F_{tot} = N \Rightarrow |F_{tot} = N, m_{F_{tot}}\rangle$

All fully ferromagnetic states and their macroscopic superpositions.

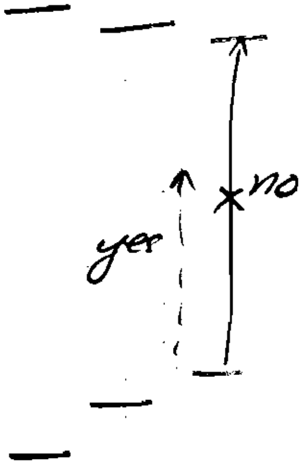
antiferro: $c_1^{(1)} > 0$, min $F_{tot} = 0 \Rightarrow |\Psi\rangle = |F_{tot} = 0, M = 0\rangle$

unique singlet state solution!

preserves rotational symmetry.

Consider typical case that $|e\rangle, |g\rangle$ come in families of spin manifolds, for which the energies are $\sim$ identical (w.r.t. large detuning

$$\Rightarrow \chi = \mathrm{Tr}\Big(\hat\rho \sum_{\substack{\text{sets of}\\ e,g}} \underbrace{\overset{\overset{\text{scalars}}{\uparrow}}{\frac{1}{\hbar\Delta_{eg}}}\ \vec d\,(\Sigma |e\rangle\langle e|)\,\vec d\,(\Sigma|g\rangle\langle g|}_{\text{dyad} \,\cdots\, \text{composed of ranks } 0,1,2}$$



$$\chi_{ij} = \chi^{(0)}\delta_{ij}\, n(\vec r)$$
$$+\chi^{(1)}\, i\varepsilon_{ijk} M_k(\vec r)$$
$$+\chi^{(2)}\, N_{ij}(\vec r)$$

$\chi^{(0)}, \chi^{(1)}, \chi^{(2)}$ give strength of response to density, magnetization, nematicity

For F=1 spinors: χ carries all information

PPT

F=2, F=3 . .

b. Exact Many-body ground states:

Question: should we expect mean-field state, Hartree wavefunction, to be true many-body ground state?

Recall scalar BEC Bogoliubov (1947)

∃ collision terms in $\mathcal{H}$: $\frac{4\pi\hbar^2 a}{m}\cdot\frac{1}{V}\left[b^\dagger_{q} b^\dagger_{-q} b_0 b_0 + h.c.\right].$

can be expressed as follows.

Let $\hat{A} = \dfrac{(\hat{\psi}_0^2 - 2\hat{\psi}_1\hat{\psi}_{-1})}{\sqrt{3}}$ F=1 singlet amplitude

$$|\Psi\rangle = (\quad)\underbrace{\hat{A}^{\dagger(N/2)}|0\rangle}_{\text{mean-field BEC of singlets.}}$$

observables: get populations $N_1 = N_{-1}$ exactly.

and $\langle N_1\rangle = \langle N_{-1}\rangle = \langle N_0\rangle$ on every realization

compare to

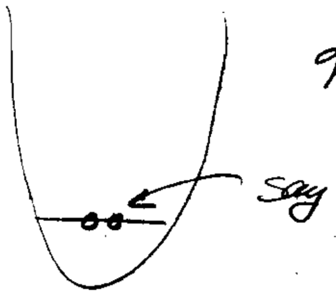
$$\underset{\uparrow}{\hat{R}_{random}}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\langle N_1\rangle = \langle N_{-1}\rangle$ but not exactly and get large fluctuations in populations.

6. Spin dynamics.

a. Microscopic

Consider just 2 F=1 atoms in ground state of trap



$$\mathcal{H} = \text{const} + \frac{4\pi\hbar^2}{m}\langle n\rangle\left[a_0\vec{P}_0 + a_2\vec{P}_2\right]$$

say we start in

$$|\Psi(0)\rangle = |0,2,0\rangle \qquad m_F = 0.$$

Superposition of

$$|F_{tot}=2, m=0\rangle = \sqrt{\tfrac{2}{3}}\,|0,2,0\rangle + \sqrt{\tfrac{1}{3}}\,|1,0,1\rangle$$

$$|F_{tot}=0, m=0\rangle = -\sqrt{\tfrac{1}{3}}\,|0,2,0\rangle + \sqrt{\tfrac{2}{3}}\,|1,0,1\rangle.$$

So we expect temporal oscillations of the spin composition

$$|\Psi(t)\rangle = \sqrt{\tfrac{2}{3}}\,e^{-i\omega_2 t}\,|F=2\rangle - \sqrt{\tfrac{1}{3}}\,e^{-i\omega_0 t}\,|F=0\rangle$$

$$= \left(\tfrac{2}{3}e^{-i\omega_2 t} + \tfrac{1}{3}e^{-i\omega_0 t}\right)|0,2,0\rangle$$

$$+ \left(\tfrac{\sqrt{2}}{3}e^{-i\omega_2 t} - \tfrac{\sqrt{2}}{3}e^{-i\omega_0 t}\right)|1,0,1\rangle.$$

"spin mixing oscillations" PPT.

b. SMA, mean-field dynamics

Approach: Write out $\mathcal{H} = \ldots$

from $i\hbar\frac{\partial}{\partial t}\hat{\psi}_i = \frac{\delta\mathcal{H}}{\delta\psi_i^\dagger}$ get eqs of motion

Lots of papers...

Simplified by Zhang et al PRA 72, 013602 (2005)

$$E^{(1)} = \frac{c_1^{(1)} n}{2} \langle F\rangle^2 + p\langle F_z\rangle + q\langle F_z^2\rangle$$

let $\psi_j = \sqrt{\rho_j}\, e^{-i\theta_j}$

pops. $\begin{cases} \rho_0 + \rho_1 + \rho_{-1} = 1 \\ \rho_1 - \rho_{-1} = m \end{cases}$

phases: note $\bar{\theta} = \theta_1 + \theta_0 + \theta_{-1}$ is irrelevant

$\theta_{1,-1} = \theta_1 - \theta_{-1}$ is also (just rotate about z)

so only: $\theta = \theta_+ + \theta_- - 2\theta_0$ is important

controls mag. vs. nematicity,

so enters in $\langle F_x\rangle^2 + \langle F_y\rangle^2$ term

$$\Rightarrow E_1^{(1)} = c_1^{(1)} n \left[\rho_0(1-\rho_0) + \rho_0\sqrt{(1-\rho_0)^2 - m^2}\cos\theta \right] + pm + q(1-\rho_0)$$

More importantly find

$$\dot{\rho}_0 = -\frac{2}{\hbar}\frac{\partial E}{\partial \theta}, \quad \dot{\theta} = \frac{2}{\hbar}\frac{\partial E}{\partial \rho_0}$$

PPT

Features:

- High q: single particle physics

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} e^{iqt/\hbar} \\ 1/2 \end{pmatrix} \qquad \text{ferro} \longrightarrow \text{polar} \longrightarrow \text{ferro...}$$

as $\dot{\theta} = \frac{2q}{\hbar}$

- zero q: Consider initial state: $\rho_0 = \frac{1}{2} + \delta\rho$ $(m=0)$

$$\theta = 0 + \delta\theta$$

$$\frac{\partial\theta}{\partial t} = \frac{2}{\hbar}\frac{\partial E}{\partial\rho_0} = \frac{2}{\hbar} \times c_1^{(1)} n\,(1-2\rho_0)(1+\cos\theta) \simeq -\frac{8}{\hbar} c_1^{(1)} n\, \delta\rho$$

$$\frac{\partial^2\theta}{\partial t^2} = -8\frac{c_1^{(1)} n}{\hbar} \times -\frac{2}{\hbar} \times c_1^{(1)} n\, \rho_0(1-\rho_0) \times -\sin\theta$$

$$= -\left(\frac{2c_1^{(1)} n}{\hbar}\right)^2 \theta \quad \nwarrow \text{ Harmonic at } \frac{2c_1^{(1)} n}{\hbar} \text{ freq}$$

- separatrix: b/w pendulum + running solutions
  ↑ slow motion "spin mixing resonance"

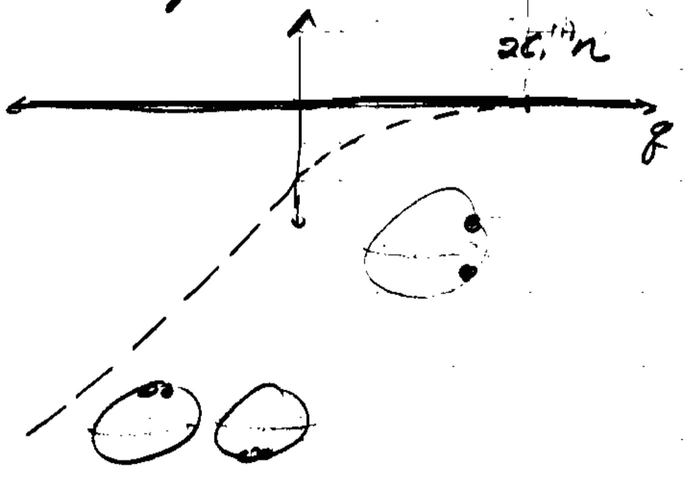
<u>PPT</u> Chapman Nature Physics 2005

spin mixing resonance + separatrix also examined by Liu et al., PRL <u>102</u>, 125301 (2009)

C. Spin mixing instability

Specific example: $m_z = 0$ state

ferro: high E extremum → min E



mean field dynamics
→ stationary

Consider perturbation $\psi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \psi_x \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} + \psi_y \begin{pmatrix} i/\sqrt{2} \\ 0 \\ i/\sqrt{2} \end{pmatrix}$

or $$\hat{\Psi} = \underset{\substack{\uparrow \\ \text{mean-field}}}{\psi_z} + \underbrace{\hat{\psi}_x + \hat{\psi}_y}_{\text{fluctuations.}}$$

Bogoliubov's linear stability analysis

$$\mathcal{H} = \sum_{\beta = x,y} \Big(\underset{\substack{\uparrow \\ \text{difference in} \\ \text{single particle} \\ \text{energy of} \\ \text{mode}}}{\varepsilon + q} + c_1^{(1)} n \Big) \hat{\psi}_\beta^\dagger \hat{\psi}_\beta - \frac{c_1^{(1)} n}{2} \left(\psi_\beta^{\dagger 2} + \psi_\beta^2 \right)$$

Identify $\hat{Z}_\beta = \dfrac{\hat{\psi}_\beta^\dagger + \hat{\psi}_\beta}{2}$, $\hat{P}_\beta = \dfrac{i(\psi_\beta^\dagger - \psi_\beta)}{2}$

$$\mathcal{H} = \sum_\beta (\varepsilon + q) Z^2 + (\varepsilon + q + 2c_1^{(1)} n) P^2$$

	ferro	polar
Stable (H.O.-like)	$\varepsilon + q > \lvert 2c_1^{(1)} n \rvert = q_0$	$\varepsilon + q > 0$
Stable (rotates in opposite sense)	$\varepsilon + q < 0$	$\varepsilon + q < -2c_1^{(1)} n$
Unstable	— middle range —	

PPT → Klempt

~~Outside SMA: $\varepsilon = \varepsilon_k = \frac{\hbar^2 k^2}{2m}$~~

Obtain spectrum of spin excitations (2 pole.)

recall, for HO.

$$\mathcal{H} = \frac{1}{2} m\omega^2 \sqrt{\frac{\hbar}{2m\omega}}^{\,2} \overbrace{\frac{(a^\dagger + a)^2}{4} \times 4}^{\hat{z}^2} + \frac{1}{2m} \sqrt{\frac{\hbar m\omega}{2}}^{\,2} \overbrace{\left(\frac{i(a^\dagger - a)}{4} \right)^2}^{\hat{p}^2}$$

mult each $\hbar^2 \omega^2$

Similarly $E^2 = (\varepsilon + q)(\varepsilon + q + 2c_1^{(1)} n)$

in unstable regime, $E_k^2 < 0$

get solutions $\sim e^{-|\omega|t}$, $e^{+|\omega|t}$

squeezing amplification

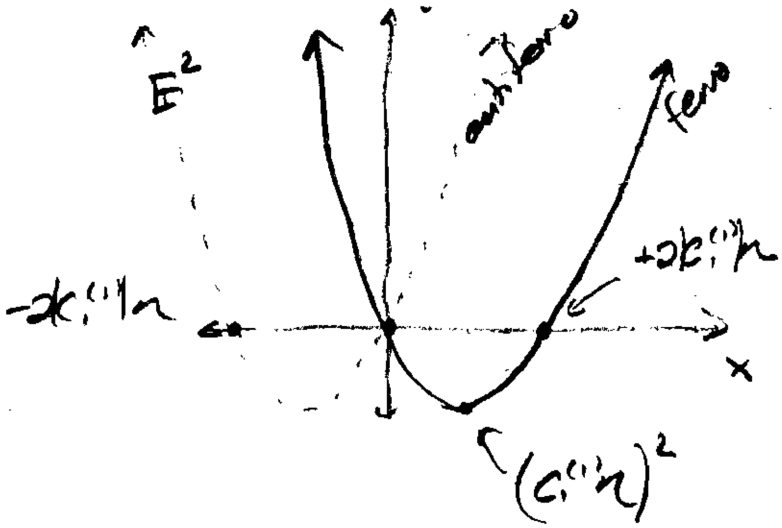
For spatially extended sample. ∃ many spatial modes

$$\varepsilon = \varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$E^2(k)$ is new spectrum of spin excitations

PPT

Let $x = \varepsilon + q$



stable $E^2 > 0$

unstable $E^2 < 0$

7 Topological defects.

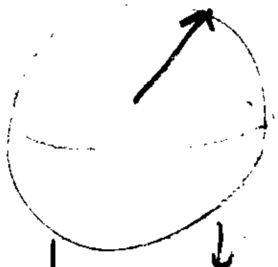
a. Observed so far

1. polar core spin vortex

Explain spin currents picture

2. 2D skyrmion

Consider order parameter defined on sphere

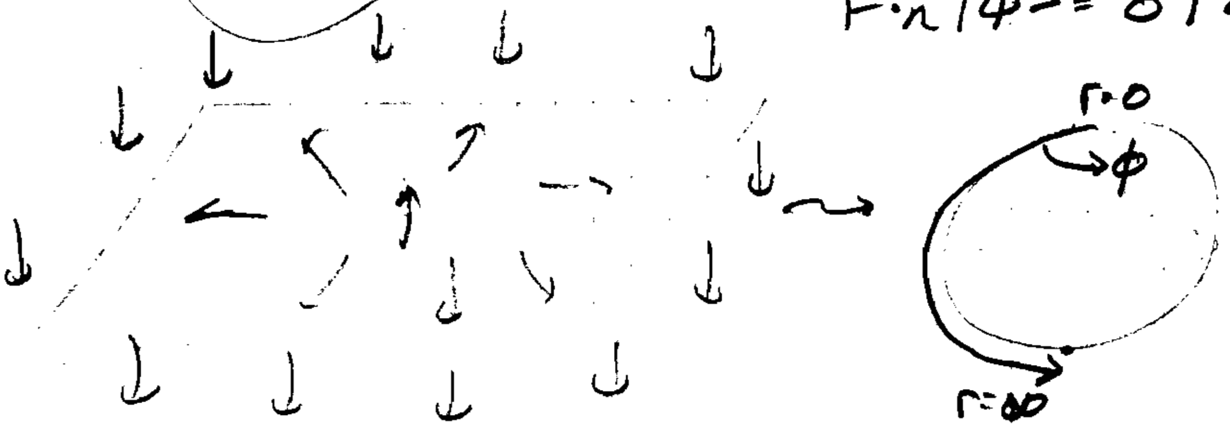


e.g. ferromagnet's magnetization

$\hat{F}\cdot\vec{n}\,|\psi\rangle = \hbar F|\psi\rangle$

polar state director

$\hat{F}\cdot\vec{n}\,|\psi\rangle = 0\,|\psi\rangle$



For F=1

$$|\psi(r,\phi)\rangle = |\psi(\underset{\substack{\uparrow \\ \text{goes } 0\to\pi \\ \text{for } r=0\to\infty}}{\theta},\phi)\rangle = \hat{R}(\theta,\phi)\,|\psi(0,0)\rangle$$

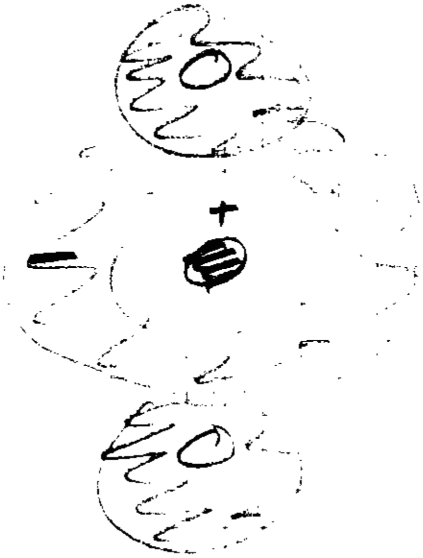
$$\hat{R}(\theta,\phi) = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{-\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{-\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix}$$

Ferro : m = 1

0 $e^{-i\phi}$

-1 $e^{-2i\phi}$

polar



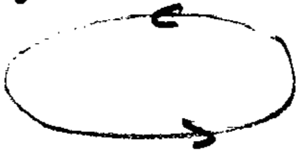
$e^{i\phi}$

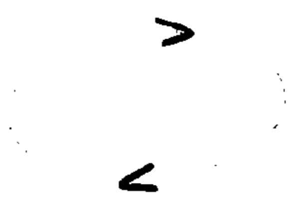
0

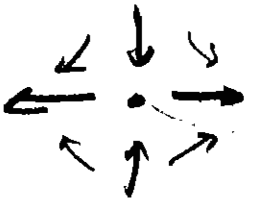
$e^{-i\phi}$.

Make using mag field

Squad.





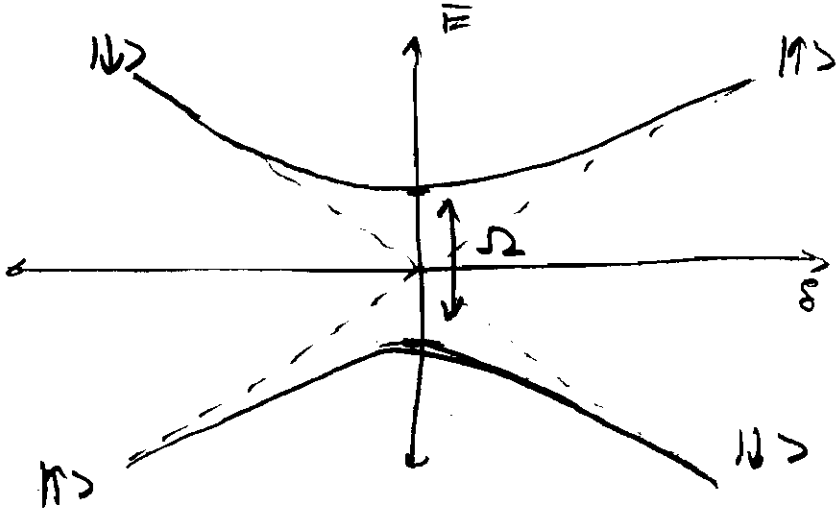


$$B = -2B'z\hat{z} + B'\rho\hat{\rho}$$

Plunge zero through gas.

For atom fixed spatially, this is a Landau-Zener transition.

Simpler picture of 2 levels $\mathcal{H} = \frac{1}{2}\begin{pmatrix}\delta & \Omega \\ \Omega & -\delta\end{pmatrix}$



units s^{-2}

Now $\delta = \dot{\delta} \times t$

if $\left(\text{time crossing resonance}\right) \gg \Omega^{-1}$ $(\Gamma \to \infty)$ adiabatic $|\downarrow\rangle \to |\uparrow\rangle$

$\ll$ $(\Gamma \to 0)$ diabatic $|\downarrow\rangle \to |\downarrow\rangle$

2 experiments:

Choi et al. PRL 108, 035301 (2012)

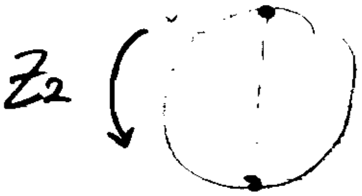
Berkeley, in progress

b. Fractional vortices.

Consider spinor states w/discrete geometric symmetry

May be accompanied by discrete phase increment

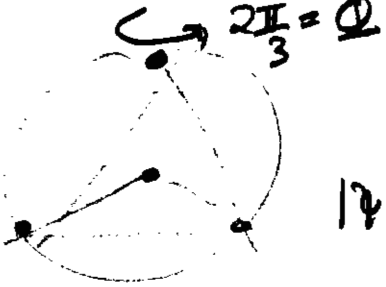
Z_2



$\rightsquigarrow \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\-1\\0\end{pmatrix}$

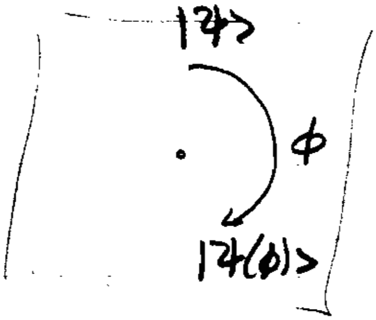
Ⓗ

or



$$|\psi\rangle \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{\sqrt{3}} e^{i4\pi/3} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}}\, e^{-2\pi/3} \\ 0 \end{pmatrix} = e^{-\frac{2\pi i}{3}} |\psi\rangle$$

implies fractional vortices



$$|\psi(\phi)\rangle = \overbrace{e^{-i\Theta\frac{\phi}{2\pi}}}^{\text{fractional circulation}} R\left(\Phi \cdot \frac{\phi}{2\pi}\right)$$

so that $|\psi(0)\rangle = |\psi(2\pi)\rangle$.

F=1 polar : $\frac{1}{2}$ vortices

F=2 tetrahedral : $\frac{1}{3}$ vortices

biaxial nematic : $\frac{1}{2}$ vortices.

Implications for 2D physics