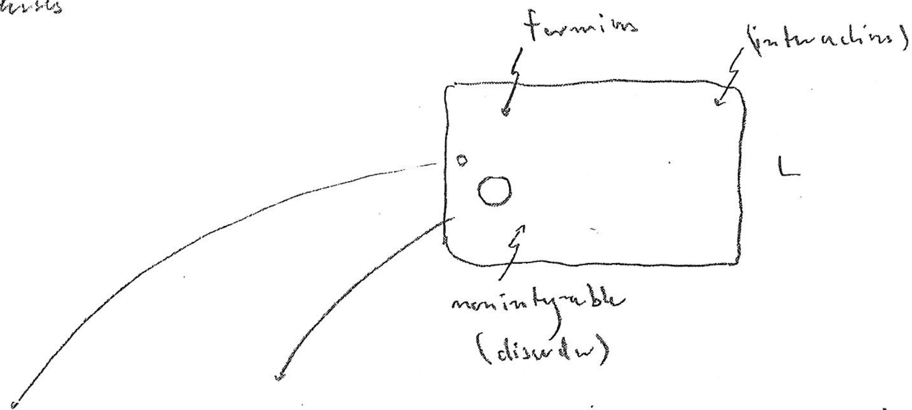
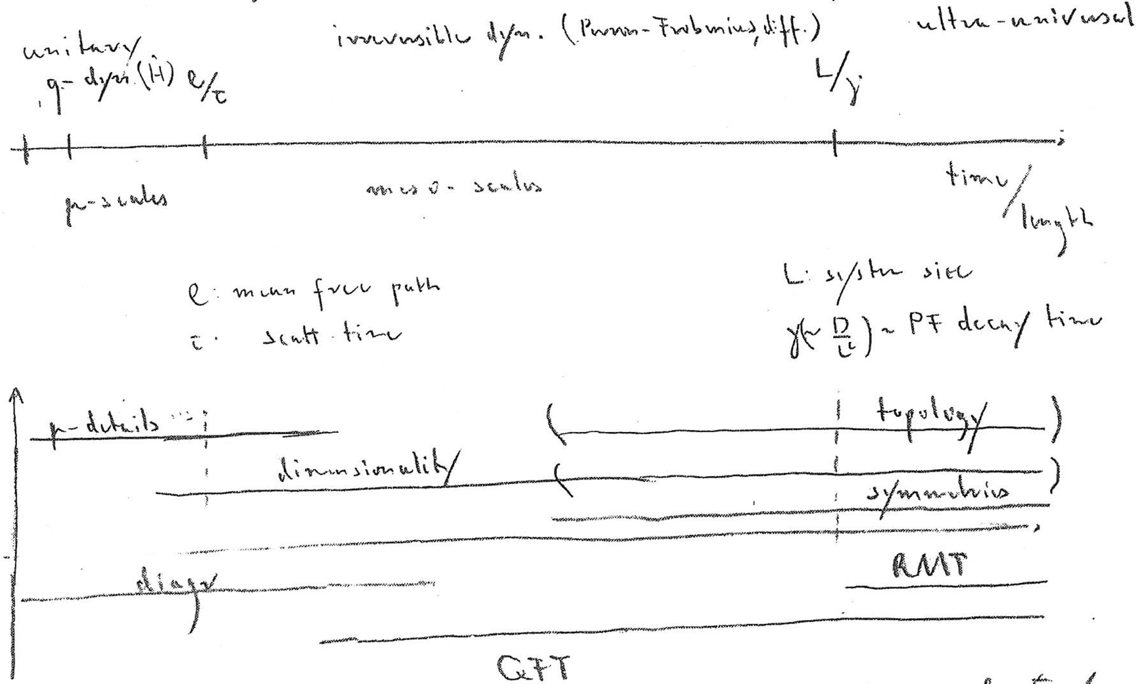
· Symmetry classes





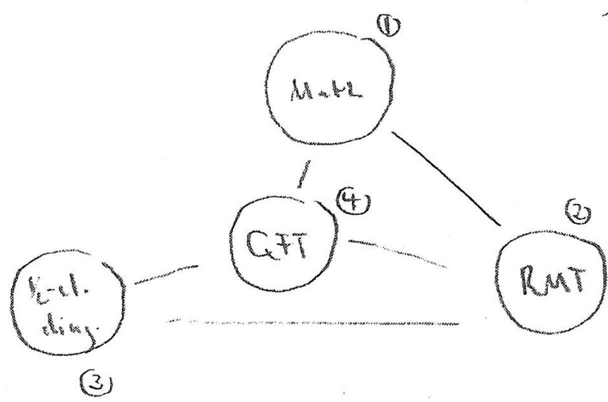
Goals: understanding sym.

· sym. → phenomenology

· sym. → field theories

· sym. → topology

Understanding sym.:



· Symmetries in QM (cf Zirnbauer/0404058)

· Given · Hilbert space $\mathcal{H}$

Hamiltonian $\hat{H}$

· Symmetry group: unitary or anti-unitary transformations $g: \mathcal{H} \to \mathcal{H}$

satisfying $g \hat{H} g^{-1} = \hat{H}$, norm preserving.

· unitary g: $\langle g\psi, g\psi' \rangle = \langle \psi, \psi' \rangle$

· transformations continuously connected to $\mathbb{1}$ (exmpl.: rotations)

· " not " " " (exmpl: reflection)

· antiunitary: $\tilde{g}$: $\langle \tilde{g}\psi, \tilde{g}\psi' \rangle = \langle \psi', \psi \rangle = \overline{\langle \psi, \psi' \rangle}$

$\tilde{g}(z\psi) = \bar{z}\, \tilde{g}\psi \qquad z \in \mathbb{C}$

· exmpl.: time reversal $\tilde{g} = T$ $\left(\tilde{g}\, i\partial_t = i\partial_{(-t)} = -i\partial_t \right)$

☺ general structure of sym. group G

· Def.: G_0 = subgroup of unitary trafos in G

G_1 : subset of anti-unitary " " G

$\tilde{g}, \tilde{h} \in G_1 \Rightarrow \tilde{g}\tilde{h} \in G_0 \rightsquigarrow$ pick one element $T \in G_1 \rightsquigarrow G_1 = T G_0$. $G = G/_{G_0} \cdot G_0$

assume T is an inversion (such as time reversal, charge conjugation)

$T^2 = z\, \mathbb{1}$ $\qquad T^3 = T^2 T = T T^2 = zT = Tz = \bar{z}T \Rightarrow z = \bar{z} \Rightarrow z = \pm 1$

$z = e^{i\phi}$

· Guiding philosophy: · assume G given (and compact)

· classify Hamiltonians commutative with G

· focus on $U = \exp(-iH)$. Manifold of symmetry compatible

time evolutions: M

· Warmup: $G_1 = \{\}$ ~ $G = G_0$. G_0 irrelevant for classification program.

Idea of argument: assume $\mathcal{H} \ni \psi = \{\psi_{a,n}\}$ where $G_0 \leftrightarrow \{a a'\}$

acts on a-index. (example: wave functions $\psi_{\underbrace{l,m}_{a}}(\underset{\underset{n}{\uparrow}}{r})$ of central problem)

$g \hat H g^{-1} = \hat H \Rightarrow \hat H = \{\hat H_{n,n'}\}$ $\qquad \mathcal{H} = \underset{\substack{\uparrow \\ \text{representation space of } G_0}}{\mathcal{H}_s} \otimes h$

$\hat H = \mathbb{1} \otimes \underset{\substack{\uparrow \\ \text{no symmetries.}}}{\hat h}$

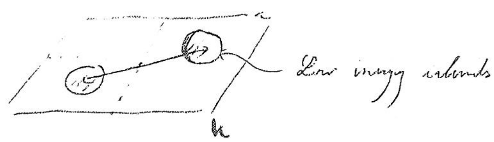
With $\dim h = n$: $U(n)$ = set of symmetry compatible time evolutions.

· More generally: $\mathcal{H} = \underbrace{\mathcal{H}_s^1 \otimes h^1}_{\mathcal{H}^1} \oplus \underbrace{\mathcal{H}_s^2 \otimes h^2}_{\mathcal{H}^2} \oplus \cdots$

$\hat H = \hat h^1 \oplus \hat h^2 \oplus$

$M = U(n_1) \times U(n_2) \times \cdots \qquad \begin{pmatrix} U(n_1) & & \\ & U(n_2) & \\ & & \ddots \end{pmatrix}$

Example.



$M = U(n) \times U(n) = U_1 \times U_2$

· Time reversal: $G_1 = T G_0 + \{0\}$

· $\hat H \xrightarrow{\tau} T \hat H T^{-1} = \hat H$ 'adjoint rep. of T'

· $\hat U \xrightarrow{\tau} T \hat U T^{-1} = T \exp(-i \hat H t) T = \exp(i \hat H t) = \hat U^{-1}$

· $\tau^2 = \mathbb{1}$ involution

· $T \mathcal{H}^i = \mathcal{H}^{\bar i}$ *) Important question: $i = \bar i$?

· $i \neq \bar i$: e.g. $i = 1, \bar i = 2$. $\quad M = U_1 \times U_2 \ni \hat U \begin{pmatrix} \hat U_1 & \\ & \hat U_2 \end{pmatrix} \Big| \begin{matrix} \mathcal{H}^1 \\ \mathcal{H}^2 \end{matrix}$

$\hat U_1 T \mathcal{H}^i \overset{?}{\in} T \mathcal{H}^i \Leftrightarrow \underbrace{T \hat U T}_{\hat U'} \mathcal{H}^i \in T^2 \mathcal{H}^i = \mathcal{H}^i$ ✓

$$\hat{\mathcal{U}} = \tau \begin{pmatrix} \hat{U}_1 & \\ & \hat{U}_2 \end{pmatrix} = \begin{pmatrix} & \tau(\hat{U}_2) \\ \tau(\hat{U}_1) & \end{pmatrix} \doteq \begin{pmatrix} \hat{U}_1^{-1} & \\ & \hat{U}_2^{-1} \end{pmatrix}$$

$$\rightsquigarrow \hat{U} = \begin{pmatrix} \hat{U}_1 & \\ & \tau(\hat{U}_1)^{-1} \end{pmatrix}$$

Conclusion: $M = U_1 \times \tau(U_1)^{-1}$
$\}$! unitary time evolution.

. i = i . Setting: . $\hat{H} : \mathcal{H} \to \mathcal{H}$ $\quad h \leftrightarrow \mathcal{H}$, $\hat{h} \leftrightarrow \hat{H}$

. unitary symmetries ✓

. anti-unitary geometry τ : $\psi \xrightarrow{\tau} T\psi$
$\hat{X} \xrightarrow{\tau} T\hat{X}T^{-1}$

$$. \; T = \underset{\substack{\uparrow \\ \text{complex conj.}}}{\overset{\substack{\text{unit.} \\ \downarrow}}{U K}}$$

. $T^2 = \begin{cases} +1 \\ -1 \end{cases}$

both situations realised in nature

. spinless states) $\exists$ Basis $\psi \to K\psi$ $\quad \zeta^2 - T^2 = 1$

. spin-ful states . $T \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} \eta_\uparrow \bar{\psi}_\downarrow \\ \eta_\downarrow \bar{\psi}_\uparrow \end{pmatrix} \cdot \underbrace{\begin{pmatrix} & \eta_\uparrow \\ \eta_\downarrow & \end{pmatrix}}_{U} \begin{pmatrix} \bar{\psi}_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix}$

phases

$= UK \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$

. $T^2 = -1 \rightsquigarrow U = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i\sigma_2$

. Classify good time evolutions:

$\tau(\mathcal{U}) = \mathcal{U}^{-1}$ $\qquad \mathcal{U} = \exp(i\hat H)$ $\qquad \tau(\hat H) = \hat H$

$T^2 = \mathbb{1} \rightsquigarrow T = K \rightsquigarrow \quad \hat H^* = \hat H$

$\hat X \equiv i\hat H \qquad \hat X^* = -\hat X \Rightarrow \hat X \in u(\overset{\dim X}{n})/o(n)$

$$\frac{\hat X^\dagger = -\hat X}{\hat X^T = \hat X} \qquad \hat{\mathcal U} \in U(n)/O(n)$$

$T^2 = -\mathbb{1} \rightsquigarrow T = i\sigma_2 K \rightsquigarrow$

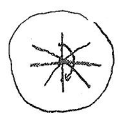
$(i\sigma_2)\hat X^*(i\sigma_2)^{-1} = -\hat X \Rightarrow \hat X \in u(n)/sp(n)$

$$\frac{\hat X^\dagger = -\hat X}{(i\sigma_2)\hat X^T(i\sigma_2) = +\hat X} \qquad \hat{\mathcal U} \in U(n)/Sp(n)$$

. Summary:

phys	time reversal	T^2	τ	$\mathcal U$	invariance group G	sym. space
unitary	$\emptyset$	–	0	$U(n)$	$U(n)$	A
orth.	+	$\mathbb 1$	1	$U(n)/O(n)$	$O(n)$	AI
sympl	+	$-\mathbb 1$	-1	$U(n)/Sp(n)$	$Sp(n)$	AII

. Symmetric spaces



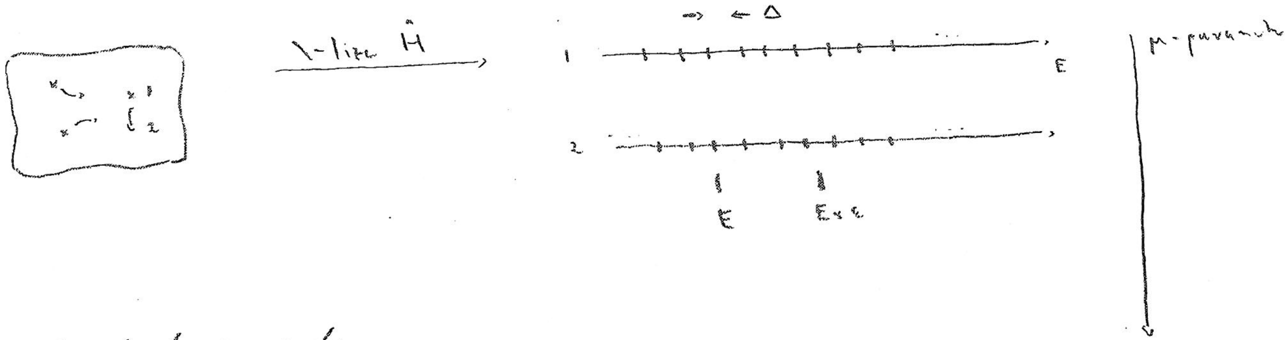
. algebraic: G a connected Lie group. Symmetric space: homogeneous space G/H, where $H \subset G$ is invariant under involution $\tau: G \to G$

. Riemannian geometry: A Riemannian mf. whose geometry is uniform; curvature R covariantly constant, $R > 0$ compact, $R < 0$ non compact.

$\exists$ 10 families of sym spaces, A, AI, AII, C, CI, AIII, ...

classical groups: $G = U, O, Sp$ or coset spaces thereof G/H

· symmetries and ergodic regime



statistical approach:

· $\langle \rho(E)\rangle_\mu = -\frac{1}{\pi}\,\mathrm{Im}\,\langle \mathrm{tr}\, G^+(E)\rangle_\mu$ $G^\pm(E) = (E^\pm - \hat H)^{-1}$

· $R_2(\varepsilon) = \langle \rho(E)\,\rho(E+\varepsilon)\rangle_\mu$

· striking universality: correlation functions depend on i) dimensionless parameter ε/Δ
ii) symmetry class but iii) not on system type
(same for heavy nuclei, complex molecules)

· analytic approach
construct reference ensemble of $\hat H$'s subject to conditions:
i) fix N iv) no 'information'
ii) fix average Δ
iii) fix symmetry

i) & ii) ~ $\langle \mathrm{tr}(H^2)\rangle \overset{!}{=} \lambda^2$ λ: width of spectrum ~ $\Delta \sim \lambda/N$

and: $P(H)\,dH$ probability measure

Idea: $H_{\mu\nu}$ independently Gaussian distributed (subject to i) – iii))

$$P(H)\,dH = \exp\left(-\frac{\beta N}{\lambda^2}\,\mathrm{tr}\, H^2\right) dH$$

β:	M
1	A
2	AI
4	AII

$P(H)dH$ maximizes information entropy! small

· Understanding physical consequences of symmetries : RMT (universal statistics)

· RMT : · old · powerful · simple · natural

· RMT in a nutshell

· put invariant probability measure on M (G explains philosophy)

$$P(U)\,dU = P(gUg^{-1})\,dU \qquad g \in G, \text{ e.g.}$$

· $P(U)\,dU = \text{const.}\; dU$ 'circular ensemble'

· $P(H)\,dH = \text{const.} \times \exp\left(-\frac{\beta N}{\lambda^2}\operatorname{tr}(H^2)\right) dH$ 'Gaussian ensembles'

β	M
2	A
1	AI
4	AII

$N = \dim M$

rational : choose P by maximum entropy principle subject to constraint $\langle \operatorname{tr} H^2 \rangle = \lambda^2$

· computing observables

· $\rho(E) = -\frac{1}{\pi} \operatorname{Im} \operatorname{tr} G^+(E) \qquad G^+(E) = (E^+ - H)^{-1}$

$$\langle \rho(E) \rangle, \quad R_2(\epsilon) = \langle \rho(E)\,\rho(E+\epsilon) \rangle,$$

$$X$$

$$\langle X \rangle = \int P(H)\, X(H)\, dH \qquad H = gDg^{-1} \quad g \in G$$

$$= \int P(D)\, X(D)\, dD\, J(D)$$

results $\rho(E) = \frac{N}{\pi\lambda}\left(1 - \left(\frac{E}{2\lambda}\right)^2\right)^{1/2} \qquad \Delta = \frac{\pi\lambda}{N}$

$$R_2(s) \overset{\beta=2}{=} \frac{\sin^2 s}{s^2} \qquad s = \frac{\pi\epsilon}{\Delta}$$

$$R_2(s) \overset{\beta=1}{=} \frac{\sin^2 s}{s^2} + \frac{d}{ds}\left(\frac{\sin^2 s}{s^2}\right) \int_s^\infty \frac{\sin^2 s'}{s'^2}\, ds'$$

· Discussion . · universal result $R_2(s)$ $s \sim$ energy $\sim$ (time)$^{-1}$

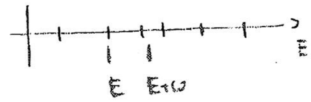
· nonanalytic in s

· short time ($s \gg 1$) perturbation expansion around $s = \infty$ ...

option: e.g.

$$R_2(s) \sim \begin{cases} \frac{1}{s^2} & \beta = 2 \\ \frac{2}{s^2} + \dots & \end{cases}$$

· $s \to 0$ $R_2(s) \to 0$



· RMT perturbation theory ☺ · physics

· link to other approaches

· geometries ↔ symmetry ↔ geometry

· Diagrammatic work (M = A, AI)

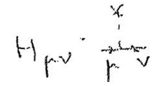
· compute $\mathrm{tr}(G^\pm(E))$

$\mathrm{tr}(G^+(E+\omega))\ \mathrm{tr}(G^-(E))$

· expansion in $H = \{H_{\mu\nu}\}$ around $G^{0\pm} = E^{-1}$

$$\langle H_{\mu\nu} H_{\nu'\mu'} \rangle = \begin{cases} \delta_{\mu\mu'}\delta_{\nu\nu'} \frac{\lambda^2}{N} & \mathrm{A} \\ (\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'}) \frac{\lambda^2}{N} & \mathrm{AI} \end{cases}$$

· $G_0^\pm = \xrightarrow{\mu}$ $H_{\mu\nu}$ =



$\mathrm{tr}\, G^\pm$ = ... $\langle \mathrm{tr}\, G^\pm \rangle$ =

· Expansion parameter s, N^{-1}

$\langle G \rangle = \Rightarrow = \underset{G^0}{\longrightarrow} + \underset{\Sigma}{\longrightarrow}$ Bethe Salpeter (Dyson) eq.

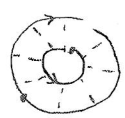
$$\Rightarrow = \frac{1}{(\rightarrow)^{-1} - \Sigma} \qquad \Sigma^{\pm} = \lambda^2 \, \mathrm{tr}\, G^{\pm}$$

$$G^{\pm} = \left(E^{\pm} - \lambda^2 \, \mathrm{tr}\, G^{\pm}\right)^{-1} \qquad \Sigma^{\pm} = \frac{\lambda^2}{E^{\pm} - \Sigma^{\pm}} \Rightarrow \Sigma^{\pm} = \frac{E}{2} \mp i \left(\lambda^2 - \left(\frac{E}{2}\right)^2\right)^{1/2}$$

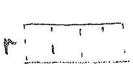
$|E| < 2\lambda$

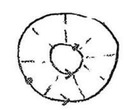
- $\rho(E)$: semicircle law.
- "spontaneous symmetry breaking" infinitesimal, δ, $E \pm i\delta$ triggers sign of $\mathrm{Im}\,\Sigma$.
- no contracted path

- $L\left(G^{+}\left(E + \frac{\omega}{2}\right)\right) \; L\left(G^{-}\left(E - \frac{\omega}{2}\right)\right)$



A, AI

μ  $\nu = i \frac{\lambda^2}{N} \frac{1}{\omega} \equiv P_{\mu\nu}$



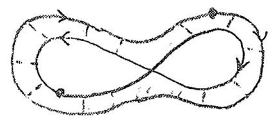
AI

- $R_2(\omega) \sim \frac{1}{\omega^2}$ ✓
- $P_{\mu\nu}$: classical probability to propagate $\mu \to \nu$ in time $\sim \omega^{-1}$

$P_{\mu\nu}(t) \propto \Theta(t)$
↑ Fourier transf.

- and spatial generalization: $P_{\mu\nu} \to P(x, x')$: Green functions.
e.g. $P|_{\text{local}} \to$ diffusion equation.

- higher order corrections

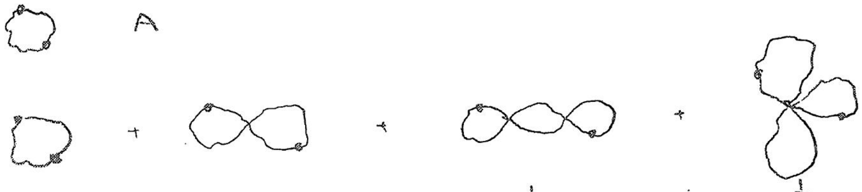

$\sim \frac{C}{N^3}$

AI

- interpretation: traversal of different 'paths' of many identical scattering phases. (decoherence ~ 1.)

- compactified notation: $\boxed{||||}$ = ~~~

$R_2(\omega) \longrightarrow$  A

$R_1(\omega) \longrightarrow$  + ...

$\sim \frac{1}{N^2}$ $\quad \frac{1}{N^3}$ $\quad \frac{1}{N^4}$ $\quad \frac{1}{N^4}$

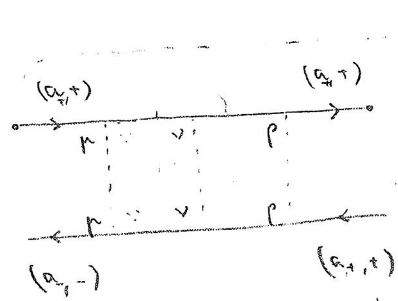
- asymptotic series.

→ geometry? {
- structure of diagrammatic code determined by topology (combinat.)
- ~ quasiparticle of ? mass and subject to nonlinear interaction
- Goldstone mode?

☺ Low energy theory: nonlinear sigma model? cf. describing disordered magnets from magnons search for a symmetry!

- label $\xrightarrow{(a_+,+)}$ $\quad a_+ \in 1,\dots,n_+$ $\quad$ ~ 'flavor space' $\mathbb{C}^{n_+ + n_-} \equiv \mathcal{H}_f$

$\xrightarrow[(a_-,-)]{}$ $\quad a_- \in 1,\dots,n_-$ $\quad$ 'color space' $\mathbb{C}^N \equiv \mathcal{H}_c$



$\psi \in \mathcal{H}_f \otimes \mathcal{H}_c$

time evolution $\mathbb{1} \otimes U$ $\quad$ 'flavor singlet' ~ color eigenspace } cf. 'Howe pairs'

low energy theory: $\tilde{U} \otimes \mathbb{1}$ $\quad$ 'color singlet' ~ ?

Hence pair duality made quantization (A)

$$Z[X] = \int \mathcal{D}(\bar\psi,\psi)\, e^{-\bar\psi^a_\mu\left((\varepsilon_a + i\delta s_a)\delta_{\mu\nu} - H_{\mu\nu} - X^a_{\mu\nu}\right)\psi^a_\nu}$$

- assume $Z[0] = 1$ (e.g. replica trick and $n_+, n_- \to 0$ at end of calculation)

- $$\left.\frac{\delta Z[X]}{\delta X^{a_1}_{\mu\nu}}\right|_{X=0} = \left(\varepsilon_{a_1} \pm i\delta - H\right)^{-1}_{\nu\mu}$$

- throughout $X=0$ for simplicity; ψ^a_μ Grassmann variables

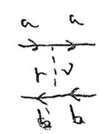
- $S_{\hat\varepsilon}[\psi,\bar\psi] = \bar\psi^a_\mu\left((\varepsilon_a + i\delta s_a)\delta_{\mu\nu} - H_{\mu\nu}\right)\psi^a_\nu \qquad \hat\varepsilon = \mathrm{diag}(\varepsilon_{a_+}, \varepsilon_{a_-})$

invariance: $S_{\varepsilon \mathbb{1}}[T\psi, \bar\psi T^{-1}] = S_{\varepsilon\mathbb{1}}[\psi,\bar\psi]$

$T \in U(n_+ + n_-)$ - flavor sym.

$$\bar Z[0] = \int \mathcal{D}H\, e^{-\frac{N}{2\lambda^2}\mathrm{tr}(H^2)} \int \mathcal{D}(\bar\psi,\psi)\, e^{-S[\bar\psi,\psi]} = \int \mathcal{D}(\bar\psi,\psi)\, e^{-S^0_{\hat\varepsilon}[\bar\psi,\psi] + \frac{\lambda^2}{2N}\bar\psi^a_\mu \psi^a_\nu \bar\psi^b_\nu \psi^b_\mu}$$

$$S^0 = S|_{\hat H = 0}$$



$$= \int \mathcal{D}(\bar\psi,\psi)\, e^{-S^0_{\hat\varepsilon}[\bar\psi,\psi] - \frac{\lambda^2}{N}\psi^b_\mu \bar\psi^a_\mu \psi^a_\nu \bar\psi^b_\nu}$$

$$= \int \mathcal{D}A\, e^{+\frac{N}{2\lambda^2}\mathrm{tr}A^2} \int \mathcal{D}(\bar\psi,\psi)\, e^{-\tilde S[\bar\psi,\psi]}$$

$$\tilde S[\bar\psi,\psi] = \bar\psi^a_\mu\left((\varepsilon_a + i\delta s_a)\delta^{ab} - A^{ab}\right)\psi^b_\mu$$

- Symmetries of action

- $\chi \to \hat{U} \chi \qquad \hat{U} \in U(n \cdot N)$

S inv. under $\quad \chi \to U_c \chi \qquad U_c \in \mathbb{1} \times U(N) \subset U(n\, N)$

$\tilde{S}$ " " $\quad \chi \to U_f \chi \qquad U_f \in U(n) \times \mathbb{1} \subset U(n \cdot N)$

- Identification of flavor Goldstone modes.

- $\int \chi \to \quad \bar{Z}[0] = \int \mathcal{D}A \; e^{\mp \frac{N}{2\lambda^2} \operatorname{tr} A^2 + N \operatorname{tr} \ln(\hat{\varepsilon} + i\delta \tau_3 - A)}$

$\left[(\tau_3)_{aa'} = \delta_{aa'} s_a \right.$

- $\frac{\delta}{\delta A} S[A] = 0 \iff A = + \frac{\lambda^2}{\hat{\varepsilon} + i\delta\tau_3 - A}$

decoupled ansatz. $A_{aa'} = \Sigma\big|_{E_a + i\delta s_a} \delta_{aa'} \simeq \frac{E}{2} - i s_a \left(\lambda^2 - \left(\frac{E}{2}\right)^2\right)^{1/2} \to A = \frac{E}{2} - i \frac{\pi\rho\lambda^2}{N} \tau_3$

$E_a = E + \omega_a , \qquad \omega_a \ll E$

mean field breaks U_f - symmetry.

- Goldstone mode manifold

$$A \to \frac{E}{2} - \frac{i\pi\rho\lambda^2}{N} \underbrace{U_f \tau_3 U_f^{-1}}_{Q}$$

$$\tau_3 = \begin{array}{cc} \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} & \begin{array}{c} n_+ \\ n_- \end{array} \end{array} \qquad U_f \in U(n_+ + n_-)$$

$$Q \in U(n_+ + n_-) \big/ U(n_+) \times U(n_-)$$

sym. space of type AIII

: symmetry duality

	time evolution	averaged theory
sym. space	$A\|_N$	$AIII\|_{(n_+,n_-)}$
invariance	$U(N)$	$U(n)$

- expansion of action in symmetry breaking $\hat{\omega} = \{\omega_a \delta_{aa'}\}$

$$S_{eff}[Q] = -i\pi\rho \, \mathrm{tr}(\hat{\omega} Q) \qquad \bar{Z}[X] = \int dQ \, \exp(-S_{eff}[Q] - i\pi\rho \, L(Q,X))$$

- $\int dQ \to$ RMT results

perturbative expansion $\quad Q = T \tau_3 T^{-1}$

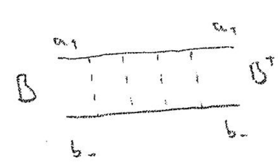
$$T = \exp W \qquad W \in U(n_+ + n_-)/U(n_+) \oplus U(n_-)$$

$$= \begin{pmatrix} & B \\ -B^\dagger & \end{pmatrix} \qquad B \in \mathrm{mat}(n_+, n_-, \mathbb{C})$$

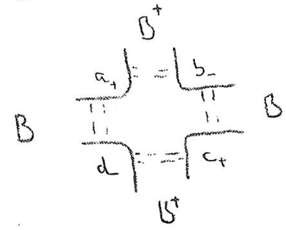
"magnon fields"

Expansion of action

$$S = i\pi\rho \, \mathrm{tr}(\hat{\omega}_+ BB^\dagger - \hat{\omega}_- B^\dagger B) + \sum_{n=2}^{\infty} S^{(2n)}[B, B^\dagger]$$



$$\sim \frac{1}{\omega_{a+} - \omega_{b-}}$$



- generalization to AI, AII:

$$\bar\chi H \chi \overset{!}{=} \frac{1}{2}\left(\bar\chi H \chi + \chi^T H \bar\chi^T\right) = \bar\Psi H \Psi$$

$$H = H^T$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi \\ \bar\chi^T\end{pmatrix} \qquad \bar\Psi = \frac{1}{\sqrt{2}}\left(\bar\chi, -\chi^T\right)$$

reality condition: $\Psi^T = \bar\Psi \begin{pmatrix} & +1 \\ -1 & \end{pmatrix} = \bar\Psi\,\tau \qquad \tau = +i\sigma$

- symmetry transformations must respect chirality

$$\Psi \to T\Psi = \Psi' \qquad \Psi'^T = \Psi^T T^T = \bar\Psi\,\tau\, T^T$$

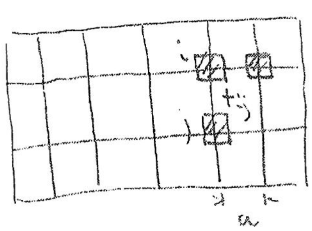
$$\bar\Psi \to \bar\Psi T^{-1} = \bar\Psi' \qquad \overset{!}{=} \quad \bar\Psi'\tau = \bar\Psi T^{-1}\tau$$

$$\Bigg\}\ T^{-1} = \tau T^T \tau^{-1} \Leftrightarrow T \in Sp(2n)\cap U(2n)$$

$$\rightsquigarrow Q = T\tau_3 T^{-1} \in Sp(2(n_+ + n_-))/Sp(2n_+)\times Sp(2n_-) = CII|_{(n_+,n_-)}$$

$$\rightsquigarrow \quad AI|_N \to CII|_{(n_+,n_-)}$$

- generalization to field theory



$$S_{eff}[Q] \to \sum_{\langle i,j\rangle} \underbrace{C}_{|t_{ij}|^2\rho} \mathrm{tr}(Q_i Q_j) - i\pi\rho \sum_i \mathrm{tr}(\hat\omega Q_i)$$

$$\left[\ \mathrm{tr}\, Q_i Q_j = \frac{1}{2}\,\mathrm{tr}\,(Q_i - Q_j)^2 + \text{const.}\right.$$

$$\to \frac{\pi\nu}{4}\int d^d r\ \mathrm{tr}\left(D(\partial Q)^2 + 2i\hat\omega Q\right)$$

$$\left[\begin{array}{l} D\nu \sim C a^{(2-d)} \\ \nu \sim \rho\, a^{-d} \end{array}\right.$$

$$S^{(2)}[B, B^\dagger] = -\frac{\pi\nu}{2}\int d^d r\ \mathrm{tr}\, B^\dagger\left(D\partial^2 + i\omega\right)B$$

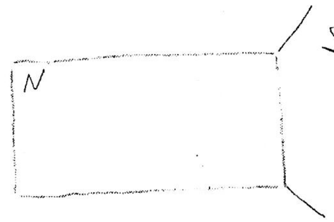
$$\left[\ \hat\omega = \omega\tau_3\right.$$

$-D\partial^2$ · Diffusion (Perron Frobenius) generator.

~ gapped spectrum $\{0, \gamma, ..\}$ $\gamma \sim \frac{D}{L^2}$ ~ 'Thouless energy'

Nonstandard symmetries.

case study:



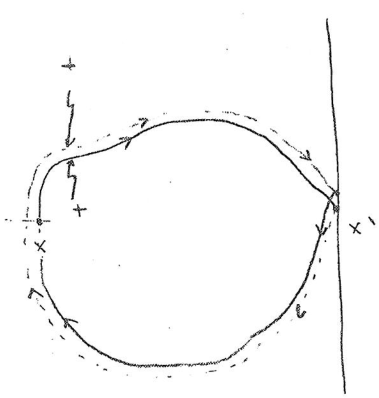
no symmetries : $\hat{H} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}$

$\Delta = -\Delta^T$ (2^{nd} q.u. $c^\dagger_\alpha c^\dagger_\beta \Delta_{\alpha\beta}$)

· semiclassical approach: $G^+(\epsilon) = \begin{pmatrix} \epsilon^+ - h & -\Delta \\ \Delta^* & \epsilon^+ + h^T \end{pmatrix}^{-1}$

· quasiparticle DOS in N

$\rho = \frac{1}{\pi} \mathrm{Im}\, \mathrm{tr}\left(G^+(\epsilon)\sigma_3 \right)$



$+ : \langle x' | \frac{1}{\epsilon^+ - h} | x \rangle = g^+(x', x, \epsilon)$

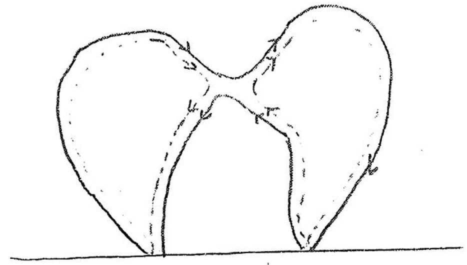
$- \quad \langle x' | \frac{1}{\epsilon^+ + h^T} | x \rangle =$

$\langle x | \frac{1}{\epsilon^+ + h} | x' \rangle =$

$= \overline{\langle x' | \frac{1}{\epsilon^- + h} | x \rangle} = -\overline{g^+(x', x, -\epsilon)}$

phase coherence at $\epsilon \to 0$ ~ DOS affected by quantum interference

· higher order quantum interference



· conclusions: · quantum interference in single particle Green function *)

· singularities for $\varepsilon \to 0$

· expect novel type of 'dual geometry'

· RMT : $\rho(s) = \frac{1}{\Delta}\left(1 + \frac{\sin s}{s}\right)$ $\quad s = \frac{\pi\varepsilon}{\Delta}$

· symmetries

$\sigma_1 \hat{H}^T \sigma_1 = -\hat{H}$ $\qquad \hat{K} = \sigma_1^{1/2} \hat{H} \sigma_1^{-1/2} \Rightarrow \hat{K}^T = -\hat{K} \qquad [\hat{K}, \hat{T}]_+ \stackrel{!}{=} 0$

(Majorana rep.)

$\hat{U} \sim \exp(-i\hat{K}t) \in O(2N) = D_{2N}$

· 10 symmetry classes

☺ Why did Dyson overlook 7 of them! Find space structure

Shortcut to classification:

$\chi = \tau \circ c$

$T = U_T K \qquad \tau : \hat{H} = U_T^\dagger \hat{H}^* U_T$

$C = U_c K \qquad c : \hat{H} = -U_c^\dagger \hat{H}^* U_c$

$\hat{H} = \tau(-U_c^\dagger \hat{H}^* U_c)$ 'chiral'

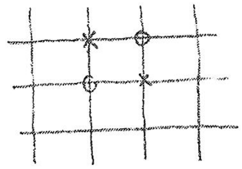
$= -U_T^\dagger U_c^T \hat{H} \underbrace{U_c^* U_T}_{\text{unit}}$

T^2	τ	C^2	c
	0		0
$\mathbb{1}$	1	$\mathbb{1}$	1
$-\mathbb{1}$	−1	$-\mathbb{1}$	−1

} 9 variants

*) expect 'level correlation' in DOS at $\varepsilon \to 0$.

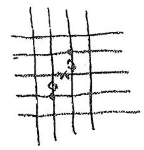
Can study: sublattice models



$$\hat H = \begin{pmatrix} & X \\ X^\dagger & \end{pmatrix} \quad \begin{matrix} \circ \\ \times \end{matrix} \qquad \sigma_3 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$$

$$[\hat H, \sigma_3] = 0 \qquad \mathrm{AIII}\big|_N : \; U(2N)/U(N)\times U(N)$$

semiclassics:



$$G^+(x,x',\epsilon) = \langle x' | \frac{1}{\epsilon^+ - \hat H} | x \rangle$$

$$G^+(x',x,\epsilon) = \langle x | \frac{1}{\epsilon^+ - H} | x' \rangle =$$

$$= \underbrace{\rho(x)\rho(x')}_{\text{sublattice parity}} \langle x | \sigma_3 \frac{1}{\epsilon^+ - H} \sigma_3 | x' \rangle$$

$$= -\rho(x)\rho(x') \langle x | \frac{1}{(-\epsilon)^- - H} | x' \rangle =$$

$$= G^-(x,x',-\epsilon)$$

Field theory (schematic)

$$(\bar\psi_L \; \bar\psi_R) \begin{pmatrix} & X \\ X^\dagger & \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

invariant under $\psi_c \to U_c \psi_c$, $\bar\psi_c \to \bar\psi_c U_c^\dagger$ $\quad c = L, R$

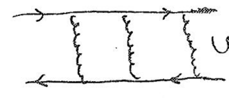
symmetry $U_L(N) \times U_R(N)$

disorder av: $\sim \bar\psi_L \psi_L \bar\psi_R \psi_R \sim - \psi_R \bar\psi_L \, \psi_L \bar\psi_R$

$$(\bar\psi_L \; \bar\psi_R) \begin{pmatrix} A_{LR} & X_0 \\ X_0^\dagger & A_{RL} \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_R \end{pmatrix} \xrightarrow{\text{stat. phase}} (\bar\psi_L \; \bar\psi_R) \begin{pmatrix} \pm i\gamma & X_0 \\ X_0^\dagger & \pm i\gamma \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

spontaneous breaking of chiral symmetry

[cf. QCD

q  U $\sim$ $\langle \bar q_R q_L \rangle$ condensate
$\bar q$

under symmetry transformation

$$\rightarrow \quad (\bar\psi_L\ \ \bar\psi_R)\begin{pmatrix} \pm i y\, U_L^{-1} U_R & X_0 \\ X_0^\dagger & \pm i y\, U_R^{-1} U_L \end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$$

symmetry broken to $U = U_L^{-1} U_R$

$\sim$ chiral symmetric space AIII

Low energy field theory:

$$S[U] = \int d^d x \left(c_1\, \mathrm{tr}(\partial U \partial U^{-1}) + c_2 \left(\mathrm{tr}(U \partial U^{-1})\right)^2 + c_3\, \epsilon\, \mathrm{tr}(U + U^{-1}) \right)$$

cf. Weinberg Lagrangian for mesons, $\epsilon \sim$ quark masses

phenomenology (Gade/Wegner)

- diverging $\rho(\epsilon)$ for $\epsilon \to 0$ in $d=2$
- finite conductivity at $\epsilon = 0$.

Case study: 2d Dirac operator

c.f. graphene, anomalous superconductor (p, d wave), surface states of 3d top. insulators,

$$\hat{H}_0 = \hat{p}_i \sigma_i = \begin{pmatrix} & \hat{p}_x - i\hat{p}_y \\ \hat{p}_x + i\hat{p}_y & \end{pmatrix}$$

chiral symmetry preserving disorder. $\hat{H}_0 \to \hat{H} = (\hat{p}_i - A_i)\sigma_i$

☹ expectation for low energy field theory.

Consider: $\bar{\psi} H \psi$ (c.f. 2d disordered system, or (2+1)d system subject to fluctuating gauge field.)

Note: $A_i = \partial_i \phi + \epsilon_{ij}\partial_j \theta$. ϕ removable by gauge transformation

θ " " chiral " "

$$\psi \to e^{\theta\sigma_3}\psi$$
$$\bar{\psi} \to \bar{\psi} e^{\theta\sigma_3}$$

• Disorder can be gauged away?

• Disorder averaging $\Rightarrow$ chiral symmetry breaking A_{III}-field theory as before

Resolution to paradox: topological term

Witten 84

$$S[\bar{\psi},\psi] = \int d^2x\, \bar{\psi}(\omega - \hat{H}_0)\psi \Leftrightarrow$$

$$\frac{1}{8\pi}\int d^2x\, \mathrm{tr}(\partial_i U \partial_i U^{-1}) + \frac{1}{24\pi^2}\Gamma[U] + c\int d^2x\, \omega\, \mathrm{tr}(U + U^{-1})$$

Def: $\tilde{U}(x,t)$ = homotopic extension $\tilde{U}(x,0) = \mathbb{1}$, $\tilde{U}(x,1) = U(x)$

$$\Gamma[U] = \int_0^1 dt \int d^dx\; \epsilon_{\mu\nu\rho}\, \mathrm{tr}(\tilde{U}^{-1}\partial_\mu \tilde{U}\, \tilde{U}^{-1}\partial_\nu \tilde{U}\, \tilde{U}^{-1}\partial_\rho \tilde{U})$$

· Dissipative op: $i\int d^dx\; \mathrm{tr}(U^{-1}\partial_i U)\,\mathrm{tr}(U^{-1}\partial_i U)$ purely imaginary

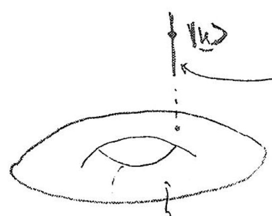
· At $\omega = 0$: $S[U]$ conformally invariant. Describes free Dirac fermions.
Protection by topological term. ⊖ Discuss role of WZW in RG

· Topological insulator primer (class A)

· TI: an insulator whose band structure contains topological structures → gapless surface excitations

bulk: N-theory
surface: n-theory

· setting:



$|h\rangle$

$[|h\rangle] = \{ g|h\rangle \mid g \in U(1)\}$
→ $U(1)$ principal bundle

$T^d = (S^1)^d$ – h space

Berry connection: $a(h) = i\langle h | d h\rangle$ $\quad (a_\mu = i\langle h|\partial_\mu h\rangle)$

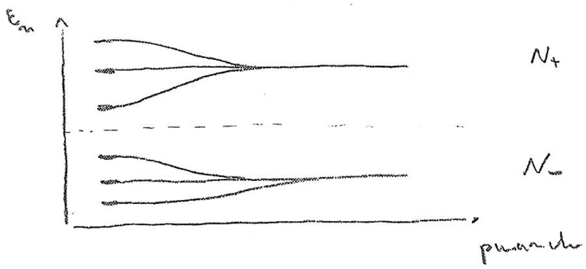
under $g = e^{i\phi}$ transformation: $a \to a + i g^{-1} dg = a - d\phi$

Berry curvature: $f = da$ $\quad (f_{\mu\nu} = i\langle \partial_\mu h | \partial_\nu h\rangle)$

First Chern number ($d=2$): $C_1 = \frac{1}{2\pi}\int_{T^2} f$ integer valued

⊖ band structures characterized by different C_1 cannot be adiabatically connected without gap closure / $C_1 = \frac{h}{e^2}\sigma_{xy}$

• alternative characterization



pick basis at k_0: $|k_0 n\rangle = e_n$ – standard unit vector

Band structure encoded in $U(k)$ $U(k) \in U(N = N_+ + N_-)$

Refined: $\underbrace{U(N)/U(N_+)\times U(N_-)}_{\text{symmetry class!}} = Gr(N_+,N)$ complex Grassmannian

topology: homotopy of maps: $T^d \to Gr(N_+,N)$ $\pi(T^d, Gr(N_+,N))$

Example: $d=2$. $Gr(1,2) = U(2)/U(1)\times U(1) = S^2$ ~ $\pi(T^2, S^2) = \mathbb{Z}$

• homotopy structure describable in terms of Dirac model Hamiltonians

homotopy classes change via band closing singularities (Dirac points)

Example: $d=2$, $N=2$ $\hat H_k = \sigma_i d_i(k)$ $\epsilon_{0,1} = \pm |d(k)|$

Def: $\phi: T^2 \to S^2$ $C_1 = \frac{1}{4\pi}\int_0^{2\pi} dk^1 dk^2\, d(k)\cdot(\partial_1 d(k) \times \partial_2 d(k))$

$k \to \frac{d(k)}{|d(k)|}$

Example: $d(k) = \begin{pmatrix} \sin k_1 \\ \sin k_2 \\ r + \cos k_1 + \cos k_2 \end{pmatrix}$ Haldane QAH

$$C_1: \overset{r<-2}{0} \xrightarrow{r=-2} 1 \xrightarrow{r=0} -1 \xrightarrow{r=2} \overset{r>2}{0}$$

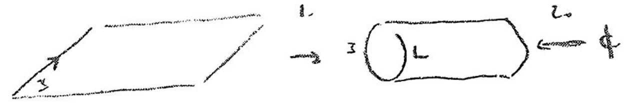
Around $r=-2$: $\hat H \simeq k_1\sigma_1 + k_2\sigma_2 + m\sigma_3$ + disorder   class A Dirac.

Can be used to diagnose changes in C_1.

· Navigating periodic table Dirac - Kaluza - Klein dimensional reduction.

Example $d: 3 \to 2 \to 1$: Consider 3d - Dirac equation $\hat{H} = \sum_{i=1}^{3} k_i \sigma_i$ (gapless)

Compactify 1 space dimension



$k_3 \in \mathbb{R}$ $\qquad$ $k_3 \in \frac{2\pi}{L}\mathbb{Z}$

$$k_3 \to \frac{(2\pi + \Phi)}{L}\mathbb{Z} \longrightarrow m$$

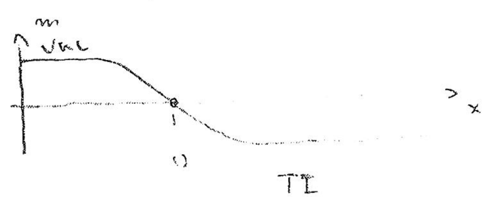
$\rightsquigarrow$ $\hat{H} = \sum_{i=1}^{2} k_i \sigma_i + m \sigma_3$ $\rightsquigarrow$ $\mathbb{Z}$ - valued Chern number

$k_2 \to 0$

$\hat{H} = k_1 \sigma_1 + m \sigma_3$ $\qquad$ $[\hat{H}, \sigma_2]_+ = 0$ $\qquad$ AIII (winding number: $\int_T a$

· Gapless boundary theories

Example: 1d/AIII



$$\hat{H} = -i\partial_x \sigma_1 + m\frac{x}{a}\sigma_3 \simeq -i\partial_x \sigma_2 + m\frac{x}{a}\sigma_1 = \begin{pmatrix} & -\partial_x + m\frac{x}{a} \\ \partial_x + \frac{mx}{a} & \end{pmatrix}$$

$$\chi_0(x) = \begin{pmatrix} e^{-\frac{x^2}{2a}m} \\ 0 \end{pmatrix}$$ 0 energy wave function.

boundary theories in higher dimensions: σ-models perturbed by top terms.

example: AIII cl. Bulk theory (real space)

$$S[Q] = \frac{1}{8}\int d^2x \left(\sigma_{11}\,\mathrm{tr}(\partial_i Q\,\partial_i Q) - \sigma_{12}\,\epsilon_{ij}\,\mathrm{tr}(Q\,\partial_i Q\,\partial_j Q)\right)$$

expect: $\sigma_{11} \to 0$

σ_{12} quantized. $\quad \mathrm{tr}(Q\, dQ \wedge dQ) = 4\, d\left(T \tau_3\, d T^{-1}\right)$

$$\rightsquigarrow \quad S[Q] \rightsquigarrow S[T] = -\sigma_{12} \oint dx\, \mathrm{tr}(T\tau_3 \partial_x T^{-1}) \qquad \text{(1d WZW action)}$$

consider $T = \begin{pmatrix} h_+ & \\ & h_- \end{pmatrix} \rightsquigarrow Q = \tau_3 \rightsquigarrow$ pure gauge $\qquad h_\pm = e^{i\phi_\pm}$

$$S[T] = -\sigma_{12} \oint dx \sum_s s\, \mathrm{tr}(h_s \partial h_s^{-1}) = -i\sigma_{12} \oint dx \sum_s s\, \partial_x \phi_s =$$

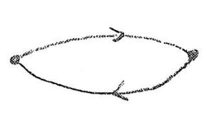
$$= -i\sigma_{12}\, 2\pi \underbrace{(W_+ - W_-)}_{\text{winding number}} \rightsquigarrow \sigma_{12} \overset{!}{\in} \mathbb{Z}$$

In the presence of finite frequency:

$$S[T] = \oint dx \left(-\sigma_{12}\, \mathrm{tr}(T\tau_3 \partial T^{-1}) + \frac{i\pi\omega\nu}{2}\, \mathrm{tr}(T\tau_3 T^{-1})\right)$$

$$\begin{cases} \nu = \dfrac{1}{\Delta L} \\[2mm] \quad = \dfrac{1}{2\pi v_F} \end{cases} \qquad \Delta = v_F \frac{2\pi}{L} \qquad T = e^{\begin{pmatrix} & B \\ -B^\dagger & \end{pmatrix}}$$

$$\simeq \oint dx \left(B\left(2\sigma_{12}\partial + \frac{\omega}{2v_F}\right) B^\dagger\right) + \dots$$



ballistic motion at speed v_F