# I. PLATEAU TRANSITION AS CRITICAL POINT

The IQHE plateau transitions are examples of quantum critical points. What sort of theoretical description should we look for? Recall Anton Andreev's lectures, in which we saw how the localisation problem of a quantum particle moving in a random potential can be represented by a suitable sigma model. So what is required is 'just' to include a magnetic field in this description . Here we will instead take a phenomenological approach.

## A. Scaling flow diagram

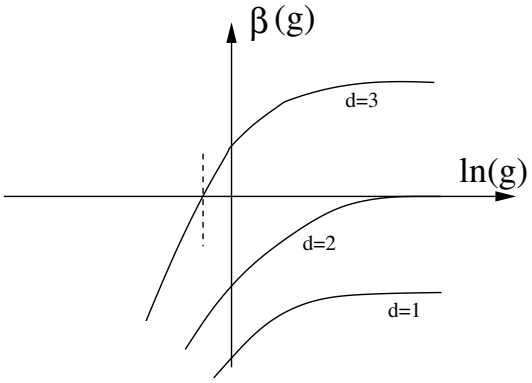
To start, we review the renormalisation group (RG) treatment of the localisation transition in the absence of magnetic field. The coupling constant is the dimensionless conductance $g(L)$ of a system at length scale $L$, related to the conductivity $\sigma_{xx}(L)$ in $d$ dimensions by

$$g(L) = \frac{h}{e^2}\sigma_{xx}(L)L^{d-2}\,.$$

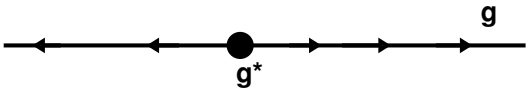
The central quantity in an RG description of the transition is

$$\beta(g) = \frac{\partial \ln g}{\partial \ln L}\,.$$

In a good metal $\sigma_{xx}(L)$ is independent of $L$ and so $g(L) \propto L^{d-2}$ implying $\beta(g) = d-2$, while in an Anderson insulator $g(L) \sim e^{-L/\xi}$ and so $\beta(g)$ is negative. These considerations lead to a form for $\beta(g)$

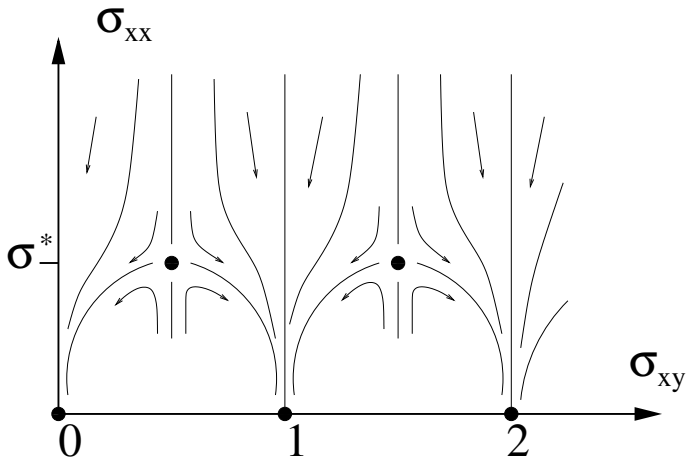



which can also be represented for $d > 2$ as



Since two is the lower critical dimension for the localisation transition in the simplest symmetry class, the critical conductance $g^*$ diverges as $d \to 2$. For $d = 2$, flow in the absence of magnetic field (or special symmetries) is towards the insulator from all starting points $g$.

The question we face is how things change in the presence of a magnetic field. The answer is first that $\sigma_{xy}$ appears as a second coupling constant in the sigma model [see: H. Levine, S. B. Libby, and A. M. M. Pruisken, Phys. Rev. Lett. **51**, 1915 (1983)], and second that the scaling flow (now in the $\sigma_{xx}$—$\sigma_{xy}$ plane) is believed to have the form [D. E. Khmel'nitzkii, JETP Lett. **38**, 552 (1983)]




Note that this scaling flow diagram contains fixed points of two kinds: stable fixed points at $(\sigma_{xx}, \sigma_{xy}) = (0, ne^2/h)$, with $n = 0, 1, 2 \ldots$, which represent Hall plateaus, and unstable fixed points at $\sigma_{xx}, \sigma_{xy} = (\sigma^*, [n + 1/2]e^2/h)$, which represent plateau transitions. This form for the scaling flow diagram implies that the Hall conductance should change at transitions only in single steps of $e^2/h$. It also implies that the width of plateau transitions should shrink to zero as they are probed on longer length scales – for example, by going to lower temperature in experiment.

### B. Critical behaviour

Let's examine this critical behaviour. Suppose the behaviour of the localisation length $\xi(E)$ as a function of energy $E$ has the form $\xi(E) \propto |E - E^*|^{-\nu}$ or at fixed fermi energy as a function of magnetic field $\xi \propto |\Delta B|^{-\nu}$. If we allow for inelastic scattering with a dephasing time $\tau_\varphi \sim T^{-p}$ and hence (assuming diffusive motion) a dephasing length $l_\varphi \sim (D\tau_\varphi)^{1/2} \sim T^{-p/2}$, then the transition width, set by $\xi \sim l_\varphi$, will vary as

$$|\Delta E| \sim |\Delta B| \sim T^{p/2\nu} .$$

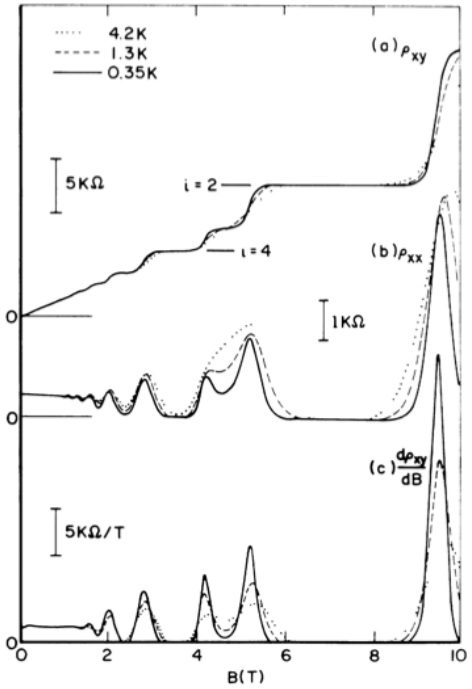
This view of the transition is closely tied to a single-particle description, with dephasing as an added extra idea. It is more appropriate to describe the transition using the general scaling framework for a zero-temperature quantum critical point. Then associated with the divergent correlation length $\xi$ there is in general a correlation time $\xi_\tau$ related via the dynamical scaling exponent $z$ by $\xi_\tau \sim \xi^z$. For a zero temperature quantum critical point we expect $\xi_\tau = \hbar/k_{\rm B}T$. Taking $\xi \sim |\Delta B|^{-\nu}$ we have scaling for the elements of the resistivity tensor in the form

$$\rho_{ij}(\Delta B, T) = f_{ij}(|\Delta B|/T^{1/z\nu})$$

and this has been tested in a series of experiments, as we now illustrate.

### C. Experiments

Scaling with temperature [H P Wei, D C Tsui, M A Paalanen and A M M Pruisken, PRL **61**, 1294 (1988)]



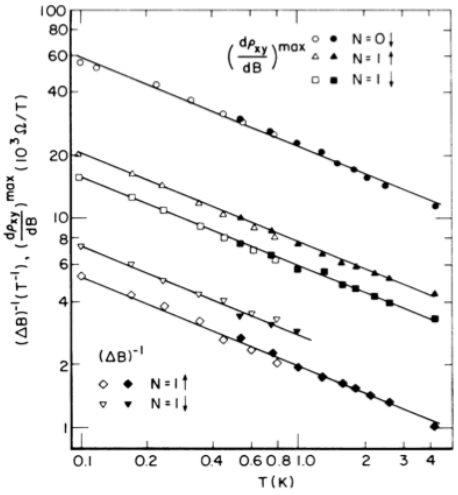
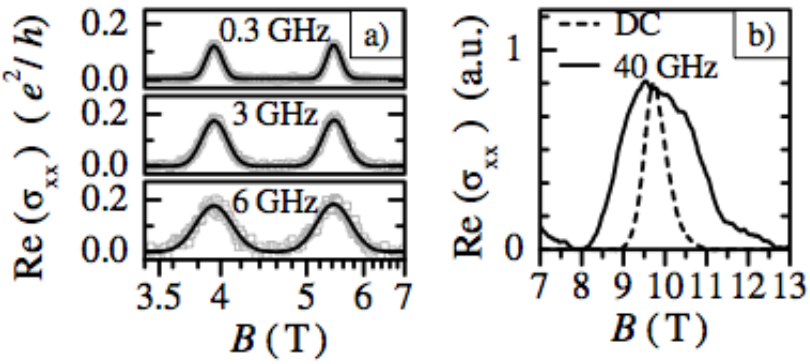
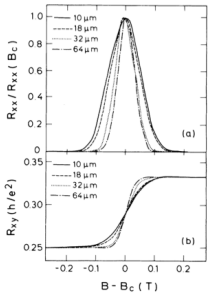




FIG. 2. The upper portion shows the $T$ dependence of $(d\rho_{xy}/dB)^{\rm max}$ for Landau levels $N=0\downarrow$, $1\uparrow$, and $1\downarrow$; the lower portion shows the $T$ dependence of $1/\Delta B$ for the $N=1\uparrow$ and $1\downarrow$ Landau levels. The open symbols are data taken in a dilution refrigerator, whereas the filled symbols are data taken in a $^3$He system. The slope of the straight lines gives $(d\rho_{xy}/dB)^{\rm max} \sim T^{-\kappa}$ and $\Delta B \sim T^{\kappa}$ with $\kappa = 0.42 \pm 0.04$. The typical uncertainty in $T$ is $\sim$0.02 K at 0.4 K.

Scaling with frequency [F Hohls, U Zeitler, R J Haug, R. Meisels, K. Dybko, and F. Kuchar, PRL **89**, 276801 (2002)]




Scaling with sample size [S Koch, R J Haug, K v Klitzing and K. Ploog, PRL **67**, 883 (1991)]



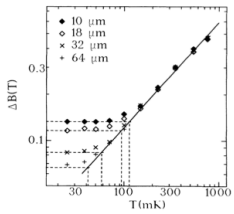




FIG. 2. The half-width $\Delta B$ as a function of temperature for sample 1 in Landau level $N=1\downarrow$ for the four sample widths. Horizontal dashed lines: extrapolation to zero temperature. The vertical dashed lines correspond to the characteristic temperatures $T_c$. Solid line: high-temperature fit.

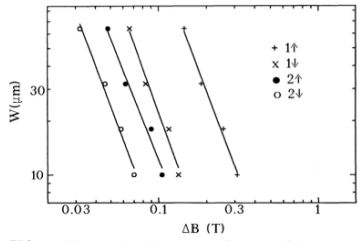
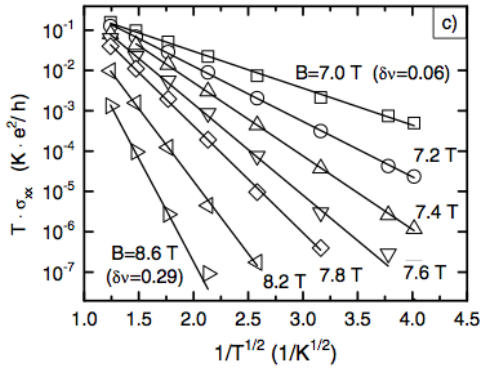
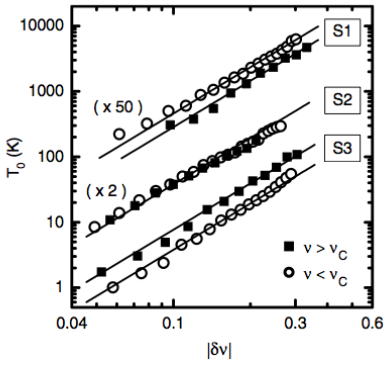



FIG. 3. The sample width $W$ as a function of the saturation half-width $\Delta B$ for sample 1 in Landau levels $N=1$ and $N=2$. Straight lines: fits.

Direct measurement of localisation length via study of hopping transport [F Hols, U Zeitler and R J Haug, PRL **88**, 036802 (2002)]




Note that in the right hand graph the data is compared to the result from the theory of hopping conduction: $\sigma(T) = \sigma_0 exp(-\sqrt{T_0/T})$ where $T_0$ is related to the localisation length via

$$k_B T_0 = C \frac{e^2}{4\pi\epsilon\epsilon_0\xi}.$$

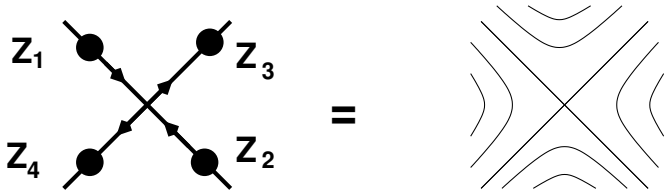
### D. Network model

We now turn to a theoretical description of the localisation problem in the IQHE. Ideally one would like to discuss the effects of both disorder and interactions but we will restrict ourselves to the single-particle problem.

One obvious approach would be to consider the full Hamiltonian $\mathcal{H} = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 + V(x, y)$, but instead we will build on the classical percolation picture discussed in Lecture I.

Suppose we carry out a semi-classical quantisation of the guiding centre motion along contours of the potential. This gives us quantum eigenstates associated with closed classical orbits. One of the main corrections to the semiclassical picture comes from mixing of spatially separated semi-classical states associated with different minima of the potential. Such mixing happens because of quantum tunnelling between disjoint classical contours at the same (or nearly the same) energy, which is important near saddle points in the potential.

The network model aims to encode this physics. First consider tunnelling at a single saddle-point and represent the incident and outgoing states by current amplitudes $z_i$ as illustrated.



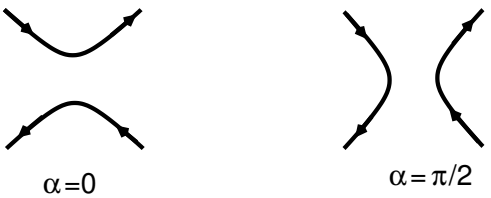

Then scattering theory leads us to write

$$\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \mathrm{U} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and current conservation requires that the $2 \times 2$ scattering matrix $\mathrm{U}$ must obey $\mathrm{U}^\dagger \mathrm{U} = \mathbb{1}$. Note that we can factorize $\mathrm{U}$ as

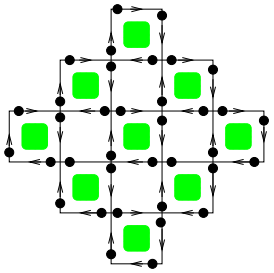
$$\mathrm{U} = \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}$$

Here the parameter $\alpha \in [0, \pi/2]$ is the crucial one, with limiting cases



To describe a complete sample we connect up a large number of such scattering nodes. In principle there are three sources of randomness: $(i)$ in the distances along equipotentials between saddle-points, which will lead to randomness in the phase accumulated; $(ii)$ in the values of the tunnelling parameter $\alpha$ at each node; and $(iii)$ in the topology of the resulting network. For simplicity we retain only $(i)$, expecting that this will be enough to access universal behaviour.

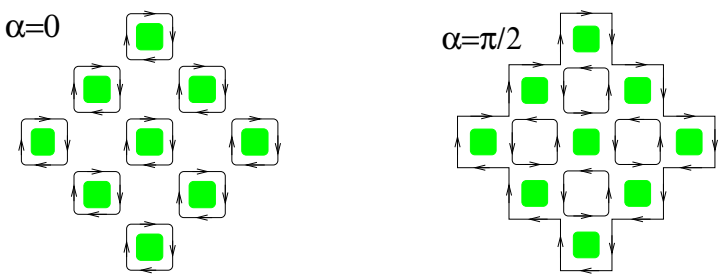
Hence we arrive at a model of the form



in which every node is characterised by the same parameter $\alpha$ and the phases associated with links are independent, uniformly distributed random variables. The model can be represented by a unitary matrix, which is a time evolution operator $\mathcal{U}$ for a time-step in which electrons propagate from the start of one link, along a link and through a node, to the start of the next link. For a system of $N$ links the matrix $\mathcal{U}$ is $N \times N$. It acts on the wave function of the system, represented by the $N$-component vector $(z_1, z_2, \ldots z_N)^{\mathrm{T}}$. It can be broken down into two factors: $\mathcal{U} = \mathcal{U}_2\mathcal{U}_1$. The factor $\mathcal{U}_1$ represents the effect of links, and is a diagonal matrix with random phases on the diagonal. The factor $\mathcal{U}_2$ represents the nodes and is composed of $2 \times 2$ blocks, each of the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} .$$

Varying the parameter $\alpha$ between $\alpha = 0$ and $\alpha = \pi/2$ is equivalent to sweeping in energy across a Landau level, as we see by considering the limiting cases




The transition has so far been studied only numerically: the localisation length $\xi$ diverges on approaching $\alpha = \pi/4$ as $\xi \sim |\alpha - \pi/4|^{-\nu}$ with an exponent $\nu \approx 2.5$.