Quasiparticle thermal Hall transport in d-wave superconductors

Oskar Vafek

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Contents

1 Overview of the main results

In the previous lecture, we derived the eigenvalue equations for the nodal d-wave quasiparticles in the vortex (lattice) state. We wrote the model on the tight-binding lattice and found:

$$
\begin{aligned}
E\begin{pmatrix}\psi_{\mathbf{r}\uparrow}\\ \psi_{\mathbf{r}\downarrow}\end{pmatrix} &= \sum_{\delta=\pm\hat{x},\pm\hat{y}} z^{(2)}_{\mathbf{r},\mathbf{r}+\delta}\begin{pmatrix} -te^{i\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\left(\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}\right)} & \Delta_\delta \\ \Delta_\delta & te^{-i\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\left(\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}\right)}\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{r}+\delta\uparrow}\\ \psi_{\mathbf{r}+\delta\downarrow}\end{pmatrix} \\
&+ \begin{pmatrix}-\mu-h_Z & 0\\ 0 & \mu-h_Z\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{r}\uparrow}\\ \psi_{\mathbf{r}\downarrow}\end{pmatrix} \qquad (1)\\
\Rightarrow E\begin{pmatrix}\psi_{\mathbf{r}\uparrow}\\ \psi_{\mathbf{r}\downarrow}\end{pmatrix} &= \hat{H}_{BdG}\begin{pmatrix}\psi_{\mathbf{r}\uparrow}\\ \psi_{\mathbf{r}\downarrow}\end{pmatrix} \qquad (2)
\end{aligned}
$$

We then argued that $\frac{1}{2}\nabla\theta - \frac{e}{\hbar c}\mathbf{A}$ is periodic with the periodicity of the vortex lattice, and that we can always choose the branch-cut term $z^{(2)}_{\mathbf{r},\mathbf{r}+\delta}$ to be periodic, although with a larger unit cell than the unit cell of the vortex lattice, by connecting pairs of vortices. Therefore, we can explicitly take out the $e^{i\mathbf{k}\cdot\mathbf{r}}$ factor from the Bloch wavefunctions, and have the resulting Hamiltonian act on periodic functions:

$$\hat{H}_{BdG}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{H}_{BdG}e^{i\mathbf{k}\cdot\mathbf{r}} \qquad (3)$$

where $\mathbf{k}$ resides in the magnetic Brillouin zone.

In the previous lecture, we looked at the density of eigenvalues of $\hat{H}_{BdG}(\mathbf{k})$ and made connection with the semi-classical results. The purpose of this lecture is to use our ability to find the wavefunctions to discuss the thermal Hall effect. So, let $|n\mathbf{k}\rangle$ be the eigenfunction of $\hat{H}_{BdG}(\mathbf{k})$ with energy $E_n(\mathbf{k})$:

$$\hat{H}_{BdG}(\mathbf{k})|n\mathbf{k}\rangle = E_n(\mathbf{k})|n\mathbf{k}\rangle. \qquad (4)$$

Then, the thermal Hall conductivity at temperature T will be shown to be given by

$$\kappa_{xy} = \frac{1}{\hbar T}\int_{-\infty}^{\infty} d\xi\xi^2\left(-\frac{\partial f(\xi)}{\partial\xi}\right)\tilde{\sigma}_{xy}(\xi) \qquad (5)$$

where

$$f(\xi) = \frac{1}{e^{\xi/(k_BT)}+1}, \qquad (6)$$

and

$$\tilde{\sigma}_{xy}(\xi) = \frac{1}{i}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\sum_{E_m(\mathbf{k})<\xi<E_n(\mathbf{k})}\frac{\left\langle m\mathbf{k}\left|\frac{\partial\hat{H}(\mathbf{k})}{\partial k_x}\right|n\mathbf{k}\right\rangle\left\langle n\mathbf{k}\left|\frac{\partial\hat{H}(\mathbf{k})}{\partial k_y}\right|m\mathbf{k}\right\rangle - (x\leftrightarrow y)}{(E_m(\mathbf{k})-E_n(\mathbf{k}))^2}. \qquad (7)$$

It is well known that the above formula can be written as the sum over occupied bands' $\mathbf{k}$-space integral over the Berry curvature:

$$\tilde{\sigma}_{xy}(\xi) = \frac{1}{2\pi}\sum_n\left(\frac{1}{2\pi i}\int_{E_n(\mathbf{k})<\xi} d^2\mathbf{k}\left(\hat{z}\cdot\nabla_{\mathbf{k}}\times\hat{A}_n(\mathbf{k})\right)\right) = \frac{C}{2\pi}. \qquad (8)$$

For each fully occupied band, the integral extends over the entire magnetic Brillouin zone, and the occupied band contribution to C is an integer called the first Chern number (see for example M. Kohmoto ANNALS OF PHYSICS 160, 343-354 (1985)).

Therefore, if we can determine the energy dependence of the $\mathbf{k}$-space integral over the Berry curvature, we can determine the temperature dependence of the thermal Hall conductivity.

2 What is thermal Hall conductivity?

The experimental setup for the thermal Hall conductivity is shown in the Fig. (1). A heater, which is typically a well characterized electrical resistor, is placed above the sample. A known amount of heat per second is injected into the sample which held in a vacuum, and which is thermally attached to stage (cold finger). The temperature is measured both perpendicular and parallel to the heat current flow.

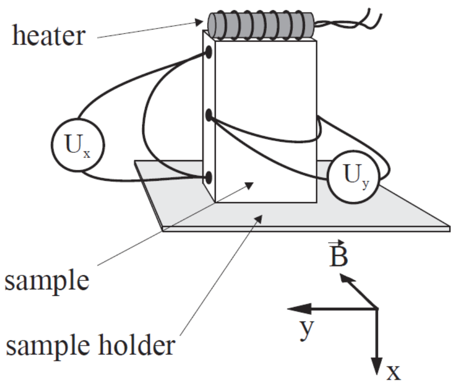

Figure 1: Schematic of the experimental setup for the thermal Hall conductivity experiment. From: Zeini et.al. Eur. Phys. J. B 20, 189 (2001)

The constitutive relation (Fourier law) in a superconductor is

$$j^Q_\mu = -\kappa_{\mu\nu}\nabla_\nu T, \tag{9}$$

where j^Q_μ is the energy current in the direction μ and $\kappa_{\mu\nu}$ is the thermal conductivity tensor. Under the experimental conditions shown in the Fig. (1) there is no heat current in the y-direction. Therefore,

$$j^Q_y = 0 = -\kappa_{yy}\nabla_y T - \kappa_{yx}\nabla_x T \tag{10}$$

$$\kappa_{xy} = -\kappa_{yx} = \kappa_{yy}\frac{\nabla_y T}{\nabla_x T}. \tag{11}$$

So, if we measure the temperature gradients and the longitudinal thermal conductivity κ_{yy}, we obtain κ_{xy}. Although κ_{yy} contains (large) phonon contribution, but κ_{xy} is essentially purely electronic in origin.

3 How to calculate thermal Hall conductivity?: Luttinger's 'pseudo-gravitational' field; from statistical response to dynamical response via hydrodynamics

How should we calculate the 'response' to a spatially varying temperature? In fact, how should we even think about a 'local temperature' within quantum statistical mechanics?

The key to answering the second question is the (standard) hydrodynamic assumption made in any theoretical description of transport coefficients. When the deviation from the equilibrium is small and the sample is macroscopically large, then we can investigate the response to small external perturbation, which vary slowly in time and space. The basic hydrodynamic assumption is the existence of a microscopically short time scale, τ_m, within which the system, perturbed by a slow perturbation, relaxes to local equilibrium.

"All properties of interest may be described in terms of an expansion about local equilibrium. More particularly, one identifies a set of conserved quantities, which in the systems of interest to us are the energy E and particle number N, and one defines corresponding conserved densities, such as the energy density $\epsilon(\mathbf{r})$

and particle density $n(\mathbf{r})$. On time scales large compared to τ_m, one assumes that all physical quantities localized near a point $\mathbf{r}$ relax to values which are determined by the values of the conserved densities and their low-order spatial derivatives in the vicinity of $\mathbf{r}$.

On the other hand, one cannot in general assume that the conserved quantities themselves relax to their equilibrium values in a microscopic time scale. The conservation laws relate the time derivatives of conserved quantities to the divergence of associated transport currents, and these time derivatives may be very small if the length scale of the system is large. In systems with short-range forces, the slowest relaxations are typically characterized by a diffusion coefficient D, so that the slowest relaxation time for the conserved densities is given by $\tau_M \approx L^2/D$, where L is either the size of the system or the wavelength of the perturbation, whichever is shorter. Clearly, if L is very large, τ_M may be very much larger than τ_m. Although the overall response to an external perturbation may be quite different in the limits where the frequency is large or small compared to τ_M^{-1}, the hydrodynamic equations themselves are assumed to apply for time scales τ large compared to τ_m, regardless of whether t is large or small compared to t_M." (Cooper, Halperin, Ruzin PRB 55, 2344, 1997)

This answers the question of how to think about the local temperature: the hydrodynamic assumption allows us to write the total energy and entropy as a functionals of the energy density, since we will assume that this is the only conserved quantity in a realistic situation:

$$\mathcal{E} = \int d^D\mathbf{r}\epsilon(\mathbf{r}). \tag{12}$$

The accuracy of this approximation is limited to terms of order $(\nabla\epsilon(\mathbf{r}))^2$.

Instead of $\epsilon(\mathbf{r})$ we can define a more convenient local independent thermodynamic variable:

$$\frac{1}{T(\mathbf{r})} = \frac{\partial s}{\partial\epsilon(\mathbf{r})}|_n. \tag{13}$$

Luttinger's idea is that there is a complete analogy between diffusion in a system with particle number conservation, which is a 'response' to the buildup of the local chemical potential i.e. particle number density, and thermal conductivity or thermopower, which is a response to the buildup of local temperature i.e. energy. Both of these are statistical responses, as opposed to the mechanical responses. An example of a mechanical response would be applying an external electric field and computing a current. There is of course a deep relation between the electrical conductivity (mechanical response) and diffusion. Luttinger therefore introduced a fictitious field, $\chi(\mathbf{r})$, which couples to the local hamiltonian density. He then related the (mechanical) response to this field to the statistical response. Once the mechanical response is computed, the fictitious field can be dispensed with and we are left with the desired result.

So,

$$\mathcal{H} \to \mathcal{H} = \int d^D\mathbf{r}(1+\chi(\mathbf{r}))\hat{h}(\mathbf{r}).$$

When $\chi(\mathbf{r})$ is time independent, there can be no transport currents because the system is in thermodynamic equilibrium. In effect, the statistical and the mechanical forces cancel. If, in addition, $\chi(\mathbf{r})$ is slowly varying with $\mathbf{r}$, then

$$\mathcal{E} = \int d^D\mathbf{r}\,(1+\chi(\mathbf{r}))\,\epsilon(\mathbf{r}) \tag{14}$$

$$\mathcal{S} = \int d^D\mathbf{r}s\,[\epsilon(\mathbf{r})]\,. \tag{15}$$

and maximizing $\mathcal{S} - \beta\mathcal{E}$, where β is $\mathbf{r}$-independent since the system is in equilibrium, with respect to $\epsilon(\mathbf{r})$ we find:

$$\frac{1}{T(\mathbf{r})} = \beta(1+\chi(\mathbf{r})) \Rightarrow \nabla\frac{1}{T(\mathbf{r})} = \beta\nabla\chi(\mathbf{r}) \Rightarrow T\nabla\frac{1}{T(\mathbf{r})} = \nabla\chi(\mathbf{r}). \tag{16}$$

Therefore, if we are away from equilibrium, the transport currents can only be a function of the combination $T\nabla\frac{1}{T(\mathbf{r})} - \nabla\chi(\mathbf{r})$.

If the external mechanical perturbation periodically drives the system rapidly, relative to the diffusion rate, then the system does not have a chance to respond by building up gradients of temperature, and then the response is purely due to $\nabla\chi$. As long as we take the limit of frequency of χ to zero after the $\mathbf{q}$ is taken to zero, then the response will be in this regime.

4 Literature

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A Derivation of the thermal current for a general tight-binding Hamiltonian

Consider a general non-interacting tight-binding Hamiltonian, unperturbed by the Luttinger's $\chi_{\mathbf{r}}$-field:

$$H = \sum_{\mathbf{r}} \left(\psi^\dagger_{\mathbf{r},a} M_{ab}(\mathbf{r}) \psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}} \left(\psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + \psi^\dagger_{\mathbf{r}+\delta,b} T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \right) \right). \tag{17}$$

In the above, the generalized hopping matrix elements satisfy:

$$T^*_{ab}(\mathbf{r},\mathbf{r}+\delta) = T_{ba}(\mathbf{r}+\delta,\mathbf{r}). \tag{18}$$

The hamiltonian density is

$$\begin{aligned} h_{\mathbf{r}} = \psi^\dagger_{\mathbf{r},a} M_{ab}(\mathbf{r}) \psi_{\mathbf{r},b} \quad &+ \quad \frac{1}{2} \sum_{\delta=\hat{x},\hat{y}} \left(\psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + \psi^\dagger_{\mathbf{r}+\delta,b} T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \right) \\ &+ \quad \frac{1}{2} \sum_{\delta=\hat{x},\hat{y}} \left(\psi^\dagger_{\mathbf{r}-\delta,a} T_{ab}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},b} + \psi^\dagger_{\mathbf{r},b} T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,a} \right) \end{aligned} \tag{19}$$

We can obtain the energy current from the equation of motion:

$$\frac{\partial h_{\mathbf{r}}}{\partial t} = \frac{i}{\hbar} [H, h_{\mathbf{r}}] \equiv -\nabla \cdot \mathbf{j}_E(\mathbf{r}). \tag{20}$$

In order to introduce the pseudo gravitational potential χ coupling to the energy density, we can take

$$\psi_{\mathbf{r},a} \to \tilde{\psi}_{\mathbf{r},a} = \left(1 + \frac{1}{2}\chi_{\mathbf{r}}\right) \psi_{\mathbf{r},a}, \tag{21}$$

and work to linear order in $\chi_{\mathbf{r}}$.

Note that, to linear order, the $\chi_{\mathbf{r}}$-field enters the perturbed Hamiltonian is

$$\begin{aligned} \tilde{H} \quad &= \quad H + \sum_{\mathbf{r}} \left(\chi_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} M_{ab}(\mathbf{r}) \psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}} \frac{1}{2} \left(\chi_{\mathbf{r}} + \chi_{\mathbf{r}+\delta} \right) \left(\psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + \psi^\dagger_{\mathbf{r}+\delta,b} T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \right) \right) \\ &= \quad H + \sum_{\mathbf{r}} \Bigg(\chi_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} M_{ab}(\mathbf{r}) \psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}} \frac{1}{2} \chi_{\mathbf{r}} \Big(\psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + \psi^\dagger_{\mathbf{r}+\delta,b} T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \\ &\quad + \psi^\dagger_{\mathbf{r}-\delta,a} T_{ab}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},b} + \psi^\dagger_{\mathbf{r},b} T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,a} \Big) \Bigg) \\ &= \quad H + \sum_{\mathbf{r}} \chi_{\mathbf{r}} h_{\mathbf{r}}. \end{aligned} \tag{22}$$

To proceed, we need to find the current operator $\tilde{\mathbf{j}}_E(\mathbf{r})$. We can obtain it from the continuity equation:

$$\frac{\partial \tilde{h}_{\mathbf{r}}}{\partial t} = \frac{i}{\hbar} \left[\tilde{H}, \tilde{h}_{\mathbf{r}} \right] \quad \equiv \quad -\nabla \cdot \tilde{\mathbf{j}}_E(\mathbf{r}) = \tilde{j}_{Ex}(\mathbf{r}-\hat{\mathbf{x}}) - \tilde{j}_{Ex}(\mathbf{r}) + \tilde{j}_{Ey}(\mathbf{r}-\hat{\mathbf{y}}) - \tilde{j}_{Ey}(\mathbf{r}). \tag{23}$$

where we used the lattice definition of the divergence of a (lattice) vector field.

The expression for $\tilde{\mathbf{j}}_E(\mathbf{r})$ is derived in detail in the Appendix and reads

$$\tilde{j}_{E\delta}(\mathbf{r}) \quad = \quad \frac{i}{\hbar} \left(A_\delta(\mathbf{r}) + B_\delta(\mathbf{r}) + C_\delta(\mathbf{r}) \right) \tag{24}$$

where, for $\delta = \hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$,

$$\begin{aligned} A_\delta(\mathbf{r}) \quad &= \quad \frac{1}{2} \Bigg(\left(1 + \frac{1}{2}\chi_{\mathbf{r}} + \frac{3}{2}\chi_{\mathbf{r}+\delta}\right) \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r},\mathbf{r}+\delta) M_{bc}(\mathbf{r}+\delta) \psi_{\mathbf{r}+\delta,c} \\ &- \quad \left(1 + \frac{1}{2}\chi_{\mathbf{r}+\delta} + \frac{3}{2}\chi_{\mathbf{r}}\right) \psi^\dagger_{\mathbf{r}+\delta,a} T^*_{ba}(\mathbf{r},\mathbf{r}+\delta) M_{bc}(\mathbf{r}) \psi_{\mathbf{r},c} \Bigg) - H.c. \end{aligned} \tag{25}$$

$$
\begin{aligned}
B_\delta(\mathbf{r}) &= \frac{1}{2}\left(\sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}+\delta+\delta'}\right)\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)T_{bc}(\mathbf{r}+\delta,\mathbf{r}+\delta+\delta')\psi_{\mathbf{r}+\delta+\delta',c}\right. \\
&- \left.\sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^\dagger_{\mathbf{r}+\delta,c}T^*_{ab}(\mathbf{r}-\delta',\mathbf{r})T^*_{bc}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}-\delta',a}\right) - H.c. \quad (26)
\end{aligned}
$$

$$
\begin{aligned}
C_{\hat{\mathbf{x}}}(\mathbf{r}) &= \frac{1}{2}\left(\left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}\right)\psi^\dagger_{\mathbf{r}+\hat{\mathbf{y}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}})T^*_{ab}(\mathbf{r}+\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{x}},a}\right. \\
&- \left.\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}\right)\psi^\dagger_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a}T^*_{cb}(\mathbf{r},\mathbf{r}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{y}})\psi_{\mathbf{r},c}\right) - H.c. \quad (27) \\
C_{\hat{\mathbf{y}}}(\mathbf{r}) &= \left(\left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}\right)\psi^\dagger_{\mathbf{r}+\hat{\mathbf{x}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}})T^*_{ab}(\mathbf{r}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}},a}\right. \\
&- \left.\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}}\right)\psi^\dagger_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}},a}T^*_{cb}(\mathbf{r},\mathbf{r}-\hat{\mathbf{x}})T_{ab}(\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}})\psi_{\mathbf{r},c}\right) - H.c. \quad (28)
\end{aligned}
$$

The expression for $\tilde{\mathbf{j}}_E(\mathbf{r})$ contains a term which is $\chi_{\mathbf{r}}$-independent and a term which is linear in $\chi_{\mathbf{r}}$. Therefore, the response consists of **two** terms: the first will involve the usual linear response (i.e. Kubo formula), and the second term, coming from the linear in $\chi_{\mathbf{r}}$ contribution to the thermal current, which acquires a ground state expectation value and cancels a significant part of the response coming from the χ-independent part of the current.

We can define the one-particle Hamiltonian operator using the Heisenberg equations of motion:

$$
\begin{aligned}
i\hbar\frac{\partial\psi_{\mathbf{r}}}{\partial t} = [\psi_{\mathbf{r}},H] \equiv \hat{H}\psi &= \sum_{\mathbf{r}'}H_{\mathbf{r}\mathbf{r}'}\psi_{\mathbf{r}'} \quad (29) \\
&= M_{ab}(\mathbf{r})\psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}}\left(T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b}+T^*_{ba}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,b}\right) \\
&= M_{ab}(\mathbf{r})\psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}}\left(T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b}+T_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,b}\right). \quad (30)
\end{aligned}
$$

A.1 Particle number current

From the Eqns.(24)-(28) we can read off the particle number current, since we could have substituted $\psi^\dagger_{\mathbf{r},a}\psi_{\mathbf{r},a}$ (instead of $h_{\mathbf{r}}$) into the Heisenberg equation of motion, and use the continuity equation, to obtain the particle number current. The part of $h_{\mathbf{r}}$ which contains $M_{ab}(\mathbf{r})$ can be used for this purpose if we temporarily set $M_{ab}(\mathbf{r}) = \delta_{ab}$. Therefore, from the expression for $A_\delta(\mathbf{r})$ we have

$$
j_\delta(\mathbf{r}) = \frac{i}{\hbar}\left(\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} - \psi^\dagger_{\mathbf{r}+\delta,a}T^*_{ba}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},b}\right). \quad (31)
$$

Note that we would obtain exactly the same result for $j_\delta(\mathbf{r})$ if we gauged the fields $\psi_{\mathbf{r},a} \to e^{i\theta_{\mathbf{r}}}\psi_{\mathbf{r},a}$ and then took the functional derivative with respect to the link differences in θ:

$$
j_\delta(\mathbf{r}) = \frac{1}{\hbar}\frac{\delta H}{\delta(\theta_{\mathbf{r}+\delta}-\theta_{\mathbf{r}})}. \quad (32)
$$

Since we will be dealing with spatial averages of currents (in order to cancel out the contribution from the magnetization currents), it is useful to define the velocity operator as

$$
\begin{aligned}
\sum_{\mathbf{r}} j_\delta(\mathbf{r}) &= \frac{i}{\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r},a}\left(T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} - T_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,b}\right) \quad (33) \\
&\equiv \sum_{\mathbf{r},\mathbf{r}'}\psi^\dagger_{\mathbf{r}}V^\delta_{\mathbf{r}\mathbf{r}'}\psi_{\mathbf{r}'}. \quad (34)
\end{aligned}
$$

A.2 Energy current

Using the above definitions of the one-particle Hamiltonian operator, $H_{\mathbf{rr}'}$ (Eq.30), and the one particle velocity operator $V^{\delta}_{\mathbf{rr}'}$ (Eq.33), we can write the spatial average of the energy current operator as

$$\begin{aligned}\sum_{\mathbf{r}} \tilde{j_{E\delta}}(\mathbf{r}) &= \frac{1}{2}\sum_{\mathbf{rr}'\mathbf{r}''} \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c \\ &+ \frac{1}{4}\sum_{\mathbf{rr}'\mathbf{r}''} \psi^{\dagger}_{\mathbf{r}}\chi_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c \\ &+ \frac{1}{2}\sum_{\mathbf{rr}'\mathbf{r}''} \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'}\chi_{\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c \\ &+ \frac{1}{4}\sum_{\mathbf{rr}'\mathbf{r}''} \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\chi_{\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\end{aligned} \tag{35}$$

The technical details are shown in the Appendix E.

B Magnetization currents and 'diathermal' contribution to response: Invalidity of Kubo formula for the thermal Hall transport

As mentioned, the expression for $\tilde{\mathbf{j}}_E(\mathbf{r})$ contains a term which is $\chi_{\mathbf{r}}$-independent and a term which is linear in $\chi_{\mathbf{r}}$. Therefore,

$$\begin{aligned}\left\langle \sum_{\mathbf{r}} \tilde{j_{E\delta}}(\mathbf{r})\right\rangle &= \frac{1}{2}\left\langle \sum_{\mathbf{rr}'\mathbf{r}''} \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c\right\rangle \\ &+ \frac{1}{4}\left\langle \sum_{\mathbf{rr}'\mathbf{r}''} \psi^{\dagger}_{\mathbf{r}}\chi_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + 2\psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'}\chi_{\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\chi_{\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\right\rangle\end{aligned} \tag{36}$$

where the averages are to be performed with respect to the ensable given by the density matrix of the Hamiltonian: $H + e^{st}\sum_{\mathbf{r}}\chi_{\mathbf{r}} h_{\mathbf{r}}$, where the perturbation is turned on adiabatically in the distant past (i.e. $s \to 0^{+}$).

For periodic boundary conditions, the first term on the right hand side of Eq.(36) vanishes when $\chi_{\mathbf{r}} = 0$. Non-zero expectation value is obtained in the first order in time-dependent perturbation theory, which is obtained by Kubo formula.

The expectation value of the second term on the right hand side of Eq.(36) vanishes when $\chi_{\mathbf{r}}$ is $\mathbf{r}$-independent (for the same reason as the first term at $\chi_{\mathbf{r}} = 0$), but for general $\chi_{\mathbf{r}}$, it does not vanish. It is analogous to the 'diamagnetic' contribution to the current operator and is sometimes called the 'diathermal' current. It is important in that it significantly changes the final answer for the response, compared to the one obtained just from the first, Kubo-like, term.

$$\psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'} H_{\mathbf{r}'\mathbf{r}''}\chi_{\mathbf{r}''}\psi_{\mathbf{r}''} = \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'}\chi_{\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + \psi^{\dagger}_{\mathbf{r}} V^{\delta}_{\mathbf{rr}'}\left(H_{\mathbf{r}'\mathbf{r}''}\chi_{\mathbf{r}''} - \chi_{\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\right)\psi_{\mathbf{r}''} \tag{37}$$

Unlike Smrcka and Streda, I would like to work with space periodic operators; therefore let $\chi_{\mathbf{r}} = \chi e^{i\mathbf{q}\cdot\mathbf{r}}$. We will be interested in taking the limit of $\mathbf{q} \to 0$ at the end of the calculation, but would like to keep this form throughout in order to maintain periodic boundary conditions.

$$e^{i\mathbf{q}\cdot\mathbf{r}} H_{\mathbf{rr}'} - H_{\mathbf{rr}'} e^{i\mathbf{q}\cdot\mathbf{r}'} \equiv \left[e^{i\mathbf{q}\cdot\mathbf{r}}, \hat{H}\right] \tag{38}$$

$$
\begin{aligned}
[\psi_{\mathbf{r},a}, H] &= M_{ab}(\mathbf{r})\psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}} \left(T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + T^*_{ba}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,b}\right) \\
&= M_{ab}(\mathbf{r})\psi_{\mathbf{r},b} + \sum_{\delta=\hat{x},\hat{y}} \left(T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + T_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,b}\right) \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
& e^{i\mathbf{q}\cdot\mathbf{r}} H_{\mathbf{r}\mathbf{r}'}\psi_{\mathbf{r}'} - H_{\mathbf{r}\mathbf{r}'} e^{i\mathbf{q}\cdot\mathbf{r}'}\psi_{\mathbf{r}'} = \\
& e^{i\mathbf{q}\cdot\mathbf{r}} M_{ab}(\mathbf{r})\psi_{\mathbf{r},b} + e^{i\mathbf{q}\cdot\mathbf{r}} \sum_{\delta=\hat{x},\hat{y}} \left(T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + T_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,b}\right) \\
-\; & e^{i\mathbf{q}\cdot\mathbf{r}} M_{ab}(\mathbf{r})\psi_{\mathbf{r},b} - \sum_{\delta=\hat{x},\hat{y}} \left(e^{i\mathbf{q}\cdot(\mathbf{r}+\delta)} T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + e^{i\mathbf{q}\cdot(\mathbf{r}-\delta)} T_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,b}\right) \\
=\; & e^{i\mathbf{q}\cdot\mathbf{r}} \sum_{\delta=\hat{x},\hat{y}} \left(\left(1-e^{i\mathbf{q}\cdot\delta}\right) T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b} + \left(1-e^{-i\mathbf{q}\cdot\delta}\right) T_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,b}\right) \qquad (40)
\end{aligned}
$$

In the limit of $\mathbf{q} \to 0$, either of the terms in the parenthesis can be written as

$$
(1 - e^{\pm i\mathbf{q}\cdot\delta}) \approx \mp i\mathbf{q}\cdot\delta + \mathcal{O}(q^2)
$$

Therefore to linear order in $\mathbf{q}$, the overall term $e^{i\mathbf{q}\cdot\mathbf{r}}$ can be dropped. So

$$
\left[e^{i\mathbf{q}\cdot\mathbf{r}}, \hat{H}\right] = -\hbar q_\mu \hat{V}^\mu + \mathcal{O}(q^2). \qquad (41)
$$

This ensures that we are working with periodic expressions.

Returning to the expectation value of the energy current:

$$
\begin{aligned}
\left\langle \sum_{\mathbf{r}} \tilde{j}_{E\mu}(\mathbf{r}) \right\rangle &= \frac{1}{2} \left\langle \sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} V^\mu_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c \right\rangle \\
&+ \frac{1}{4} \left\langle \sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} \chi_{\mathbf{r}} V^\mu_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + 3\psi^\dagger_{\mathbf{r}} V^\mu_{\mathbf{r}\mathbf{r}'} \chi_{\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c. \right\rangle \\
&+ \frac{1}{4} \left\langle \sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \hbar q_\nu \chi \left(\psi^\dagger_{\mathbf{r}} V^\mu_{\mathbf{r}\mathbf{r}'} V^\nu_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} - \psi^\dagger_{\mathbf{r}} V^\nu_{\mathbf{r}\mathbf{r}'} V^\mu_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''}\right) \right\rangle \qquad (42)
\end{aligned}
$$

Expanding the $\psi_{\mathbf{r}}$ fields in the eigenstates of $\hat{H}$, we find

$$
\begin{aligned}
\left\langle \sum_{\mathbf{r}} \tilde{j}_{E\mu}(\mathbf{r}) \right\rangle &= \frac{1}{2} \left\langle \sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} V^\mu_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c \right\rangle \\
&+ \sum_M \epsilon_M n_F(\epsilon_M) \langle M | \chi_{\mathbf{r}} V^\mu + V^\mu \chi_{\mathbf{r}} | M \rangle \\
&+ (-iq_\nu \chi) \frac{i\hbar}{4} \sum_{m,n,\mathbf{k}} \frac{1}{e^{\beta\epsilon_m(\mathbf{k})} + 1} \left(V^\mu_{mn}(\mathbf{k}) V^\nu_{nm}(\mathbf{k}) - V^\nu_{mn}(\mathbf{k}) V^\mu_{nm}(\mathbf{k})\right). \qquad (43)
\end{aligned}
$$

B.1 Kubo contribution to the response

Standard first order time-dependent perturbation theory gives

$$
\langle \mathcal{O}_2 \rangle(t) = -\frac{i}{\hbar} \int_{-\infty}^{t} dt' e^{st'} \operatorname{Tr}\left(\hat{\rho}_0 \left[\mathcal{O}_2^{(H)}(t), \mathcal{O}_1^{(H)}(t')\right]\right) \qquad (44)
$$

where the factor e^{st} assures that the perturbation has been turned on slowly in the distant past, and

$$\begin{aligned} \mathcal{O}_i^{(H)}(t) &= e^{\frac{i}{\hbar}Ht}\mathcal{O}_i e^{-\frac{i}{\hbar}Ht} && (45)\\ \hat{\rho}_0 &= \frac{e^{-\beta H}}{\mathrm{Tr}\left(e^{-\beta H}\right)}. && (46)\end{aligned}$$

For our purposes,

$$\begin{aligned} \mathcal{O}_1 &= \sum_{\mathbf{r}} \chi_{\mathbf{r}} h_{\mathbf{r}} && (47)\\ \mathcal{O}_2 &= \frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi_{\mathbf{r}}^\dagger V_{\mathbf{r}\mathbf{r}'}^{\mu} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c. && (48)\end{aligned}$$

In order to remain in the 'rapid' regime described by Luttinger, we need to take $\mathbf{q} \to 0$ limit before we take the $s \to 0$ limit.

Now, using the cyclic property of trace

$$\mathrm{Tr}\left(A\left[B,C\right]\right) = \mathrm{Tr}\left(ABC - ACB\right) = \mathrm{Tr}\left(CAB - ACB\right) = \mathrm{Tr}\left(\left[C,A\right]B\right) \quad (49)$$

we can write

$$\langle \mathcal{O}_2\rangle(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} dt' e^{st'}\,\mathrm{Tr}\left(\left[\mathcal{O}_1^{(H)}(t'),\hat{\rho}_0\right]\mathcal{O}_2^{(H)}(t)\right). \quad (50)$$

Note that

$$\begin{aligned} \hat{\rho}_0\int_0^\beta d\lambda e^{\lambda H}\left[\mathcal{O}_1^{(H)}(t'),H\right]e^{-\lambda H} &= -\hat{\rho}_0\int_0^\beta d\lambda \frac{d}{d\lambda}\left(e^{\lambda H}\mathcal{O}_1^{(H)}(t')e^{-\lambda H}\right)\\ &= -\hat{\rho}_0\left(e^{\beta H}\mathcal{O}_1^{(H)}(t')e^{-\beta H} - \mathcal{O}_1^{(H)}(t')\right)\\ &= -\mathcal{O}_1^{(H)}(t')\hat{\rho}_0 + \hat{\rho}_0\mathcal{O}_1^{(H)}(t')\\ &= \left[\hat{\rho}_0,\mathcal{O}_1^{(H)}(t')\right]. && (51)\end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathcal{O}_2\rangle(t) &= -\frac{i}{\hbar}\int_0^\beta d\lambda\int_{-\infty}^{t} dt' e^{st'}\,\mathrm{Tr}\left(\hat{\rho}_0 e^{\lambda H}\left[H,\mathcal{O}_1^{(H)}(t')\right]e^{-\lambda H}\mathcal{O}_2^{(H)}(t)\right)\\ &= -\frac{i}{\hbar}\int_0^\beta d\lambda\int_{-\infty}^{t} dt' e^{st'}\,\mathrm{Tr}\left(\hat{\rho}_0 e^{\lambda H}\left[H,\sum_{\mathbf{r}}\chi_{\mathbf{r}} h_{\mathbf{r}}^{(H)}(t')\right]e^{-\lambda H}\mathcal{O}_2^{(H)}(t)\right)\\ &= -\frac{i}{\hbar}\int_0^\beta d\lambda\int_{-\infty}^{t} dt' e^{st'}\,\mathrm{Tr}\left(\hat{\rho}_0 e^{\lambda H}\sum_{\mathbf{r}}\chi_{\mathbf{r}} i\hbar\nabla\cdot\mathbf{j}_E^{(H)}(\mathbf{r},t')e^{-\lambda H}\mathcal{O}_2^{(H)}(t)\right) && (52)\end{aligned}$$

Using the following identity for the sum over the lattice divergence

$$\begin{aligned} \sum_{\mathbf{r}}\chi_{\mathbf{r}}\nabla\cdot\mathbf{j}_E^{(H)}(\mathbf{r},t') &= \sum_{\mathbf{r}}\chi_{\mathbf{r}}\left(j_{E_x}^{(H)}(\mathbf{r},t') - j_{E_x}^{(H)}(\mathbf{r}-\hat{\mathbf{x}},t') + j_{E_y}^{(H)}(\mathbf{r},t') - j_{E_y}^{(H)}(\mathbf{r}-\hat{\mathbf{y}},t')\right)\\ &= -\sum_{\mathbf{r}}\left(j_{E_x}^{(H)}(\mathbf{r},t')\left(\chi_{\mathbf{r}+\hat{\mathbf{x}}}-\chi_{\mathbf{r}}\right) + j_{E_y}^{(H)}(\mathbf{r},t')\left(\chi_{\mathbf{r}+\hat{\mathbf{y}}}-\chi_{\mathbf{r}}\right)\right) && (53)\\ &\to -iq_\nu\chi\sum_{\mathbf{r}} j_{E_\nu}^{(H)}(\mathbf{r},t'). && (54)\end{aligned}$$

$$\langle \mathcal{O}_2\rangle(t) = -iq_\nu\chi\int_0^\beta d\lambda\int_{-\infty}^{t} dt' e^{st'}\,\mathrm{Tr}\left(\hat{\rho}_0 e^{\lambda H}\sum_{\mathbf{r}} j_{E_\nu}^{(H)}(\mathbf{r},t')e^{-\lambda H}\mathcal{O}_2^{(H)}(t)\right) \quad (55)$$

where

$$\sum_{\mathbf{r}} j_{E_\nu}^{(H)}(\mathbf{r},t') = e^{\frac{i}{\hbar}Ht'}\left(\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}}^\dagger V_{\mathbf{r}\mathbf{r}'}^{\nu} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\right)e^{-\frac{i}{\hbar}Ht'}. \tag{56}$$

Therefore, at $t=0$ we find:

$$\begin{aligned}\left\langle \frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}}^\dagger V_{\mathbf{r}\mathbf{r}'}^{\mu} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\right\rangle &= -iq_\nu\chi\int_0^\beta d\lambda\int_{-\infty}^0 dt' e^{st'}\,\mathrm{Tr}\left(\hat\rho_0 e^{\lambda H}\sum_{\mathbf{r}'} j_{E_\nu}^{(H)}(\mathbf{r}',t')e^{-\lambda H}\sum_{\mathbf{r}} j_{E_\mu}^{(H)}(\mathbf{r},0)\right)\\ &= -iq_\nu\chi\int_0^\beta d\lambda\int_0^\infty dt' e^{-st'}\,\mathrm{Tr}\left(\hat\rho_0 e^{\lambda H}\sum_{\mathbf{r}'} j_{E_\nu}^{(H)}(\mathbf{r}',-t')e^{-\lambda H}\sum_{\mathbf{r}} j_{E_\mu}^{(H)}(\mathbf{r},0)\right)\end{aligned} \tag{57}$$

Now,

$$\begin{aligned}\sum_{\mathbf{r}} j_{E_\nu}^{(H)}(\mathbf{r},-t') &= e^{-\frac{i}{\hbar}Ht'}\left(\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}}^\dagger V_{\mathbf{r}\mathbf{r}'}^{\nu} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\right)e^{\frac{i}{\hbar}Ht'} &(58)\\ &= \frac{1}{2}\sum_{m,n} d_m^\dagger d_n e^{-\frac{i}{\hbar}\epsilon_m t'}e^{\frac{i}{\hbar}\epsilon_n t'}\left(\epsilon_n+\epsilon_m\right)V_{mn}^{\nu} &(59)\end{aligned}$$

therefore

$$e^{\lambda H}\sum_{\mathbf{r}} j_{E_\nu}^{(H)}(\mathbf{r},-t')e^{-\lambda H} = \frac{1}{2}\sum_{m,n} d_m^\dagger d_n e^{\epsilon_m\left(\lambda-\frac{i}{\hbar}t'\right)}e^{\epsilon_n\left(\frac{i}{\hbar}t'-\lambda\right)}\left(\epsilon_n+\epsilon_m\right)V_{mn}^{\nu}. \tag{60}$$

And

$$\begin{aligned}&\left\langle \frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}}^\dagger V_{\mathbf{r}\mathbf{r}'}^{\mu} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\right\rangle = -iq_\nu\chi\int_0^\beta d\lambda\int_0^\infty dt' e^{-st'}\sum_{m,n}\sum_{m',n'}\frac{1}{4}\left(\epsilon_n+\epsilon_m\right)\left(\epsilon_{n'}+\epsilon_{m'}\right)\\ &\times\quad e^{(\epsilon_m-\epsilon_n)\left(\lambda-\frac{i}{\hbar}t'\right)}V_{mn}^{\nu}V_{m'n'}^{\mu}\,\mathrm{Tr}\left(\hat\rho_0 d_m^\dagger d_n d_{m'}^\dagger d_{n'}\right)\\ &=\quad -iq_\nu\chi\sum_{m,n}\sum_{m',n'}\frac{1}{4}\left(\epsilon_n+\epsilon_m\right)\left(\epsilon_{n'}+\epsilon_{m'}\right)\frac{-i\hbar\left(e^{\beta(\epsilon_m-\epsilon_n)}-1\right)}{(\epsilon_m-\epsilon_n)(\epsilon_m-\epsilon_n-i0^+)}V_{mn}^{\nu}V_{m'n'}^{\mu}\,\mathrm{Tr}\left(\hat\rho_0 d_m^\dagger d_n d_{m'}^\dagger d_{n'}\right)\end{aligned} \tag{61}$$

The terms with $m=n$ vanishes for the same reason as the ground state expectation value of the spatial average of the energy current vanishes. Therefore we have

$$\begin{aligned}&\left\langle \frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}}^\dagger V_{\mathbf{r}\mathbf{r}'}^{\mu} H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''} + H.c.\right\rangle = -iq_\nu\chi\sum_{m,n,\mathbf{k}}\frac{1}{4}\left(\epsilon_n+\epsilon_m\right)^2\\ &\times\quad\frac{-i\hbar\left(e^{\beta(\epsilon_m-\epsilon_n)}-1\right)}{(\epsilon_m-\epsilon_n)(\epsilon_m-\epsilon_n-i0^+)}V_{mn}^{\nu}V_{nm}^{\mu}\frac{1}{e^{\beta\epsilon_m}+1}\left(1-\frac{1}{e^{\beta\epsilon_n}+1}\right) &(62)\\ &=\quad -iq_\nu\chi\sum_{m,n}\frac{1}{4}\left(\epsilon_n+\epsilon_m\right)^2\frac{-i\hbar V_{nm}^{\mu}V_{mn}^{\nu}}{(\epsilon_m-\epsilon_n)(\epsilon_m-\epsilon_n-i0^+)}\left((1-f_m)f_n-f_m(1-f_n)\right) &(63)\\ &=\quad -iq_\nu\chi\sum_{m,n}\frac{-i\hbar}{4}\left(\epsilon_n+\epsilon_m\right)^2\frac{V_{nm}^{\mu}V_{mn}^{\nu}}{(\epsilon_n-\epsilon_m+i0^+)^2}\left(f_n-f_m\right). &(64)\end{aligned}$$

where $f_m = 1/\left(e^{\beta\epsilon_m}+1\right)$.

C Derivation of Smrčka-Středa formula

We have arrived to the following result in the previous sections:

$$\begin{aligned}\left\langle \sum_{\mathbf{r}} \tilde{j_{E\mu}}(\mathbf{r}) \right\rangle &= -iq_\nu \chi \sum_{m,n} \frac{-i\hbar}{4} (\epsilon_n + \epsilon_m)^2 \frac{V^\mu_{nm} V^\nu_{mn}}{(\epsilon_n - \epsilon_m + i0^+)^2} (f_n - f_m) \\ &+ \sum_m \epsilon_m f_m (\chi_{\mathbf{r}} V^\mu + V^\mu \chi_{\mathbf{r}})_{mm} \\ &+ (-iq_\nu \chi) \frac{i\hbar}{4} \sum_{m,n} f_n \left(V^\mu_{nm} V^\nu_{mn} - V^\nu_{nm} V^\mu_{mn} \right). \end{aligned} \tag{65}$$

Using

$$(\epsilon_n + \epsilon_m)^2 = \left(\epsilon_n - \epsilon_m \pm i0^+\right)^2 + 4\epsilon_m \epsilon_n + \mathcal{O}(0^+) \tag{66}$$

we see that the portion of the first term cancels the third term, and

$$\begin{aligned}\left\langle \sum_{\mathbf{r}} \tilde{j_{E\mu}}(\mathbf{r}) \right\rangle &= -iq_\nu \chi \sum_{m,n} (-i\hbar) \epsilon_n \epsilon_m \frac{V^\mu_{nm} V^\nu_{mn}}{(\epsilon_n - \epsilon_m + i0^+)^2} (f_n - f_m) \\ &+ \sum_m \epsilon_m f_m (\chi_{\mathbf{r}} V^\mu + V^\mu \chi_{\mathbf{r}})_{mm} \end{aligned} \tag{67}$$

$$\begin{aligned} &= -iq_\nu \chi \sum_{m,n} (-i\hbar) \epsilon_n \epsilon_m f_n \left(\frac{V^\mu_{nm} V^\nu_{mn}}{(\epsilon_n - \epsilon_m + i0^+)^2} - \frac{V^\mu_{mn} V^\nu_{nm}}{(\epsilon_n - \epsilon_m - i0^+)^2} \right) \\ &+ \int_{-\infty}^{\infty} d\eta \eta f(\eta) \mathrm{Tr} \left(\delta(\eta - \hat{H}) (\chi_{\mathbf{r}} V^\mu + V^\mu \chi_{\mathbf{r}}) \right) \end{aligned} \tag{68}$$

$$\begin{aligned} &= -iq_\nu \chi \sum_{m,n} (-i\hbar) \int_{-\infty}^{\infty} d\eta \eta f(\eta) \delta(\eta - \epsilon_n) \epsilon_m \left(\frac{V^\mu_{nm} V^\nu_{mn}}{(\eta - \epsilon_m + i0^+)^2} - \frac{V^\mu_{mn} V^\nu_{nm}}{(\eta - \epsilon_m - i0^+)^2} \right) \\ &+ \int_{-\infty}^{\infty} d\eta \eta f(\eta) \mathrm{Tr} \left(\delta(\eta - \hat{H}) (\chi_{\mathbf{r}} V^\mu + V^\mu \chi_{\mathbf{r}}) \right) \end{aligned} \tag{69}$$

When dealing with integrals of the form

$$\int_{-\infty}^{\infty} d\eta f(\eta) \ell(\eta) = \left[f(\xi) \int_{-\infty}^{\xi} d\eta \ell(\eta) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} d\xi \frac{df}{d\xi} \int_{-\infty}^{\xi} d\eta \ell(\eta) \tag{70}$$

$$= \int_{-\infty}^{\infty} d\xi \left(-\frac{df}{d\xi} \right) \int_{-\infty}^{\xi} d\eta \ell(\eta). \tag{71}$$

we see that it is sufficient to study the quantity $\int_{-\infty}^{\xi} d\eta \ell(\eta)$. Define the one particle Green's functions:

$$G_\pm(\eta) = \left(\eta - \hat{H} \pm i0^+ \right)^{-1} \tag{72}$$

So, we have:

$$\begin{aligned} &-iq_\nu \chi \sum_{m,n} (-i\hbar) \int_{-\infty}^{\xi} d\eta \eta \delta(\eta - \epsilon_n) \epsilon_m \left(\frac{V^\mu_{nm} V^\nu_{mn}}{(\eta - \epsilon_m + i0^+)^2} - \frac{V^\mu_{mn} V^\nu_{nm}}{(\eta - \epsilon_m - i0^+)^2} \right) \\ + &\int_{-\infty}^{\xi} d\eta \eta \mathrm{Tr} \left(\delta(\eta - \hat{H}) (\chi_{\mathbf{r}} V^\mu + V^\mu \chi_{\mathbf{r}}) \right) = \\ &-iq_\nu \chi (-i\hbar) \int_{-\infty}^{\xi} d\eta \eta \mathrm{Tr} \left(\delta(\eta - \hat{H}) \left(\hat{V}^\mu \hat{H} \left(-\frac{dG_+(\eta)}{d\eta} \right) \hat{V}^\nu - \hat{V}^\nu \hat{H} \left(-\frac{dG_-(\eta)}{d\eta} \right) \hat{V}^\mu \right) \right) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^{\mu}+V^{\mu}\chi_{\mathbf{r}}\right)\right) = \\
&-iq_{\nu}\chi(-i\hbar)\int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^{\mu}\left(-G_{+}(\eta)-\eta\frac{dG_{+}(\eta)}{d\eta}\right)\hat{V}^{\nu}-\hat{V}^{\nu}\left(-G_{-}(\eta)-\eta\frac{dG_{-}(\eta)}{d\eta}\right)\hat{V}^{\mu}\right)\right) \\
&+ \int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^{\mu}+V^{\mu}\chi_{\mathbf{r}}\right)\right) = \\
&-iq_{\nu}\chi\hbar\left(\int_{-\infty}^{\xi} d\eta\eta^{2}A(\eta)+\int_{-\infty}^{\xi} d\eta\eta B(\eta)\right)+\int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^{\mu}+V^{\mu}\chi_{\mathbf{r}}\right)\right). \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
A(\eta) &= i\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^{\mu}\frac{dG_{+}(\eta)}{d\eta}\hat{V}^{\nu}-\hat{V}^{\nu}\frac{dG_{-}(\eta)}{d\eta}\hat{V}^{\mu}\right)\right) \qquad (74)\\
B(\eta) &= i\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^{\mu}G_{+}(\eta)\hat{V}^{\nu}-\hat{V}^{\nu}G_{-}(\eta)\hat{V}^{\mu}\right)\right) \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\xi} d\eta\eta^{2}A(\eta)+\int_{-\infty}^{\xi} d\eta\eta B(\eta) &= \int_{-\infty}^{\xi} d\eta\eta^{2}A(\eta)+\left[\frac{\eta^{2}}{2}B(\eta)\right]_{-\infty}^{\xi}-\frac{1}{2}\int_{-\infty}^{\xi} d\eta\eta^{2}\frac{dB(\eta)}{d\eta} \\
&= \int_{-\infty}^{\xi} d\eta\eta^{2}A(\eta)+\frac{1}{2}\int_{-\infty}^{\xi} d\eta\left(\xi^{2}-\eta^{2}\right)\frac{dB(\eta)}{d\eta} \qquad (76)\\
&= \xi^{2}\int_{-\infty}^{\xi} d\eta A(\eta)+\int_{-\infty}^{\xi} d\eta\left(\eta^{2}-\xi^{2}\right)\left(A(\eta)-\frac{1}{2}\frac{dB(\eta)}{d\eta}\right) \qquad (77)
\end{aligned}
$$

where we used the fact that $B(\eta\to-\infty)$ vanishes (much) faster than η^{-2} due to the spectrum being bounded from below.

$$
\begin{aligned}
&-iq_{\nu}\chi\sum_{m,n}(-i\hbar)\int_{-\infty}^{\xi} d\eta\eta\delta(\eta-\epsilon_{n})\epsilon_{m}\left(\frac{V^{\mu}_{nm}V^{\nu}_{mn}}{(\eta-\epsilon_{m}+i0^{+})^{2}}-\frac{V^{\mu}_{mn}V^{\nu}_{nm}}{(\eta-\epsilon_{m}-i0^{+})^{2}}\right) \\
&+ \int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^{\mu}+V^{\mu}\chi_{\mathbf{r}}\right)\right) = \\
&-iq_{\nu}\chi\hbar\left(\xi^{2}\int_{-\infty}^{\xi} d\eta A(\eta)+\int_{-\infty}^{\xi} d\eta\left(\eta^{2}-\xi^{2}\right)\left(A(\eta)-\frac{1}{2}\frac{dB(\eta)}{d\eta}\right)\right) \\
&+ \int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^{\mu}+V^{\mu}\chi_{\mathbf{r}}\right)\right). \qquad (78)
\end{aligned}
$$

Use:

$$
\begin{aligned}
\hbar\left(A(\eta)-\frac{1}{2}\frac{dB(\eta)}{d\eta}\right) &= -\frac{i}{2q}\mathrm{Tr}\left(\left(e^{iqr_{\mu}}V^{\nu}-e^{iqr_{\nu}}V^{\mu}\right)\frac{d\delta(\eta-\hat{H})}{d\eta}\right) \qquad (79)\\
&= \frac{i}{2q}\mathrm{Tr}\left(\left(e^{iqr_{\nu}}V^{\mu}+V^{\mu}e^{iqr_{\nu}}\right)\frac{d\delta(\eta-\hat{H})}{d\eta}\right) \qquad (80)
\end{aligned}
$$

Integrating the second term in Eq.(81) by parts we have

$$
\begin{aligned}
&-iq_{\nu}\chi\sum_{m,n}(-i\hbar)\int_{-\infty}^{\xi} d\eta\eta\delta(\eta-\epsilon_{n})\epsilon_{m}\left(\frac{V^{\mu}_{nm}V^{\nu}_{mn}}{(\eta-\epsilon_{m}+i0^{+})^{2}}-\frac{V^{\mu}_{mn}V^{\nu}_{nm}}{(\eta-\epsilon_{m}-i0^{+})^{2}}\right) \\
&+ \int_{-\infty}^{\xi} d\eta\eta \mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^{\mu}+V^{\mu}\chi_{\mathbf{r}}\right)\right) =
\end{aligned}
$$

$$
\begin{aligned}
&-iq_\nu\chi\hbar\left(\xi^2\int_{-\infty}^{\xi}d\eta A(\eta)+\int_{-\infty}^{\xi}d\eta\left(\eta^2-\xi^2\right)\frac{i}{2\hbar q}\mathrm{Tr}\left(\left(e^{iqr_\nu}V^\mu+V^\mu e^{iqr_\nu}\right)\frac{d\delta(\eta-\hat{H})}{d\eta}\right)\right)\\
+&\int_{-\infty}^{\xi}d\eta\eta\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^\mu+V^\mu\chi_{\mathbf{r}}\right)\right)\\
=&\left(-iq_\nu\chi\hbar\xi^2\int_{-\infty}^{\xi}d\eta A(\eta)+\chi\int_{-\infty}^{\xi}d\eta\left(\eta^2-\xi^2\right)\frac{1}{2}\mathrm{Tr}\left(\left(e^{iqr_\nu}V^\mu+V^\mu e^{iqr_\nu}\right)\frac{d\delta(\eta-\hat{H})}{d\eta}\right)\right)\\
+&\int_{-\infty}^{\xi}d\eta\eta\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^\mu+V^\mu\chi_{\mathbf{r}}\right)\right)\\
=&\left(-iq_\nu\chi\hbar\xi^2\int_{-\infty}^{\xi}d\eta A(\eta)-\int_{-\infty}^{\xi}d\eta\eta\mathrm{Tr}\left(\left(\chi_{\mathbf{r}}V^\mu+V^\mu\chi_{\mathbf{r}}\right)\delta(\eta-\hat{H})\right)\right)\\
+&\int_{-\infty}^{\xi}d\eta\eta\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\chi_{\mathbf{r}}V^\mu+V^\mu\chi_{\mathbf{r}}\right)\right)\\
=&-iq_\nu\chi\hbar\xi^2\int_{-\infty}^{\xi}d\eta A(\eta) \qquad (81)
\end{aligned}
$$

So, we have

$$
\left\langle\sum_{\mathbf{r}}\tilde{j_{E}}_\mu(\mathbf{r})\right\rangle = -iq_\nu\chi\hbar\int_{-\infty}^{\infty}d\xi\xi^2\left(-\frac{df(\xi)}{d\xi}\right)\int_{-\infty}^{\xi}d\eta A(\eta) \qquad (82)
$$

where

$$
\begin{aligned}
A(\eta) &= i\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\hat{V}^\nu-\hat{V}^\nu\frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) &(83)\\
&= -i\sum_{m,n}\left(\delta(\eta-\epsilon_m)\left(\frac{V^\mu_{mn}V^\nu_{nm}}{\left(\epsilon_m-\epsilon_n+i0^+\right)^2}-V^\nu_{mn}\frac{1}{\left(\epsilon_m-\epsilon_n-i0^+\right)^2}V^\mu_{nm}\right)\right) &(84)\\
&= -i\sum_{m,n}\left(\delta(\eta-\epsilon_m)-\delta(\eta-\epsilon_n)\right)\frac{V^\mu_{mn}V^\nu_{nm}}{\left(\epsilon_m-\epsilon_n+i0^+\right)^2} &(85)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle\sum_{\mathbf{r}}\tilde{j_{E}}_\mu(\mathbf{r})\right\rangle &= -iq_\nu\chi\hbar\int_{-\infty}^{\infty}d\xi\xi^2\left(-\frac{df(\xi)}{d\xi}\right)\left(-i\sum_{m,n}\left(\theta(\xi-\epsilon_m)-\theta(\xi-\epsilon_n)\right)\frac{V^\mu_{mn}V^\nu_{nm}}{\left(\epsilon_m-\epsilon_n+i0^+\right)^2}\right)\\
&= -iq_\nu\chi\hbar\int_{-\infty}^{\infty}d\xi\xi^2\left(-\frac{df(\xi)}{d\xi}\right)\left(-i\sum_{\epsilon_m<\xi<\epsilon_n}\frac{V^\mu_{mn}V^\nu_{nm}}{\left(\epsilon_m-\epsilon_n+i0^+\right)^2}+i\sum_{\epsilon_n<\xi<\epsilon_m}\frac{V^\mu_{mn}V^\nu_{nm}}{\left(\epsilon_m-\epsilon_n+i0^+\right)^2}\right)\\
&= -iq_\nu\chi\frac{\hbar}{i}\int_{-\infty}^{\infty}d\xi\xi^2\left(-\frac{df(\xi)}{d\xi}\right)\sum_{\epsilon_m<\xi<\epsilon_n}\left(\frac{V^\mu_{mn}V^\nu_{nm}}{\left(\epsilon_m-\epsilon_n+i0^+\right)^2}-\frac{V^\nu_{mn}V^\mu_{nm}}{\left(\epsilon_m-\epsilon_n-i0^+\right)^2}\right)
\end{aligned}
\qquad (86)
$$

For the vortex lattice

$$
\left\langle\sum_{\mathbf{r}}\tilde{j_{E}}_\mu(\mathbf{r})\right\rangle =
$$

$$
-iq_\nu\chi\frac{\hbar}{i}\int_{-\infty}^{\infty}d\xi\xi^2\left(-\frac{df(\xi)}{d\xi}\right)\frac{1}{\hbar^2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\sum_{\epsilon_m<\xi<\epsilon_n}\frac{\left\langle m\mathbf{k}|\frac{\partial H}{\partial k_\mu}|n\mathbf{k}\right\rangle\left\langle n\mathbf{k}|\frac{\partial H}{\partial k_\nu}|m\mathbf{k}\right\rangle-\left\langle m\mathbf{k}|\frac{\partial H}{\partial k_\nu}|n\mathbf{k}\right\rangle\left\langle n\mathbf{k}|\frac{\partial H}{\partial k_\mu}|m\mathbf{k}\right\rangle}{\left(\epsilon_m(\mathbf{k})-\epsilon_n(\mathbf{k})\right)^2}
\qquad (87)
$$

Now,

$$-iq\chi = -\nabla\chi \to T\nabla\frac{1}{T}$$

So,

$$\left\langle \sum_{\mathbf{r}} \tilde{j}_{E\mu}(\mathbf{r}) \right\rangle =$$

$$\frac{-\nabla_\nu T}{T}\frac{1}{\hbar}\int_{-\infty}^{\infty} d\xi\xi^2\left(-\frac{df(\xi)}{d\xi}\right)\frac{1}{i}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\sum_{\epsilon_m<\xi<\epsilon_n}\frac{\left\langle m\mathbf{k}|\frac{\partial H}{\partial k_\mu}|n\mathbf{k}\right\rangle\left\langle n\mathbf{k}|\frac{\partial H}{\partial k_\nu}|m\mathbf{k}\right\rangle - \left\langle m\mathbf{k}|\frac{\partial H}{\partial k_\nu}|n\mathbf{k}\right\rangle\left\langle n\mathbf{k}|\frac{\partial H}{\partial k_\mu}|m\mathbf{k}\right\rangle}{\left(\epsilon_m(\mathbf{k})-\epsilon_n(\mathbf{k})\right)^2} \tag{88}$$

C.1 Proof of the identity for $A(\eta) - \frac{1}{2}\frac{dB(\eta)}{d\eta}$

The crucial identity, which needs to be proved, and which was used in the above derivation is

$$\hbar\left(A(\eta) - \frac{1}{2}\frac{dB(\eta)}{d\eta}\right) = -\frac{i}{2q}\mathrm{Tr}\left(\left(e^{iqr_\mu}V^\nu - e^{iqr_\nu}V^\mu\right)\frac{d\delta(\eta-\hat{H})}{d\eta}\right). \tag{89}$$

Start with,

$$\begin{aligned} A(\eta) - \frac{1}{2}\frac{dB(\eta)}{d\eta} &= \frac{i}{2}\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\hat{V}^\nu - \hat{V}^\nu\frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) \\ &- \frac{i}{2}\mathrm{Tr}\left(\frac{\delta(\eta-\hat{H})}{d\eta}\left(\hat{V}^\mu G_+(\eta)\hat{V}^\nu - \hat{V}^\nu G_-(\eta)\hat{V}^\mu\right)\right) \end{aligned} \tag{90}$$

Consider the second term and use the fact that for vanishingly small q, the following identity holds:

$$\hat{V}^\mu = \frac{1}{\hbar q}\left[e^{iqr_\mu}, G_+^{-1}(\eta)\right] = \frac{1}{\hbar q}\left[e^{iqr_\mu}, G_-^{-1}(\eta)\right]. \tag{91}$$

Then,

$$\begin{aligned} A(\eta) - \frac{1}{2}\frac{dB(\eta)}{d\eta} &= \frac{i}{2}\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\hat{V}^\nu - \hat{V}^\nu\frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) \\ - &\frac{i}{2\hbar q}\mathrm{Tr}\left(\frac{\delta(\eta-\hat{H})}{d\eta}\left(\left(e^{iqr_\mu}G_+^{-1}(\eta) - G_+^{-1}(\eta)e^{iqr_\mu}\right)G_+(\eta)\hat{V}^\nu - \left(e^{iqr_\nu}G_-^{-1}(\eta) - G_-^{-1}(\eta)e^{iqr_\nu}\right)G_-(\eta)\hat{V}^\mu\right)\right) \\ = &\frac{i}{2}\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\hat{V}^\nu - \hat{V}^\nu\frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) \\ - &\frac{i}{2\hbar q}\mathrm{Tr}\left(\frac{\delta(\eta-\hat{H})}{d\eta}\left(e^{iqr_\mu}\hat{V}^\nu - e^{iqr_\nu}\hat{V}^\mu\right)\right) \\ + &\frac{i}{2\hbar q}\mathrm{Tr}\left(\frac{\delta(\eta-\hat{H})}{d\eta}\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\right) \end{aligned} \tag{92}$$

Therefore, we need to show that

$$\begin{aligned} 0 &= \frac{i}{2}\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\hat{V}^\nu - \hat{V}^\nu\frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) \\ &+ \frac{i}{2\hbar q}\mathrm{Tr}\left(\frac{\delta(\eta-\hat{H})}{d\eta}\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\right). \end{aligned} \tag{93}$$

Now we use,

$$\delta(\eta - \hat{H}) = \frac{i}{2\pi}\left(G_+(\eta) - G_-(\eta)\right) \tag{94}$$

$$\begin{aligned}
& \frac{i}{2}\mathrm{Tr}\left(\delta(\eta-\hat{H})\left(\hat{V}^\mu \frac{dG_+(\eta)}{d\eta}\hat{V}^\nu - \hat{V}^\nu \frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) \\
+ & \frac{i}{2\hbar q}\mathrm{Tr}\left(\frac{\delta(\eta-\hat{H})}{d\eta}\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\right) \\
= & -\frac{1}{4\pi}\mathrm{Tr}\left(\left(G_+(\eta)-G_-(\eta)\right)\left(\hat{V}^\mu \frac{dG_+(\eta)}{d\eta}\hat{V}^\nu - \hat{V}^\nu \frac{dG_-(\eta)}{d\eta}\hat{V}^\mu\right)\right) \\
- & \frac{1}{4\pi\hbar q}\mathrm{Tr}\left(\left(\frac{dG_+(\eta)}{d\eta} - \frac{dG_-(\eta)}{d\eta}\right)\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\right) \\
= & -\frac{1}{4\pi}\mathrm{Tr}\left(\hat{V}^\nu\left(G_+(\eta)-G_-(\eta)\right)\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\right) \\
- & \frac{1}{4\pi\hbar q}\mathrm{Tr}\left(\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\frac{dG_+(\eta)}{d\eta}\right) \\
+ & \frac{1}{4\pi}\mathrm{Tr}\left(\hat{V}^\mu\left(G_+(\eta)-G_-(\eta)\right)\hat{V}^\nu\frac{dG_-(\eta)}{d\eta}\right) \\
+ & \frac{1}{4\pi\hbar q}\mathrm{Tr}\left(\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\frac{dG_-(\eta)}{d\eta}\right).
\end{aligned} \tag{95}$$

It is enough to show that the first four terms in the last equation vanish since the remaining four are related to it by $\mu \leftrightarrow \nu$ and $+ \leftrightarrow -$.

Therefore,

$$\begin{aligned}
& -\mathrm{Tr}\left(\hat{V}^\nu\left(G_+(\eta)-G_-(\eta)\right)\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\right) - \frac{1}{\hbar q}\mathrm{Tr}\left(\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu - G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\right)\frac{dG_+(\eta)}{d\eta}\right) \\
= & -\frac{1}{\hbar q}\mathrm{Tr}\left(\hat{V}^\nu G_+(\eta)\left(e^{iqr_\mu}G_+^{-1} - G_+^{-1}e^{iqr_\mu}\right)\frac{dG_+(\eta)}{d\eta}\right) + \frac{1}{\hbar q}\mathrm{Tr}\left(\left(e^{iqr_\nu}G_-^{-1} - G_-^{-1}e^{iqr_\nu}\right)G_-(\eta)\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\right) \\
- & \frac{1}{\hbar q}\mathrm{Tr}\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu\frac{dG_+(\eta)}{d\eta}\right) + \frac{1}{\hbar q}\mathrm{Tr}\left(G_-^{-1}(\eta)e^{iqr_\nu}G_-(\eta)\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\right)
\end{aligned} \tag{96}$$

$$\begin{aligned}
= & -\frac{1}{\hbar q}\mathrm{Tr}\left(\hat{V}^\nu G_+(\eta)e^{iqr_\mu}G_+^{-1}\frac{dG_+(\eta)}{d\eta}\right) + \frac{1}{\hbar q}\mathrm{Tr}\left(\hat{V}^\nu e^{iqr_\mu}\frac{dG_+(\eta)}{d\eta}\right) \\
+ & \frac{1}{\hbar q}\mathrm{Tr}\left(e^{iqr_\nu}\hat{V}^\mu\frac{dG_+(\eta)}{d\eta}\right) - \frac{1}{\hbar q}\mathrm{Tr}\left(G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\hat{V}^\nu\frac{dG_+(\eta)}{d\eta}\right)
\end{aligned} \tag{97}$$

Since, $\frac{d}{d\eta}\left(GG^{-1}\right) = 0$, we have $\frac{dG_+}{d\eta} = -G_+^2$. Continuing,

$$\begin{aligned}
\ldots = & \frac{1}{\hbar q}\mathrm{Tr}\left(\hat{V}^\nu G_+(\eta)e^{iqr_\mu}G_+(\eta)\right) - \frac{1}{\hbar q}\mathrm{Tr}\left(G_+(\eta)\hat{V}^\nu e^{iqr_\mu}G_+(\eta)\right) \\
- & \frac{1}{\hbar q}\mathrm{Tr}\left(G_+(\eta)e^{iqr_\nu}\hat{V}^\mu G_+(\eta)\right) + \frac{1}{\hbar q}\mathrm{Tr}\left(e^{iqr_\mu}G_+(\eta)\hat{V}^\nu G_+(\eta)\right)
\end{aligned} \tag{98}$$

Substituting again

$$\hat{V}^\mu = \frac{1}{\hbar q}\left[e^{iqr_\mu}, G_+^{-1}(\eta)\right]. \tag{99}$$

and noting that the first and fourth terms are the same, we find

$$\ldots = \frac{2}{\hbar^2 qq'}\mathrm{Tr}\left(\left(e^{iq'r_\nu}G_+^{-1}(\eta) - G_+^{-1}(\eta)e^{iq'r_\nu}\right)G_+(\eta)e^{iqr_\mu}G_+(\eta)\right)$$

$$
\begin{aligned}
&- \quad \frac{1}{\hbar^2 qq'} \mathrm{Tr}\left(G_+(\eta)\left(e^{iq'r_\nu}G_+^{-1}(\eta) - G_+^{-1}(\eta)e^{iq'r_\nu}\right)e^{iqr_\mu}G_+(\eta)\right) \\
&- \quad \frac{1}{\hbar^2 qq'} \mathrm{Tr}\left(G_+(\eta)e^{iqr_\nu}\left(e^{iq'r_\mu}G_+^{-1}(\eta) - G_+^{-1}(\eta)e^{iq'r_\mu}\right)G_+(\eta)\right) \\
&= \quad -\frac{1}{\hbar^2 qq'} \mathrm{Tr}\left(G_+(\eta)e^{iq'r_\nu}G_+^{-1}(\eta)e^{iqr_\mu}G_+(\eta)\right) \\
&+ \quad \frac{1}{\hbar^2 qq'} \mathrm{Tr}\left(G_+(\eta)e^{iqr_\nu}G_+^{-1}(\eta)e^{iq'r_\mu}G_+(\eta)\right) \\
&+ \quad \frac{1}{\hbar^2 qq'} \mathrm{Tr}\left(e^{iq'r_\nu}e^{iqr_\mu}G_+(\eta)\right) - \frac{1}{\hbar^2 qq'} \mathrm{Tr}\left(G_+(\eta)e^{iqr_\nu}e^{iq'r_\mu}\right)
\end{aligned} \tag{100}
$$

Because q and q' are dummy variables, the above vanishes.

D Technical details of the current derivation

$$
\begin{aligned}
&\left[\tilde{H}, \tilde{h}_{\mathbf{r}}\right] = \\
&\sum_{\mathbf{r}'}\left[\left(\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'} + \sum_{\delta'}\left(\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'} + \tilde{\psi}^\dagger{}_{\mathbf{r}'+\delta',b'}T^*_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'}\right)\right),\right. \\
&\tilde{\psi}^\dagger{}_{\mathbf{r},a}M_{ab}(\mathbf{r})\tilde{\psi}_{\mathbf{r},b} + \frac{1}{2}\sum_{\delta}\left(\tilde{\psi}^\dagger{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b} + \tilde{\psi}^\dagger{}_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\right) \\
&\left.+\frac{1}{2}\sum_{\delta}\left(\tilde{\psi}^\dagger{}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r},b} + \tilde{\psi}^\dagger{}_{\mathbf{r},b}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r}-\delta,a}\right)\right]
\end{aligned} \tag{101}
$$

$$
\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r},a}M_{ab}(\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\right] = 0. \tag{102}
$$

$$
\begin{aligned}
&\sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b} + \tilde{\psi}^\dagger{}_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\right] = \sum_{\delta}\left(\Upsilon 1_\delta(\mathbf{r}) + \Upsilon 2_\delta(\mathbf{r})\right) \\
&\sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b}\right] + \sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r},b} + \tilde{\psi}^\dagger{}_{\mathbf{r},b}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r}-\delta,a}\right] = \sum_{\delta}\left(\Upsilon 1'_\delta(\mathbf{r}) + \Upsilon 2'_\delta(\mathbf{r})\right) \\
&\sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\right] + \sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r},b}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r}-\delta,a}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\mathbf{r}',\delta'}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'} + \tilde{\psi}^\dagger{}_{\mathbf{r}'+\delta',b'}T^*_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'}, \tilde{\psi}^\dagger{}_{\mathbf{r},a}M_{ab}(\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\right] \\
&= \quad \sum_{\delta}\left(\Upsilon 3_\delta(\mathbf{r}) + \Upsilon 4_\delta(\mathbf{r})\right)
\end{aligned} \tag{103}
$$

$$
\sum_{\delta}\Upsilon 1_\delta(\mathbf{r}) \quad = \quad \sum_{\mathbf{r}'}\sum_{\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'}, \tilde{\psi}^\dagger{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b}\right] =
$$

$$
\begin{aligned}
&\sum_{\mathbf{r}'}\sum_{\delta}\left(1+\frac{1}{2}\chi_{\mathbf{r}'}\right)^2\left(1+\frac{1}{2}\chi_{\mathbf{r}}\right)\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)M_{a'b'}(\mathbf{r}')T_{ab}(\mathbf{r},\mathbf{r}+\delta)\times\\
&\left(\delta_{\mathbf{r},\mathbf{r}'}\delta_{ab'}\psi^\dagger_{\mathbf{r}',a'}\psi_{\mathbf{r}+\delta,b}-\delta_{\mathbf{r}',\mathbf{r}+\delta}\delta_{a'b}\psi^\dagger_{\mathbf{r},a}\psi_{\mathbf{r}',b'}\right)=\\
&\sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^\dagger_{\mathbf{r},a'}M_{a'a}(\mathbf{r})T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b}\\
-\;&\sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)M_{bb'}(\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b'}
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
\sum_{\delta}\Upsilon 2_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'},\tilde{\psi}^\dagger{}_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\right]=\\
&- \sum_{\delta}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta}+\frac{3}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)M_{aa'}(\mathbf{r})\psi_{\mathbf{r},a'}+\\
&\sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r}+\delta,b'}M_{b'b}(\mathbf{r}+\delta)T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a}
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
\sum_{\delta}\Upsilon 1'_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}'}\sum_{\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'},\tilde{\psi}^\dagger{}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\right]=\\
&\sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r}-\delta,a'}M_{a'a}(\mathbf{r}-\delta)T_{ab}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},b}\\
&- \sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^\dagger_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})M_{bb'}(\mathbf{r})\psi_{\mathbf{r},b'}
\end{aligned}
\tag{106}
$$

$$
\begin{aligned}
\sum_{\delta}\Upsilon 2'_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}',\delta}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}M_{a'b'}(\mathbf{r}')\tilde{\psi}_{\mathbf{r}',b'},\tilde{\psi}^\dagger{}_{\mathbf{r},b}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r}-\delta,a}\right]=\\
&- \sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r},b}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})M_{aa'}(\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,a'}\\
&+ \sum_{\delta}\left(1+\frac{3}{2}\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^\dagger_{\mathbf{r},b'}M_{b'b}(\mathbf{r})T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,a}
\end{aligned}
\tag{107}
$$

$$
\begin{aligned}
\sum_{\delta}\Upsilon 3_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}',\delta'}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'},\tilde{\psi}^\dagger{}_{\mathbf{r},a}M_{ab}(\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\right]=\\
&\sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\frac{3}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r}-\delta',a'}T_{a'a}(\mathbf{r}-\delta',\mathbf{r})M_{ab}(\mathbf{r})\psi_{\mathbf{r},b}\\
&- \sum_{\delta'}\left(1+\frac{3}{2}\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta'}\right)\psi^\dagger_{\mathbf{r},a}M_{ab}(\mathbf{r})T_{bb'}(\mathbf{r},\mathbf{r}+\delta')\psi_{\mathbf{r}+\delta',b'}
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
\sum_{\delta}\Upsilon 4_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}',\delta'}\left[\tilde{\psi}^\dagger{}_{\mathbf{r}'+\delta',b'}T^*_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'},\tilde{\psi}^\dagger{}_{\mathbf{r},a}M_{ab}(\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\right]=\\
&\sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'}+\frac{3}{2}\chi_{\mathbf{r}}\right)\psi^\dagger_{\mathbf{r}+\delta',b'}T^*_{ab'}(\mathbf{r},\mathbf{r}+\delta')M_{ab}(\mathbf{r})\psi_{\mathbf{r},b}\\
&- \sum_{\delta'}\left(1+\frac{3}{2}\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta'}\right)\psi^\dagger_{\mathbf{r},a}T^*_{a'b}(\mathbf{r}-\delta',\mathbf{r})M_{ab}(\mathbf{r})\psi_{\mathbf{r}-\delta',a'}
\end{aligned}
\tag{109}
$$

The 1^{st} term in $\sum_\delta \Upsilon 1_\delta(\mathbf{r})$ and the 2^{nd} term in $\sum_\delta \Upsilon 3_\delta(\mathbf{r})$ cancel, as do the 1^{st} terms in $\sum_\delta \Upsilon 2_\delta(\mathbf{r})$ and $\sum_\delta \Upsilon 4_\delta(\mathbf{r})$. Moreover, the 2^{st} term in $\sum_\delta \Upsilon 1'_\delta(\mathbf{r})$ and the 1^{nd} term in $\sum_\delta \Upsilon 3_\delta(\mathbf{r})$ cancel, as do the 2^{nd} terms in $\sum_\delta \Upsilon 2'_\delta(\mathbf{r})$ and $\sum_\delta \Upsilon 4_\delta(\mathbf{r})$.

Therefore we find that

$$\sum_\delta \left(\frac{1}{2}\Upsilon 1_\delta(\mathbf{r}) + \frac{1}{2}\Upsilon 2_\delta(\mathbf{r}) + \frac{1}{2}\Upsilon 1'_\delta(\mathbf{r}) + \frac{1}{2}\Upsilon 2'_\delta(\mathbf{r}) \right) + \Upsilon 3_\delta(\mathbf{r}) + \Upsilon 4_\delta(\mathbf{r}) = \sum_\delta I1_\delta(\mathbf{r}) - I1_\delta(\mathbf{r}+\delta) \tag{110}$$

where

$$\begin{aligned} I1_\delta(\mathbf{r}) &= \frac{1}{2}\left(1 + \frac{1}{2}\chi_{\mathbf{r}-\delta} + \frac{3}{2}\chi_{\mathbf{r}}\right) \psi^\dagger_{\mathbf{r}-\delta,a} T_{ab}(\mathbf{r}-\delta,\mathbf{r}) M_{bc}(\mathbf{r}) \psi_{\mathbf{r},c} \\ &- \frac{1}{2}\left(1 + \frac{1}{2}\chi_{\mathbf{r}-\delta} + \frac{3}{2}\chi_{\mathbf{r}}\right) \psi^\dagger_{\mathbf{r},a} M_{ab}(\mathbf{r}) T^*_{cb}(\mathbf{r}-\delta,\mathbf{r}) \psi_{\mathbf{r}-\delta,c} \\ &+ \frac{1}{2}\left(1 + \frac{1}{2}\chi_{\mathbf{r}} + \frac{3}{2}\chi_{\mathbf{r}-\delta}\right) \psi^\dagger_{\mathbf{r}-\delta,a} M_{ab}(\mathbf{r}-\delta) T_{bc}(\mathbf{r}-\delta,\mathbf{r}) \psi_{\mathbf{r},c} \\ &- \frac{1}{2}\left(1 + \frac{1}{2}\chi_{\mathbf{r}} + \frac{3}{2}\chi_{\mathbf{r}-\delta}\right) \psi^\dagger_{\mathbf{r},a} T^*_{ba}(\mathbf{r}-\delta,\mathbf{r}) M_{bc}(\mathbf{r}-\delta) \psi_{\mathbf{r}-\delta,c} \end{aligned} \tag{111}$$

To evaluate

$$\sum_{\mathbf{r}'}\sum_{\delta,\delta'}\Big[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'}+\tilde{\psi}^\dagger{}_{\mathbf{r}'+\delta',b'}T^*_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'},$$
$$\tilde{\psi}^\dagger{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b}+\tilde{\psi}^\dagger{}_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\Big]=\sum_\delta\left(\Upsilon5_\delta(\mathbf{r})+\Upsilon6_\delta(\mathbf{r})+\Upsilon7_\delta(\mathbf{r})+\Upsilon8_\delta(\mathbf{r})\right).$$

and

$$\sum_{\mathbf{r}'}\sum_{\delta,\delta'}\Big[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'}+\tilde{\psi}^\dagger{}_{\mathbf{r}'+\delta',b'}T^*_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'},$$
$$\tilde{\psi}^\dagger{}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r},b}+\tilde{\psi}^\dagger{}_{\mathbf{r},b}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r}-\delta,a}\Big]=\sum_\delta\left(\Upsilon5'_\delta(\mathbf{r})+\Upsilon6'_\delta(\mathbf{r})+\Upsilon7'_\delta(\mathbf{r})+\Upsilon8'_\delta(\mathbf{r})\right).$$

start with

$$\begin{aligned}\sum_\delta\Upsilon5_\delta(\mathbf{r}) &= \sum_{\mathbf{r}'}\sum_{\delta,\delta'}\Big[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'},\tilde{\psi}^\dagger{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b}\Big] \qquad (112)\\
&= \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^\dagger_{\mathbf{r}-\delta',a}T_{ab}(\mathbf{r}-\delta',\mathbf{r})T_{bc}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,c}\\
&- \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}+\delta'}+\frac{1}{2}\chi_{\mathbf{r}+\delta+\delta'}\right)\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta')T_{bc}(\mathbf{r}+\delta',\mathbf{r}+\delta+\delta')\psi_{\mathbf{r}+\delta+\delta',c}\\
&= \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta'}\right)\psi^\dagger_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})T_{bc}(\mathbf{r},\mathbf{r}+\delta')\psi_{\mathbf{r}+\delta',c}\\
&- \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}+\delta+\delta'}\right)\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)T_{bc}(\mathbf{r}+\delta,\mathbf{r}+\delta+\delta')\psi_{\mathbf{r}+\delta+\delta',c}\end{aligned}$$

Then $\sum_\delta\Upsilon6_\delta(\mathbf{r})=\sum_{\mathbf{r}'}\sum_{\delta,\delta'}\Big[\tilde{\psi}^\dagger{}_{\mathbf{r}'+\delta',b'}T^*_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'},\tilde{\psi}^\dagger{}_{\mathbf{r}+\delta,b}T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\Big]$ is $-$ Hermitian conjugate of $\sum_\delta\Upsilon5_\delta(\mathbf{r})$ in Eq.(112).

Similarly

$$\begin{aligned}\sum_\delta\Upsilon5'_\delta(\mathbf{r}) &= \sum_{\mathbf{r}'}\sum_{\delta,\delta'}\Big[\tilde{\psi}^\dagger{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'},\tilde{\psi}^\dagger{}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r},b}\Big] \qquad (113)\\
&= \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^\dagger_{\mathbf{r}-\delta-\delta',a}T_{ab}(\mathbf{r}-\delta-\delta',\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},c}\\
&- \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^\dagger_{\mathbf{r}-\delta',a}T_{ab}(\mathbf{r}-\delta',\mathbf{r})T_{bc}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,c}\end{aligned}$$

And again

$$\sum_\delta\Upsilon6'_\delta(\mathbf{r})=-\left(\sum_\delta\Upsilon5'_\delta(\mathbf{r})\right)^\dagger$$

Note that,

$$\sum_\delta\Upsilon5_\delta(\mathbf{r})+\Upsilon6_\delta(\mathbf{r})+\Upsilon5'_\delta(\mathbf{r})+\Upsilon6'_\delta(\mathbf{r})=\sum_\delta I2_\delta(\mathbf{r})-I2_\delta(\mathbf{r}+\delta) \qquad (114)$$

$$\begin{aligned}I2_\delta(\mathbf{r}) &= \sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta'}\right)\psi^\dagger_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})T_{bc}(\mathbf{r},\mathbf{r}+\delta')\psi_{\mathbf{r}+\delta',c}\\
&- \sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta'}\right)\psi^\dagger_{\mathbf{r}+\delta',c}T^*_{ab}(\mathbf{r}-\delta,\mathbf{r})T^*_{bc}(\mathbf{r},\mathbf{r}+\delta')\psi_{\mathbf{r}-\delta,a}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\delta'} \left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta-\delta'}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}-\delta-\delta',a}T_{ab}(\mathbf{r}-\delta-\delta',\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},c} \\
&- \sum_{\delta'} \left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta-\delta'}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},c}T^{*}_{ab}(\mathbf{r}-\delta-\delta',\mathbf{r}-\delta)T^{*}_{bc}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta-\delta',a}
\end{aligned} \tag{115}
$$

Next, we have

$$
\begin{aligned}
\sum_{\delta}\Upsilon 7_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}'}\sum_{\delta,\delta'}\left[\tilde{\psi}^{\dagger}{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'},\tilde{\psi}^{\dagger}{}_{\mathbf{r}+\delta,b}T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r},a}\right]= \\
&\sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta-\delta'}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\delta-\delta',a'}T_{a'b}(\mathbf{r}+\delta-\delta',\mathbf{r}+\delta)T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \\
&- \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^{\dagger}_{\mathbf{r}+\delta,b}T_{ab'}(\mathbf{r},\mathbf{r}+\delta')T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta',b'}
\end{aligned} \tag{116}
$$

and $\sum_{\mathbf{r}'}\sum_{\delta,\delta'}\left[\tilde{\psi}^{\dagger}{}_{\mathbf{r}'+\delta',b'}T^{*}_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'},\tilde{\psi}^{\dagger}{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b}\right]$ is the $-$ Hermitian conjugate of Eq.(116). So

$$
\begin{aligned}
\sum_{\delta}\Upsilon 8_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}'}\sum_{\delta,\delta'}\left[\tilde{\psi}^{\dagger}{}_{\mathbf{r}'+\delta',b'}T^{*}_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}',a'},\tilde{\psi}^{\dagger}{}_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)\tilde{\psi}_{\mathbf{r}+\delta,b}\right]= \\
&- \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta-\delta'}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\delta-\delta',\mathbf{r}+\delta)T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta-\delta',a'} \\
&+ \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^{\dagger}_{\mathbf{r}+\delta',b'}T^{*}_{ab'}(\mathbf{r},\mathbf{r}+\delta')T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b}
\end{aligned} \tag{117}
$$

Next,

$$
\begin{aligned}
\sum_{\delta}\Upsilon 7'_{\delta}(\mathbf{r}) &= \sum_{\mathbf{r}'}\sum_{\delta,\delta'}\left[\tilde{\psi}^{\dagger}{}_{\mathbf{r}',a'}T_{a'b'}(\mathbf{r}',\mathbf{r}'+\delta')\tilde{\psi}_{\mathbf{r}'+\delta',b'},\tilde{\psi}^{\dagger}{}_{\mathbf{r},b}T^{*}_{ab}(\mathbf{r}-\delta,\mathbf{r})\tilde{\psi}_{\mathbf{r}-\delta,a}\right]= \\
&= \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta',a}T_{ab}(\mathbf{r}-\delta',\mathbf{r})T^{*}_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,c} \\
&- \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'-\delta}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\delta,\mathbf{r})T_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r}+\delta'-\delta,c}
\end{aligned} \tag{118}
$$

and since $\sum_{\delta}\Upsilon 8'_{\delta}(\mathbf{r})=-\left(\sum_{\delta}\Upsilon 7'_{\delta}(\mathbf{r})\right)^{\dagger}$

$$
\begin{aligned}
\sum_{\delta}\Upsilon 8'_{\delta}(\mathbf{r}) &= -\sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta,c}T^{*}_{ab}(\mathbf{r}-\delta',\mathbf{r})T_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta',a} \\
&+ \sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'-\delta}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\delta'-\delta,c}T_{ba}(\mathbf{r}-\delta,\mathbf{r})T^{*}_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r},a}
\end{aligned} \tag{119}
$$

Summing the Eqs. (116) and (117) we find for $\delta=\delta'$:

$$
\begin{aligned}
&\sum_{\delta}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a'}T_{a'b}(\mathbf{r},\mathbf{r}+\delta)T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \\
&- \sum_{\delta}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^{\dagger}_{\mathbf{r}+\delta,b}T_{ab'}(\mathbf{r},\mathbf{r}+\delta)T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b'}
\end{aligned}
$$

$$
\begin{aligned}
- & \sum_{\delta}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a'}T_{a'b}(\mathbf{r},\mathbf{r}+\delta)T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \\
+ & \sum_{\delta}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^{\dagger}_{\mathbf{r}+\delta,b}T_{ab'}(\mathbf{r},\mathbf{r}+\delta)T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b'}=0.
\end{aligned}
\tag{120}
$$

Summing the Eqs. (116) and (117) we find for $\delta \neq \delta'$:

$$
\begin{aligned}
& \sum_{\delta}\Upsilon 7_{\delta}(\mathbf{r})+\Upsilon 8_{\delta}(\mathbf{r}) \\
= & \sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta-\delta'}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\delta-\delta',a'}T_{a'b}(\mathbf{r}+\delta-\delta',\mathbf{r}+\delta)T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a} \\
- & \sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^{\dagger}_{\mathbf{r}+\delta,b}T_{ab'}(\mathbf{r},\mathbf{r}+\delta')T^{*}_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta',b'} \\
- & \sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta-\delta'}+\chi_{\mathbf{r}+\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\delta-\delta',\mathbf{r}+\delta)T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta-\delta',a'} \\
+ & \sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta}\right)\psi^{\dagger}_{\mathbf{r}+\delta',b'}T^{*}_{ab'}(\mathbf{r},\mathbf{r}+\delta')T_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,b}= \\
= & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r},a} \\
- & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}},b}T_{ab'}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{y}},b'} \\
- & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'} \\
+ & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}},b'}T^{*}_{ab'}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{x}},b} \\
+ & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r},a} \\
- & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}},b}T_{ab'}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{x}},b'} \\
- & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'} \\
+ & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}},b'}T^{*}_{ab'}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}},b}
\end{aligned}
\tag{121}
$$

The key is to notice that the 2^{nd} and the 8^{th} terms cancel, as do 4^{th} and 6^{th}. We can therefore change these terms at will, as long as we change simultaneously both of the canceling terms.

So,

$$
\begin{aligned}
& \sum_{\delta}\Upsilon 7_{\delta}(\mathbf{r})+\Upsilon 8_{\delta}(\mathbf{r}) \\
= & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r},a} \\
- & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}},b}T_{ab'}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{y}})T^{*}_{ab}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{y}},b'} \\
- & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'} \\
+ & \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}},b'}T^{*}_{ab'}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{y}})T_{ab}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{x}},b}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r},a} \\
&- \quad \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}},b}T_{ab'}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T^{*}_{ab}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{x}},b'} \\
&- \quad \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'} \\
&+ \quad \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}},b'}T^{*}_{ab'}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T_{ab}(\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}},b}
\end{aligned}
\tag{122}
$$

Similarly,

$$
\begin{aligned}
&\sum_{\delta}\left(\Upsilon 7'_{\delta}(\mathbf{r})+\Upsilon 8'_{\delta}(\mathbf{r})\right)\\
= &\sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta',a}T_{ab}(\mathbf{r}-\delta',\mathbf{r})T^{*}_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,c}\\
- &\sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'-\delta}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\delta,\mathbf{r})T_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r}+\delta'-\delta,c}\\
- &\sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta,c}T^{*}_{ab}(\mathbf{r}-\delta',\mathbf{r})T_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta',a}\\
+ &\sum_{\delta,\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'-\delta}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\delta'-\delta,c}T_{ba}(\mathbf{r}-\delta,\mathbf{r})T^{*}_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r},a} \qquad (123)\\
= &\sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta',a}T_{ab}(\mathbf{r}-\delta',\mathbf{r})T^{*}_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,c}\\
- &\sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'-\delta}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\delta,\mathbf{r})T_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r}+\delta'-\delta,c}\\
- &\sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta'}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta,c}T^{*}_{ab}(\mathbf{r}-\delta',\mathbf{r})T_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta',a}\\
+ &\sum_{\delta\neq\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}+\delta'-\delta}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\delta'-\delta,c}T_{ba}(\mathbf{r}-\delta,\mathbf{r})T^{*}_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r},a}\\
= &\left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})\psi_{\mathbf{r}-\hat{\mathbf{x}},c}\\
- &\left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})T_{bc}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},c}\\
- &\left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},c}T^{*}_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})T_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})\psi_{\mathbf{r}-\hat{\mathbf{y}},a}\\
+ &\left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},c}T_{ba}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})T^{*}_{bc}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}})\psi_{\mathbf{r},a}\\
+ &\left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})\psi_{\mathbf{r}-\hat{\mathbf{y}},c}\\
- &\left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})T_{bc}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},c}\\
- &\left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},c}T^{*}_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})T_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})\psi_{\mathbf{r}-\hat{\mathbf{x}},a}\\
+ &\left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},c}T_{ba}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})T^{*}_{bc}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r},a} \qquad (124)
\end{aligned}
$$

Similarly to what happened before, in the last equality, 1^{st} and 7^{th} terms cancel, as do 3^{rd} and 5^{th}.

Therefore,

$$
\begin{aligned}
&\sum_{\delta}\left(\Upsilon 7'_{\delta}(\mathbf{r})+\Upsilon 8'_{\delta}(\mathbf{r})\right)\\
=\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{x}},c}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})T_{bc}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},c}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{y}},c}\\
+\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},c}T_{ba}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r})T^{*}_{bc}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}})\psi_{\mathbf{r},a}\\
+\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{y}},c}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})T_{bc}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},c}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{x}},c}\\
+\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},c}T_{ba}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r})T^{*}_{bc}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r},a}
\end{aligned}
\tag{125}
$$

Note that

$$
\sum_{\delta}\Upsilon 7_{\delta}(\mathbf{r})+\Upsilon 8_{\delta}(\mathbf{r})+\Upsilon 7'_{\delta}(\mathbf{r})+\Upsilon 8'_{\delta}(\mathbf{r}) = I3_{\hat{\mathbf{x}}}(\mathbf{r})-I3_{\hat{\mathbf{x}}}(\mathbf{r}+\hat{\mathbf{x}})+I3_{\hat{\mathbf{y}}}(\mathbf{r})-I3_{\hat{\mathbf{y}}}(\mathbf{r}+\hat{\mathbf{y}})
$$

where

$$
\begin{aligned}
I3_{\hat{\mathbf{x}}}(\mathbf{r}) =\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r},a}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'}\\
+\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{y}},c}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{x}},c}\\
I3_{\hat{\mathbf{y}}}(\mathbf{r}) =\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r},a}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'}\\
+\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{x}},c}\\
-\;& \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{y}},c}
\end{aligned}
$$

and where we used

$$
T^{*}_{ab}(\mathbf{r},\mathbf{r}')=T_{ba}(\mathbf{r}',\mathbf{r}). \tag{126}
$$

So, putting it all together we have

$$\left[\tilde{H}, \tilde{h}_{\mathbf{r}}\right] = \sum_{\delta} \left(\left(I1_{\delta}(\mathbf{r}) + \frac{1}{2} I2_{\delta}(\mathbf{r}) + \frac{1}{2} I3_{\delta}(\mathbf{r}) \right) - \left(I1_{\delta}(\mathbf{r}+\delta) + \frac{1}{2} I2_{\delta}(\mathbf{r}+\delta) + \frac{1}{2} I3_{\delta}(\mathbf{r}+\delta) \right) \right) \tag{127}$$

where

$$\begin{aligned}
I1_{\delta}(\mathbf{r}) &= \frac{1}{2}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\frac{3}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})M_{bc}(\mathbf{r})\psi_{\mathbf{r},c} \\
&- \frac{1}{2}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\frac{3}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}M_{ab}(\mathbf{r})T^{*}_{cb}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta,c} \\
&+ \frac{1}{2}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\frac{3}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r}-\delta,a}M_{ab}(\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},c} \\
&- \frac{1}{2}\left(1+\frac{1}{2}\chi_{\mathbf{r}}+\frac{3}{2}\chi_{\mathbf{r}-\delta}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{ba}(\mathbf{r}-\delta,\mathbf{r})M_{bc}(\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,c} \qquad (128)\\
I2_{\delta}(\mathbf{r}) &= \sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta'}\right)\psi^{\dagger}_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})T_{bc}(\mathbf{r},\mathbf{r}+\delta')\psi_{\mathbf{r}+\delta',c} \\
&- \sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta}+\chi_{\mathbf{r}}+\frac{1}{2}\chi_{\mathbf{r}+\delta'}\right)\psi^{\dagger}_{\mathbf{r}+\delta',c}T^{*}_{ab}(\mathbf{r}-\delta,\mathbf{r})T^{*}_{bc}(\mathbf{r},\mathbf{r}+\delta')\psi_{\mathbf{r}-\delta,a} \\
&+ \sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta-\delta'}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}-\delta-\delta',a}T_{ab}(\mathbf{r}-\delta-\delta',\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r},c} \\
&- \sum_{\delta'}\left(1+\frac{1}{2}\chi_{\mathbf{r}-\delta-\delta'}+\chi_{\mathbf{r}-\delta}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},c}T^{*}_{ab}(\mathbf{r}-\delta-\delta',\mathbf{r}-\delta)T^{*}_{bc}(\mathbf{r}-\delta,\mathbf{r})\psi_{\mathbf{r}-\delta-\delta',a} \\
I3_{\hat{\mathbf{x}}}(\mathbf{r}) &= \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r},a} \\
&- \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}+\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},\mathbf{r}+\hat{\mathbf{y}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{y}})\psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'} \\
&+ \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{y}},c} \\
&- \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{x}},c} \\
I3_{\hat{\mathbf{y}}}(\mathbf{r}) &= \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'}T_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T^{*}_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r},a} \\
&- \left(1+\frac{1}{2}\chi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}+\hat{\mathbf{x}}}+\frac{1}{2}\chi_{\mathbf{r}}\right)\psi^{\dagger}_{\mathbf{r},a}T^{*}_{a'b}(\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{r}+\hat{\mathbf{x}})T_{ab}(\mathbf{r},\mathbf{r}+\hat{\mathbf{x}})\psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},a'} \\
&+ \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{y}},a}T_{ab}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{x}},c} \\
&- \left(1+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{y}}}+\chi_{\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}}+\frac{1}{2}\chi_{\mathbf{r}-\hat{\mathbf{x}}}\right)\psi^{\dagger}_{\mathbf{r}-\hat{\mathbf{x}},a}T^{*}_{cb}(\mathbf{r}-\hat{\mathbf{y}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})T_{ab}(\mathbf{r}-\hat{\mathbf{x}},\mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}})\psi_{\mathbf{r}-\hat{\mathbf{y}},c}
\end{aligned}$$

$$\frac{\partial \tilde{h}_{\mathbf{r}}}{\partial t} = \frac{i}{\hbar}\left[\tilde{H}, \tilde{h}_{\mathbf{r}}\right] \equiv -\nabla \cdot \tilde{\mathbf{j}}_E(\mathbf{r}) = \tilde{j_{E_x}}(\mathbf{r}-\hat{\mathbf{x}}) - \tilde{j_{E_x}}(\mathbf{r}) + \tilde{j_{E_y}}(\mathbf{r}-\hat{\mathbf{y}}) - \tilde{j_{E_y}}(\mathbf{r}). \tag{129}$$

$$\tilde{j_{E_x}}(\mathbf{r}) = \frac{i}{\hbar}\left(I1_{\hat{\mathbf{x}}}(\mathbf{r}+\hat{\mathbf{x}}) + \frac{1}{2}I2_{\hat{\mathbf{x}}}(\mathbf{r}+\hat{\mathbf{x}}) + \frac{1}{2}I3_{\hat{\mathbf{x}}}(\mathbf{r}+\hat{\mathbf{x}})\right) \tag{130}$$

$$\tilde{j_{E_y}}(\mathbf{r}) = \frac{i}{\hbar}\left(I1_{\hat{\mathbf{y}}}(\mathbf{r}+\hat{\mathbf{y}}) + \frac{1}{2}I2_{\hat{\mathbf{y}}}(\mathbf{r}+\hat{\mathbf{y}}) + \frac{1}{2}I3_{\hat{\mathbf{y}}}(\mathbf{r}+\hat{\mathbf{y}})\right). \tag{131}$$

E Spatial average of the energy current

To obtain χ-independent contribution to the spatial average of the energy current, consider

$$\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi^\dagger_{\mathbf{r}}V^\delta_{\mathbf{r}\mathbf{r}'}H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''}+H.c=\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi^\dagger_{\mathbf{r}}\left(V^\delta_{\mathbf{r}\mathbf{r}'}H_{\mathbf{r}'\mathbf{r}''}+H_{\mathbf{r}\mathbf{r}'}V^\delta_{\mathbf{r}'\mathbf{r}''}\right)\psi_{\mathbf{r}''}. \tag{132}$$

Now,

$$\begin{aligned}
&\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''}\psi^\dagger_{\mathbf{r}}V^\delta_{\mathbf{r}\mathbf{r}'}H_{\mathbf{r}'\mathbf{r}''}\psi_{\mathbf{r}''}+H.c.=\\
&\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'}\psi^\dagger_{\mathbf{r}}V^\delta_{\mathbf{r}\mathbf{r}'}\left(M_{ab}(\mathbf{r}')\psi_{\mathbf{r}',b}+\sum_{\delta'=\hat{x},\hat{y}}\left(T_{ab}(\mathbf{r}',\mathbf{r}'+\delta')\psi_{\mathbf{r}'+\delta',b}+T_{ab}(\mathbf{r}',\mathbf{r}'-\delta')\psi_{\mathbf{r}'-\delta',b}\right)\right)+H.c.=\\
&\frac{i}{2\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)M_{bc}(\mathbf{r}+\delta)\psi_{\mathbf{r}+\delta,c}-\frac{i}{2\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r}+\delta,c}M_{cb}(\mathbf{r}+\delta)T^*_{ab}(\mathbf{r},\mathbf{r}+\delta)\psi_{\mathbf{r},a}\\
-&\frac{i}{2\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}-\delta)M_{bc}(\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,c}+\frac{i}{2\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r}-\delta,c}M_{cb}(\mathbf{r}-\delta)T^*_{ab}(\mathbf{r},\mathbf{r}-\delta)\psi_{\mathbf{r},a}\\
+&\frac{i}{2\hbar}\sum_{\mathbf{r}}\sum_{\delta'=\pm\hat{x},\pm\hat{y}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)T_{bc}(\mathbf{r}+\delta,\mathbf{r}+\delta'+\delta)\psi_{\mathbf{r}+\delta'+\delta,c}+H.c.\\
-&\frac{i}{2\hbar}\sum_{\mathbf{r}}\sum_{\delta'=\pm\hat{x},\pm\hat{y}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r}+\delta'-\delta,c}+H.c.
\end{aligned} \tag{133}$$

Now,

$$\begin{aligned}
\frac{i}{\hbar}\sum_{\mathbf{r}}I1_\delta(\mathbf{r})|_{\chi=0} &= \frac{i}{2\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r}-\delta,a}T_{ab}(\mathbf{r}-\delta,\mathbf{r})M_{bc}(\mathbf{r})\psi_{\mathbf{r},c}+H.c\\
&- \frac{i}{2\hbar}\sum_{\mathbf{r}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}-\delta)M_{bc}(\mathbf{r}-\delta)\psi_{\mathbf{r}-\delta,c}+H.c.
\end{aligned} \tag{134}$$

which is equal to the first two lines in Eq.(133).

Similarly,

$$\begin{aligned}
\frac{i}{2\hbar}\sum_{\mathbf{r}}I2_\delta(\mathbf{r})|_{\chi=0} &= \frac{i}{2\hbar}\sum_{\mathbf{r}}\sum_{\delta'=\hat{\mathbf{x}},\hat{\mathbf{y}}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)T_{bc}(\mathbf{r}+\delta,\mathbf{r}+\delta'+\delta)\psi_{\mathbf{r}+\delta'+\delta,c}+H.c.\\
&- \frac{i}{2\hbar}\sum_{\mathbf{r}}\sum_{\delta'=\hat{\mathbf{x}},\hat{\mathbf{y}}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r}-\delta-\delta')\psi_{\mathbf{r}-\delta-\delta',c}+H.c.
\end{aligned} \tag{135}$$

is equal to the 3^{rd} line in Eq.(133) for $\delta'=\hat{\mathbf{x}}$ and $\delta'=\hat{\mathbf{y}}$, and to the 4^{th} line in Eq.(133) for $\delta'=-\hat{\mathbf{x}}$ and $\delta'=-\hat{\mathbf{y}}$

The remaining terms in Eq.(133) are

$$\begin{aligned}
+&\frac{i}{2\hbar}\sum_{\mathbf{r}}\sum_{\delta'=\hat{x},\hat{y}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}+\delta)T_{bc}(\mathbf{r}+\delta,\mathbf{r}-\delta'+\delta)\psi_{\mathbf{r}-\delta'+\delta,c}+H.c.\\
-&\frac{i}{2\hbar}\sum_{\mathbf{r}}\sum_{\delta'=\hat{x},\hat{y}}\psi^\dagger_{\mathbf{r},a}T_{ab}(\mathbf{r},\mathbf{r}-\delta)T_{bc}(\mathbf{r}-\delta,\mathbf{r}+\delta'-\delta)\psi_{\mathbf{r}+\delta'-\delta,c}+H.c.
\end{aligned} \tag{136}$$

If $\delta'=\delta$, then the above terms vanish because we have a purely imaginary quantity and subtracting its hermitian conjugate gives zero.

So consider $\delta = \hat{\mathbf{x}}$, then the remaining terms in Eq.(133) are

$$\begin{aligned}
&+ \quad \frac{i}{2\hbar}\sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r}, \mathbf{r}+\hat{\mathbf{x}}) T_{bc}(\mathbf{r}+\hat{\mathbf{x}}, \mathbf{r}-\hat{\mathbf{y}}+\hat{\mathbf{x}}) \psi_{\mathbf{r}-\hat{\mathbf{y}}+\hat{\mathbf{x}},c} + H.c. \\
&- \quad \frac{i}{2\hbar}\sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r}, \mathbf{r}-\hat{\mathbf{x}}) T_{bc}(\mathbf{r}-\hat{\mathbf{x}}, \mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}) \psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},c} + H.c. \\
&= \quad \frac{i}{2\hbar}\sum_{\mathbf{r}} I3_{\hat{\mathbf{x}}}(\mathbf{r})|_{\chi=0} \qquad (137)
\end{aligned}$$

The last line follows from

$$\begin{aligned}
\sum_{\mathbf{r}} I3_{\hat{\mathbf{x}}}(\mathbf{r})|_{\chi=0} &= \sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},a'} T_{a'b}(\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}, \mathbf{r}+\hat{\mathbf{y}}) T^*_{ab}(\mathbf{r}, \mathbf{r}+\hat{\mathbf{y}}) \psi_{\mathbf{r},a} - H.c. \\
&- \left(\sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r}-\hat{\mathbf{y}},a} T^*_{cb}(\mathbf{r}-\hat{\mathbf{x}}, \mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}) T_{ab}(\mathbf{r}-\hat{\mathbf{y}}, \mathbf{r}-\hat{\mathbf{x}}-\hat{\mathbf{y}}) \psi_{\mathbf{r}-\hat{\mathbf{x}},c} - H.c. \right) \\
&= \sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r}, \mathbf{r}+\hat{\mathbf{x}}) T_{bc}(\mathbf{r}+\hat{\mathbf{x}}, \mathbf{r}-\hat{\mathbf{y}}+\hat{\mathbf{x}}) \psi_{\mathbf{r}-\hat{\mathbf{y}}+\hat{\mathbf{x}},c} - H.c. \\
&- \left(\sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r}, \mathbf{r}-\hat{\mathbf{x}}) T_{bc}(\mathbf{r}-\hat{\mathbf{x}}, \mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}}) \psi_{\mathbf{r}+\hat{\mathbf{y}}-\hat{\mathbf{x}},c} - H.c. \right). \qquad (138)
\end{aligned}$$

Similarly, for $\delta = \hat{\mathbf{y}}$, the remaining terms in Eq.(133) are

$$\begin{aligned}
&+ \quad \frac{i}{2\hbar}\sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r}, \mathbf{r}+\hat{\mathbf{y}}) T_{bc}(\mathbf{r}+\hat{\mathbf{y}}, \mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}}) \psi_{\mathbf{r}-\hat{\mathbf{x}}+\hat{\mathbf{y}},c} + H.c. \\
&- \quad \frac{i}{2\hbar}\sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r},a} T_{ab}(\mathbf{r}, \mathbf{r}-\hat{\mathbf{y}}) T_{bc}(\mathbf{r}-\hat{\mathbf{y}}, \mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}}) \psi_{\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},c} + H.c. \\
&= \quad \frac{i}{2\hbar}\sum_{\mathbf{r}} I3_{\hat{\mathbf{y}}}(\mathbf{r})|_{\chi=0} \qquad (139)
\end{aligned}$$

Therefore, we arrive at an important conclusion:

$$\frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} V^\delta_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''} \psi_{\mathbf{r}''} + H.c \quad = \quad \sum_{\mathbf{r}} \tilde{j}_{E\,\delta}(\mathbf{r})|_{\chi_{\mathbf{r}}=0}. \qquad (140)$$

The full contribution to the energy current follows from the above arguments almost immediately. We find the formula

$$\begin{aligned}
\sum_{\mathbf{r}} \tilde{j}_{E\,\delta}(\mathbf{r}) &= \frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} V^\delta_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''} \psi_{\mathbf{r}''} + H.c \\
&+ \frac{1}{4}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} \chi_{\mathbf{r}} V^\delta_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''} \psi_{\mathbf{r}''} + H.c \\
&+ \frac{1}{2}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} V^\delta_{\mathbf{r}\mathbf{r}'} \chi_{\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''} \psi_{\mathbf{r}''} + H.c \\
&+ \frac{1}{4}\sum_{\mathbf{r}\mathbf{r}'\mathbf{r}''} \psi^\dagger_{\mathbf{r}} V^\delta_{\mathbf{r}\mathbf{r}'} H_{\mathbf{r}'\mathbf{r}''} \chi_{\mathbf{r}''} \psi_{\mathbf{r}''} + H.c. \qquad (141)
\end{aligned}$$

This can be seen by noticing that $\chi_{\mathbf{r}}$ is kind of laced-up between the shift operators of V and H.