

Topological Superconductivity

Liang Fu



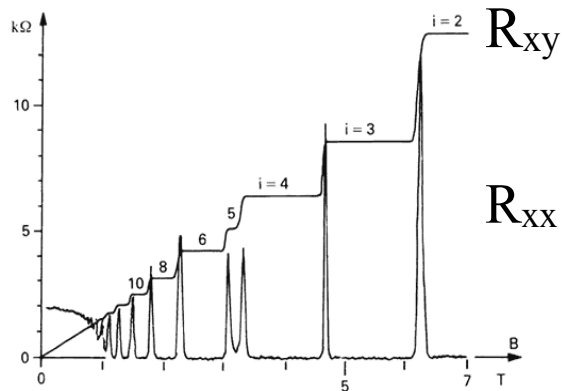
Boulder Summer 2014



Quantum Hall Effect

Quantized Hall conductance $\sigma_{xy} = Ne^2/h$ has a topological origin:

Quantum hall effect in 2D electron gas

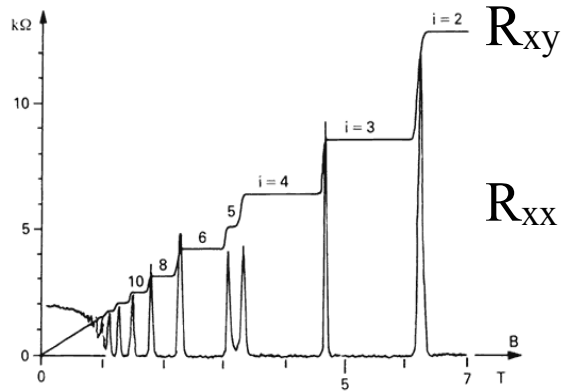


$N = \#$ of occupied Landau levels

Quantum Hall Effect

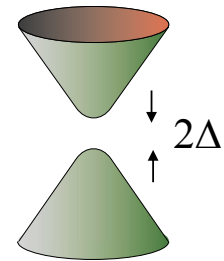
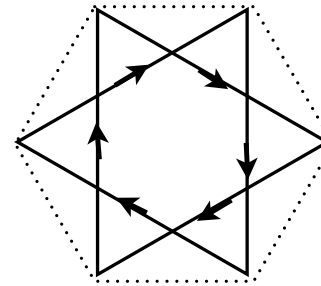
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$N = \#$ of occupied Landau levels

Model for quantum Hall effect without Landau levels [Haldane \(88\)](#)

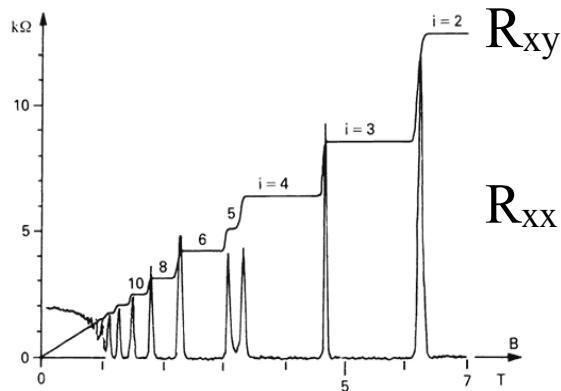


- QHE from T-breaking band structure

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Quantum hall effect in 2D electron gas

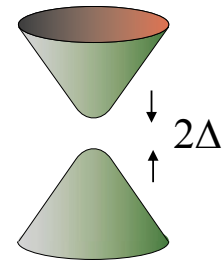
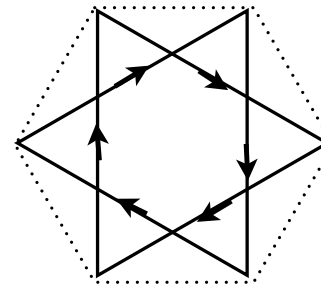


$N = \#$ of occupied Landau levels



Model for quantum Hall effect

without Landau levels [Haldane \(88\)](#)



- QHE from T-breaking band structure

$N =$ Chern number (topological invariant)

[Thouless et al \(81\)](#)

Rise of Topology

Not all insulators are equivalent: topology enters.

Topology: property of a manifold that are insensitive to smooth deformations, e.g. genus

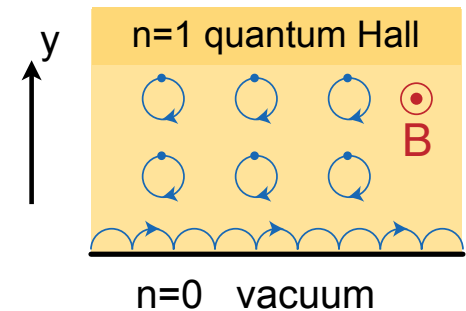
Topology of gapped quantum phases:

property of many-body ground state that is invariant under adiabatic deformation

- expressed in terms of discrete [topological index](#)
- topology cannot change unless gap closes

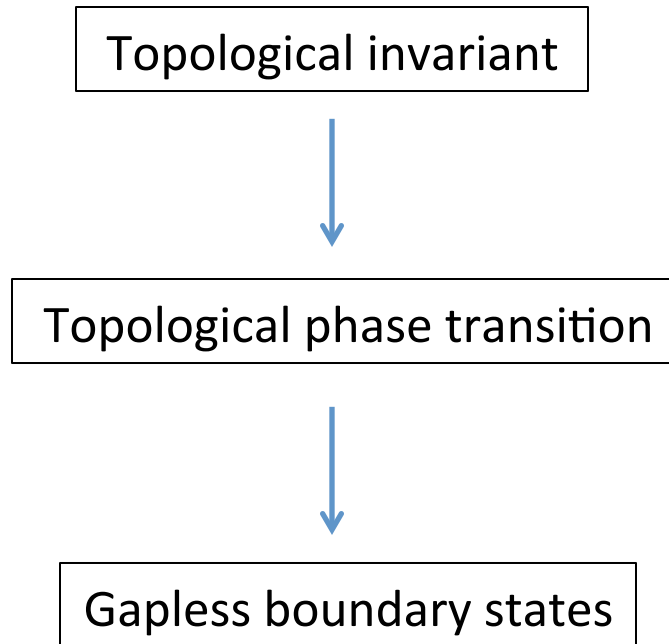
Universal manifestation: gapless excitation at interface between topologically distinct states

- dictated by topology change across interface



Topological Superconductors

Lecture I:



We will exploit the formal analogy between band electrons and Bogoliubov quasi-particles and discuss topology of insulators and superconductors in a unified framework.

BCS Theory of Superconductivity

Topology: exact result with approximate method

Mean field Hamiltonian:

$$H = \sum_{i,j} c_i^\dagger H_{ij}^0 c_j + (\Delta_{ij} c_i^\dagger c_j^\dagger + \Delta_{ij}^* c_j c_i)$$
$$= \sum_{i,j} (c_i^\dagger, c_i) \underbrace{\begin{pmatrix} H^0 & \Delta \\ \Delta^\dagger & -H^0 \end{pmatrix}}_{ij} \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix}$$

H_{BdG} : Bogoliubov de-Gennes Hamiltonian in Nambu space

- eigenvalue spectrum of H_{BdG} comes in $(E, -E)$ pairs
= excitation spectrum of quasi-particles at $E > 0$, doubled to $(E, -E)$
- this doubling is due to particle-hole redundancy (not a physical symmetry)

$$\Xi H_{\text{BdG}} \Xi^{-1} = -H_{\text{BdG}}, \quad \Xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K$$

Topology in Momentum Space

BdG Hamiltonian in k-space:

$$H = \sum_k (c_k^\dagger, c_{-k}) H_{\text{BdG}}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

- many-body ground state is constructed from eigenfunctions at $E < 0$: $\xi_k = (u_k, v_k)$

$$|G\rangle = \exp\left(\sum_k g(k) c_k^\dagger c_{-k}^\dagger\right) |vac\rangle \quad \text{where} \quad g(k) = v(k)/u(k)$$

note that $|G\rangle$ is *not* a direct product state, unlike band insulators

- topology of ground state must be encoded in **gapped** BdG Hamiltonian $H_{\text{BdG}}(k)$ in **momentum space**

- topological classification problem:

find distinct classes of $H_{\text{BdG}}(k)$ that satisfy $\tau_x H_{\text{BdG}}^*(k) \tau_x = -H_{\text{BdG}}(-k)$

1D Topological Superconductor

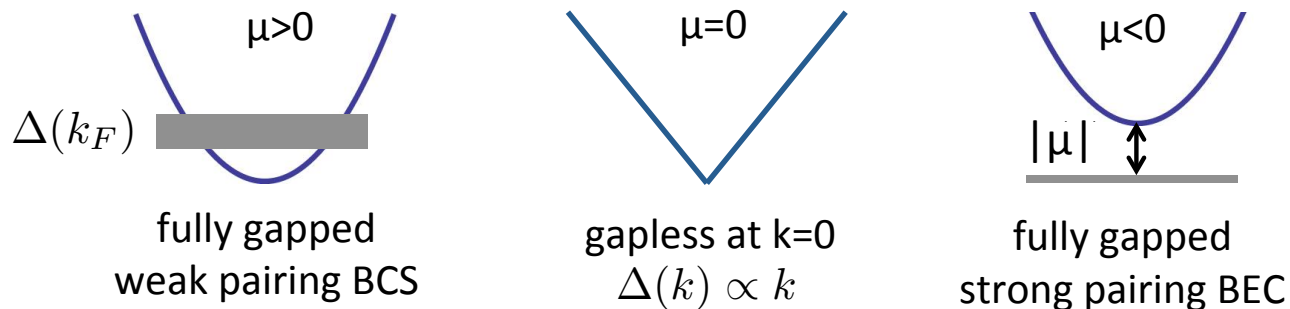
1D single band, spinless superconductor [Kitaev \(00\)](#)

$$H = \sum_k c_k^\dagger (E_k - \mu) c_k + \Delta(k) c_k^\dagger c_{-k}^\dagger + \Delta^*(k) c_{-k} c_k$$

$$H_{\text{BdG}} = \begin{pmatrix} E(k) - \mu & \Delta(k) \\ \Delta^\dagger(k) & \mu - E(k) \end{pmatrix} = \epsilon(k) \vec{n}_k \cdot \vec{\sigma}$$

- Fermi-Dirac statistics dictates p-wave pairing: $\Delta(k) = -\Delta(-k)$

Quasi-particle spectrum: $\epsilon(k) = \sqrt{(E(k) - \mu)^2 + |\Delta(k)|^2}$ [\(Leo's lecture\)](#)



- p-wave gap vanishes at $k=0 \Rightarrow$ gap closes at $k=0$ for $\mu=0$

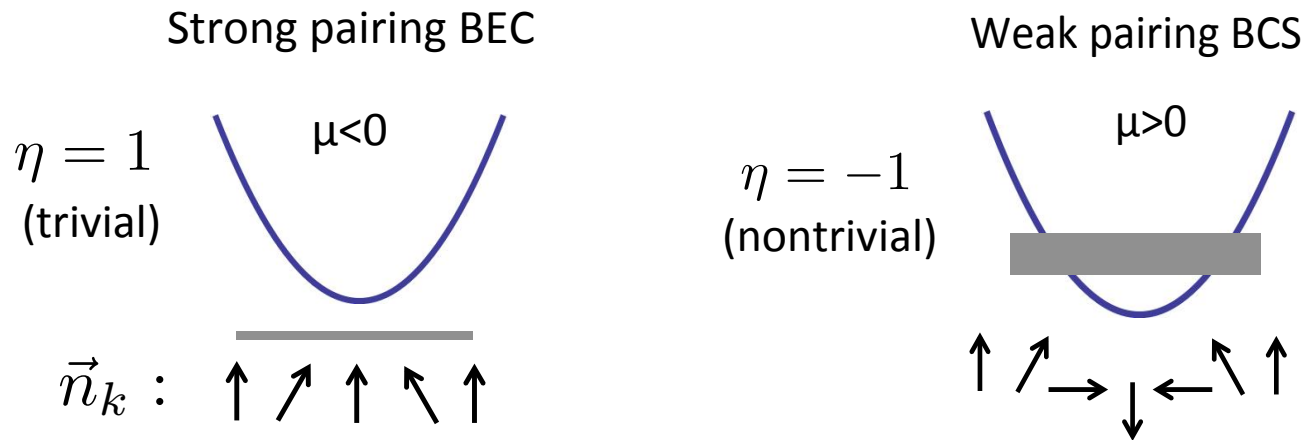
Topological Invariant

Particle-hole condition requires:

$$H_{\text{BdG}} = \epsilon(k) \vec{n}_k \cdot \vec{\sigma} \quad \text{where } (n_x, n_y, n_z)_k = (-n_x, -n_y, n_z)_{-k}$$

$$\Rightarrow \vec{n}_{k=0, \pi} \parallel \hat{z}$$

Z_2 invariant: $\eta = \text{sgn}[E(0) - \mu] \cdot \text{sgn}[E(\pi) - \mu]$ [Kitaev \(00\)](#)



The two distinct phases are separated by topological phase transition at $\mu=0$.

Majorana Mode

Semi-infinite wire:



- end of wire = domain wall between weak and strong pairing (vacuum) phase
- change of topology across the domain leads to a zero-energy localized mode
- BdG equation = 1D Dirac equation with a mass twist

$$H = \begin{pmatrix} E_k - \mu & \Delta(k) \\ \Delta(k) & \mu - E_k \end{pmatrix} \rightarrow \begin{pmatrix} -\mu(x) & -iv\partial_x \\ iv\partial_x & \mu(x) \end{pmatrix} \quad \text{for small } \mu$$

- zero-mode corresponds to a **real, local** operator

$$\gamma = \int dx u(x)c^\dagger(x) + v(x)c(x) \quad \text{where } u(x) = v^*(x) = e^{-\mu x/v}$$
$$\gamma = \gamma^\dagger : \text{Majorana zero mode}$$

in contrast, $\epsilon > 0$ solutions correspond to ordinary fermions (quasi-particles).

2D Topological Superconductor

2D single band, spinless, $\mathbf{p}+i\mathbf{p}$ superconductor [Read & Green \(00\)](#)

$$H = \sum_k c_k^\dagger (E_k - \mu) c_k + \Delta(k) c_k^\dagger c_{-k}^\dagger + \Delta^*(k) c_{-k} c_k \quad \text{where } \Delta(k) \propto (k_x + ik_y)$$

$$k = (k_x, k_y)$$

$$H_{\text{BdG}} = \begin{pmatrix} E(k) - \mu & \Delta(k) \\ \Delta^\dagger(k) & \mu - E(k) \end{pmatrix} = \epsilon(k) \vec{n}_k \cdot \vec{\sigma}$$

Quasi-particle spectrum: $\epsilon(k) = \sqrt{(E(k) - \mu)^2 + |\Delta(k)|^2}$

BCS-BEC transition at $\mu=0$: gap closes at $\mathbf{k}=(0,0)$

Low-energy theory near transition = 2D Majorana fermion mass reversal ($m=\mu$)

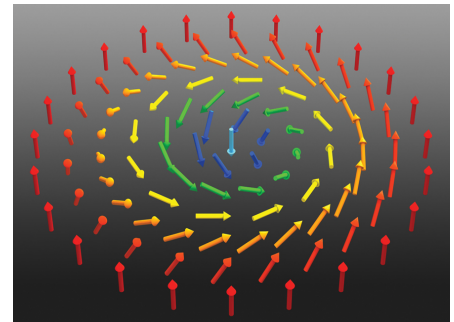
$$H = \sum_k (c_k^\dagger, c_{-k}) \begin{pmatrix} -m & v(k_x + ik_y) \\ v(k_x - ik_y) & m \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

2D Topological Superconductor

Topology of 2D gapped BdG Hamiltonian $H_{\text{BdG}} = \epsilon(k)\vec{n}_k \cdot \vec{\sigma}$

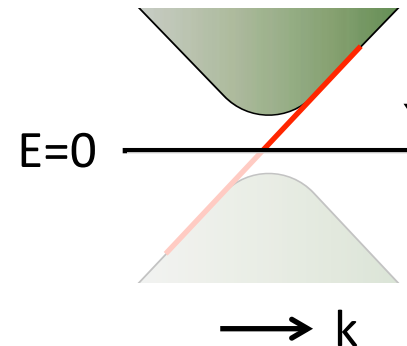
- integer topological invariant N
= # of times unit sphere is covered by
the mapping from (k_x, k_y) to unit vector n

multiband generalization: N = Chern number



Chiral Majorana fermions at edge: conduct heat, not charge

- 2D domain wall problem
reduces to 1D for each k parallel to edge
- # of chiral edge mode = Chern number:
bulk-boundary correspondence



2D chiral TSC = “half” of quantum Hall insulator

Topology Meets Symmetry

	Symmetry		Dimension		
	Time reversal	Spin rotation	d=1	d=2	d=3
D	No	No	Z_2	Z	0

A more refined topological classification taking (internal) symmetries into account has recently been achieved for insulators and superconductors.

Symmetry-protected topological phases:

- cannot be adiabatically connected to a trivial phase under symmetry-preserving deformations
- can be deformed to a trivial phase by symmetry-breaking perturbations

Topology Meets Symmetry

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Topological insulators: Hasan & Kane, Qi & Zhang ...

Topology Meets Symmetry

	Symmetry		Dimension		
	Time reversal	Spin rotation	d=1	d=2	d=3
D	No	No	Z_2	Z	0
<hr/>					
DIII	Yes	No	Z_2	Z_2	Z
C	No	Yes	0	Z	0
CI	Yes	Yes	0	0	Z

Topological Superfluid: He-3

Volovik et al

	Symmetry		Dimension		
	Time reversal	Spin rotation	d=1	d=2	d=3
D	No	No	Z_2	Z	0
DIII	Yes	No	Z_2	Z_2	Z
P-wave spin-triplet pairing:	$c_{k\uparrow}^\dagger c_{-k\uparrow}^\dagger, c_{k\downarrow}^\dagger c_{-k\downarrow}^\dagger, c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + c_{k\downarrow}^\dagger c_{-k\uparrow}^\dagger$ (Sz=1) (Sz= -1) (Sz=0)				

useful notation: d-vector $c_k^\dagger [\vec{d}(k) \cdot \vec{s}] (i s_y) c_{-k}^\dagger$

- component of Cooper pair spin along the direction of d-vector = 0

Different configurations of $d(k)$ over Fermi surface result in various phases of He-3

Topological Superfluid: He-3

Volovik et al

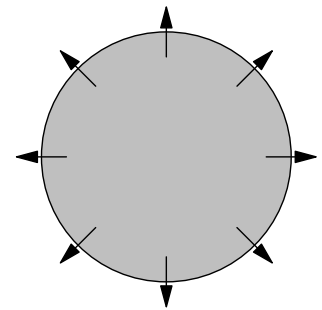
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D	No	No	Z_2	Z	0
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BW phase

Hedgehog d-vector: mapping from Fermi surface sphere to unit sphere with winding number ± 1

$$H = \sum_k \Psi_k^\dagger \begin{pmatrix} \frac{k^2}{2m} - \mu & v \sum_i k_i s_i \\ v \sum_i k_i s_i & \mu - \frac{k^2}{2m} \end{pmatrix} \Psi_k$$

$$\Psi_k^\dagger = (c_{k\uparrow}^\dagger, c_{k\downarrow}^\dagger, c_{-k\downarrow}, -c_{-k\uparrow})$$



- near $\mu \sim 0$, 3D Majorana fermion; “half” of topological insulator

Topological Superfluid: He-3

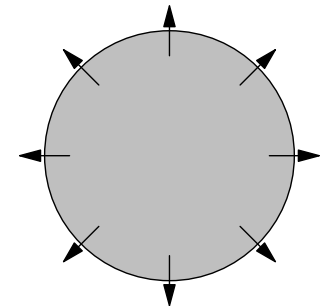
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BW phase

Hedgehog d-vector: mapping from Fermi surface sphere to unit sphere with winding number ± 1

- integer index N for **generic** T-invariant 3D superconductors
- N = winding number for superfluid He-3 BW.



Topological Superfluid: He-3

Volovik et al

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BW phase

Sketch of homotopy formula for N [Schnyder, Ryu, Furusaki & Ludwig](#)

$$\Xi H(k) \Xi^{-1} = -H(-k), \quad \Theta H(k) \Theta^{-1} = H(-k) \quad \rightarrow \quad \{H(k), \tau_z\} = 0$$

“flattened” BdG Hamiltonian: $\tilde{H}_{\text{BdG}}(k) = \begin{pmatrix} 0 & U(k) \\ U^\dagger(k) & 0 \end{pmatrix}$ U is unitary

$\pi_3(U(N)) = Z \Rightarrow$ integer classification of T-invariant superconductor in 3D

Topological Superfluid: He-3

Volovik et al

	Symmetry		Dimension		
	Time reversal	Spin rotation	d=1	d=2	d=3
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DIII	Yes	No	Z_2	Z_2	Z

ABM phase

uniaxial d-vector: $\vec{d}(k) = (k_x + ik_y)\hat{z}$

$$H = \sum_k \Psi_k^\dagger \begin{pmatrix} \frac{k^2}{2m} - \mu & v(k_x + ik_y)s_z \\ v(k_x - ik_y)s_z & \mu - \frac{k^2}{2m} \end{pmatrix} \Psi_k$$

- two copies of spinless p+ip: N=2

Search for Topological Superconductors

	Symmetry		Dimension		
	Time reversal	Spin rotation	d=1	d=2	d=3
D	No	No	Z_2	Z	0
DIII	Yes	No	Z_2	Z_2	Z
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Among thousands of known superconductors, not a single topological superconductor has been conclusively found yet.

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Among thousands of known superconductors, not a single topological superconductor has been conclusively found yet. 😊

Some are very close ...

1. Cuprate: gapping d-wave nodes by chiral d + i d order parameter => 2D class-C TSC
2. Strontium ruthenate: believed to be chiral p + i p, however is nodal.

Criterion for T-invariant Topological Superconductor with Inversion Symmetry

LF & Berg (10)

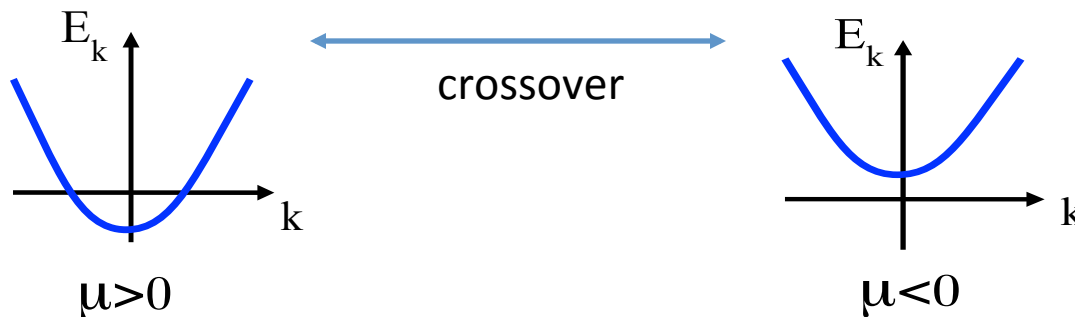
	d=1	d=2	d=3
DIII:	Z_2	Z_2	Z

Key necessary requirement: **odd-parity pairing** $\Delta(k) = -\Delta(-k)$

Simple argument:

- BEC limit is trivial: no gapless quasi-particle at boundary
- topological phase must be separated from trivial by a gap-closing transition

Even-parity superconductor: $\Delta(c_{k\alpha}^\dagger c_{-k\beta}^\dagger - c_{k\beta}^\dagger c_{-k\alpha}^\dagger)$ near band edge



Criterion for T-invariant Topological Superconductor with Inversion Symmetry

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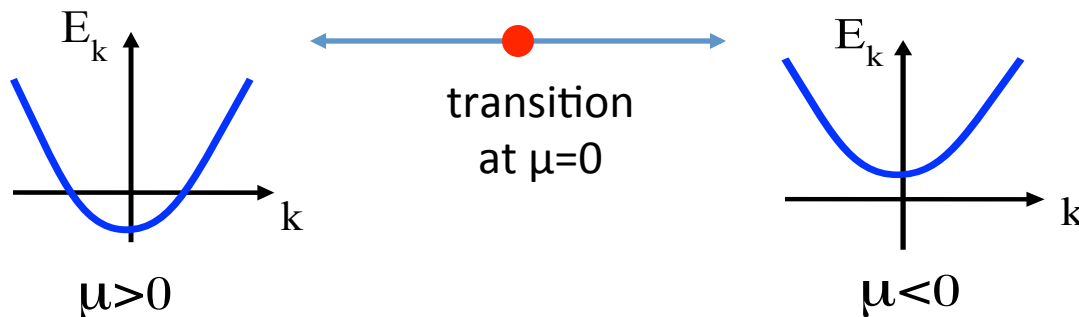
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Odd-parity superconductor: $\Delta c_k^\dagger [\vec{k} \cdot \vec{s}] (i s_y) c_{-k}^\dagger$ near band edge



Criterion for T-invariant Topological Superconductor with Inversion Symmetry

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	d=1	d=2	d=3
DIII:	Z_2	Z_2	Z

A T-invariant fully-gaped superconductor is topological if it has odd-parity pairing and an odd number of Fermi surface.

- general criterion for systems with multi-bands and spin-orbit coupling
- sufficient & necessary in d=1 & 2; sufficient in d=3
- proof is based on generalization of “parity criterion” for insulators [LF & Kane \(07\)](#)

Sketch of proof: $H_{\text{BdG}}(k) = H_0(k)\tau_z + \Delta(k)\tau_x$

$$\tilde{P}H_{\text{BdG}}(k)\tilde{P}^{-1} = H_{\text{BdG}}(-k), \quad \tilde{P} = P\tau_z$$

- Z_2 invariant = product of parity over TRIM
- here “parity” = particle or hole index
- $Z_2 = -1$ for odd number of FS



Criterion for T-invariant Topological Superconductor with Inversion Symmetry

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	d=1	d=2	d=3
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Topological superconductivity in class DIII \approx Odd-parity pairing

Role of Strong Spin-Orbit Coupling

LF & Berg (10)

Odd-parity pairing in presence of strong SOC:

- Bloch states in P- and T-invariant systems are doubly degenerate, but strongly spin-orbit entangled.
- odd-parity order parameter is defined in spin and orbital basis
- be careful with $d(\mathbf{k})$ that is defined in band basis: arbitrariness in basis choice at every \mathbf{k}

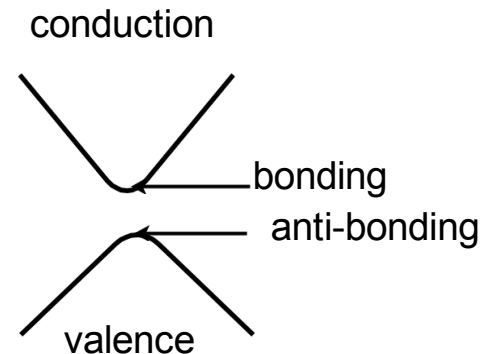
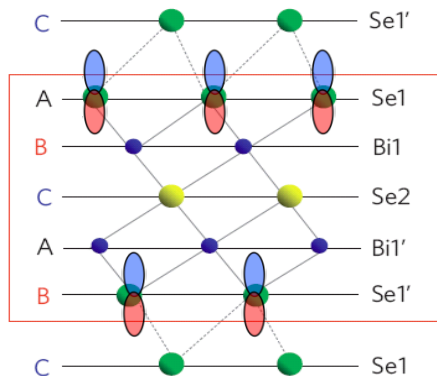
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Example: doped topological insulator $\text{Cu}_x\text{Bi}_2\text{Se}_3$...



$$H_0(\mathbf{k}) = \underbrace{m\sigma_x + v_z k_z \sigma_y}_{\text{inter-layer, spin-independent}} + \underbrace{v(k_x \sigma_z s_y - k_y \sigma_z s_x)}_{\text{intra-layer, bilayer Rashba SOC from local electric fields with opposite signs}}$$

inter-layer, spin-independent intra-layer, bilayer Rashba SOC from local electric fields with opposite signs

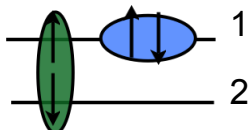
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Classification of pairing symmetry based on representation of crystal point group D_{3d} :

	pairing symmetry	C_3 rotation	inversion	mirror	
$c_{1\uparrow}c_{1\downarrow} + c_{2\uparrow}c_{2\downarrow}$ & $c_{1\uparrow}c_{2\downarrow} - c_{1\downarrow}c_{2\uparrow}$	A_{1g}	+	+	+	
$c_{1\uparrow}c_{2\downarrow} + c_{1\downarrow}c_{2\uparrow}$	A_{1u}	+	-	-	
$c_{1\uparrow}c_{1\downarrow} - c_{2\uparrow}c_{2\downarrow}$	A_{2u}	+	-	+	
$(c_{1\uparrow}c_{2\uparrow}, c_{1\downarrow}c_{2\downarrow})$	E_u	(x, y)	$(-, -)$	$(+, -)$	

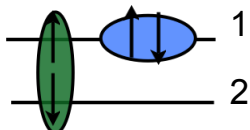
} odd-parity topological SC

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$c_{1\uparrow}c_{1\downarrow} + c_{2\uparrow}c_{2\downarrow}$ & $c_{1\uparrow}c_{2\downarrow} - c_{1\downarrow}c_{2\uparrow}$	A_{1g}	+	+	+	
$c_{1\uparrow}c_{2\downarrow} + c_{1\downarrow}c_{2\uparrow}$	A_{1u}	+	-	-	} odd-parity topological SC
$c_{1\uparrow}c_{1\downarrow} - c_{2\uparrow}c_{2\downarrow}$	A_{2u}	+	-	+	
$(c_{1\uparrow}c_{2\uparrow}, c_{1\downarrow}c_{2\downarrow})$	E_u	(x, y)	$(-, -)$	$(+, -)$	