Leonhard Euler's Elastic Curves

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# Leonhard Euler's Elastic Curves

**(De Curvis Elasticis, Additamentum I to his Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes,** Lausanne and Geneva, 1744).

Translated and Annotated
by
W. A. OLDFATHER, C. A. ELLIS, and D. M. BROWN

## PREFACE

In the fall of 1920 Mr. CHARLES A. ELLIS, at that time Professor of Structural Engineering in the University of Illinois, called my attention to the famous appendix on elastic curves by LEONHARD EULER, which he felt might well be made available in an English translation to those students of structural engineering who were interested in the classical treatises which constitute landmarks in the history of this ever increasingly important branch of scientific and technical achievement. He secured photostats of that copy of the original publication which was owned by the New York Public Library, and together we spent many delightful evenings working over the translation, and correcting the occasional errors of printing and calculation which such a first edition inevitably contained. We also examined and translated a considerable number of the notes in Dr. H. LINSENBARTH's admirable translation and commentary (Leipzig, 1910). The Ms. was practically completed when Mr. ELLIS left the University in order to enter active business in Chicago. For some time the various drafts and annotations lay in my files, until early in 1932, when I was fortunate enough to secure the

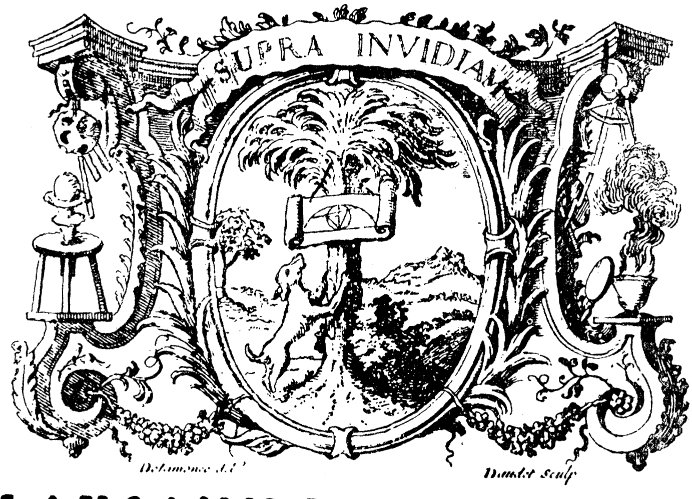
# METHODUS
## INVENIENDI
# LINEAS CURVAS
Maximi Minimive proprietate gaudentes,

*SIVE*

# SOLUTIO
## PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI

*AUCTORE*


## LEONHARDO EULERO,
*Professore Regio, & Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*



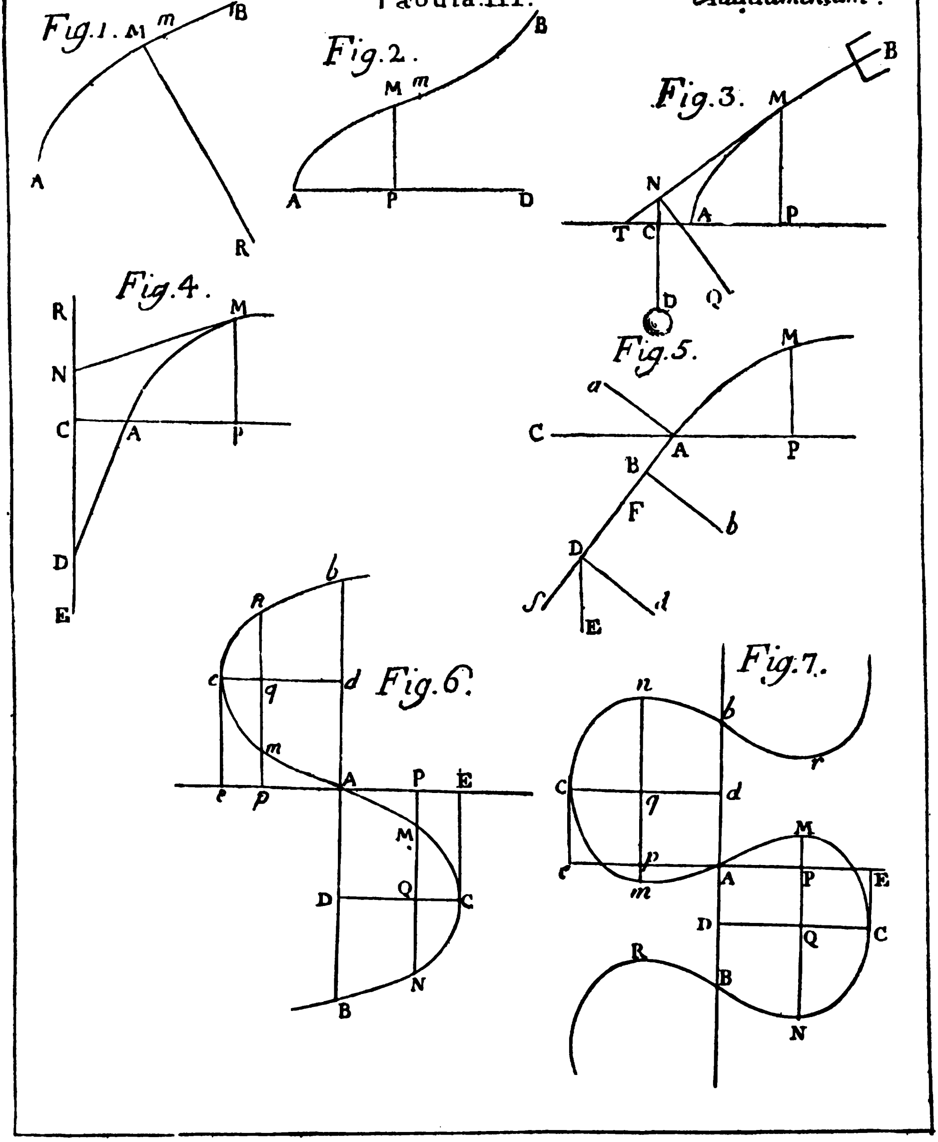
LAUSANNÆ & GENEVÆ,

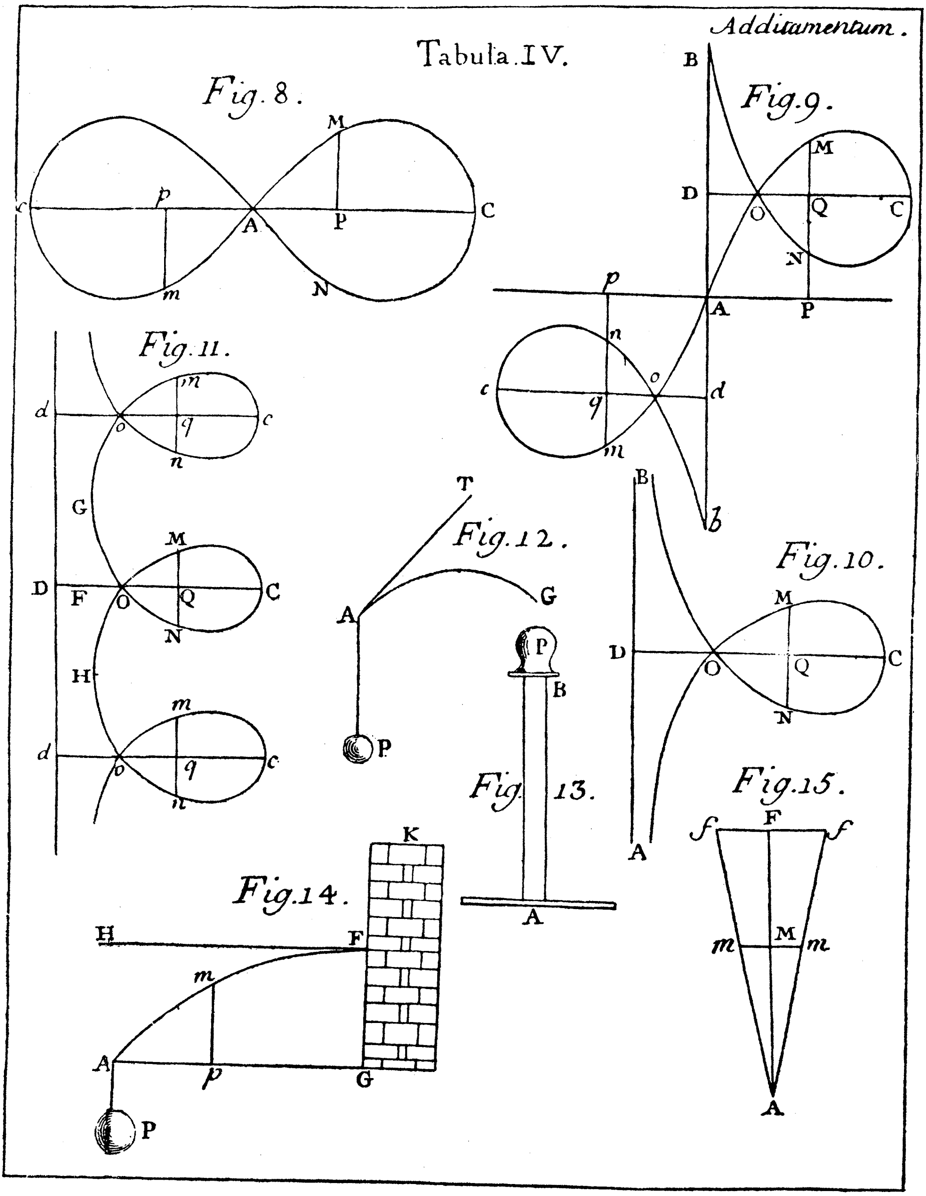
Apud MARCUM-MICHAELEM BOUSQUET & Socios,

MDCCXLIV.

Tabula III.

Additamentum.

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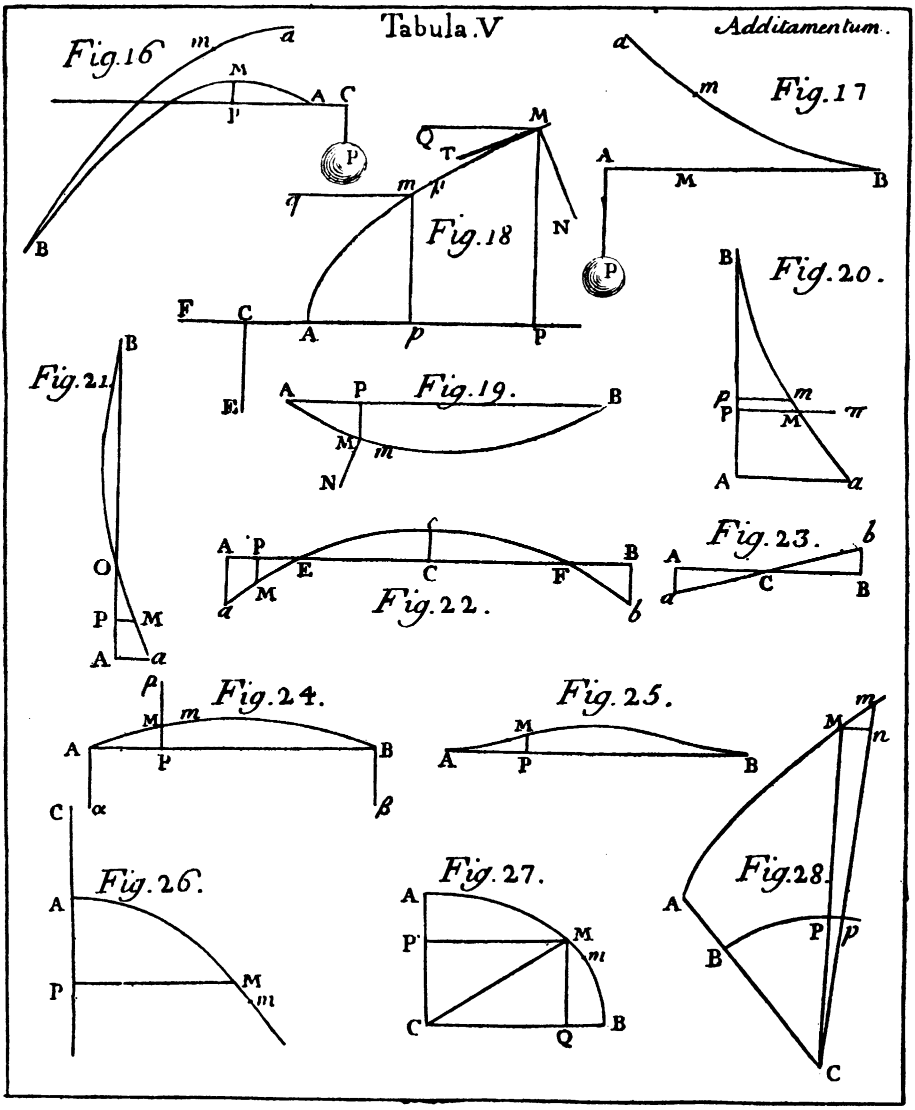
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very competent assistance of Mr. DONALD M. BROWN, formerly a student of Engineering, but at present an assistant in the Department of Mathematics of the University of Illinois, who undertook to revise the translation, together with Dr. LINSENBARTH's notes, to check all the equations and calculations, and occasionally to express the mathematical formulas in the more modern and generally current notation. It is to be hoped that the combination of an engineer, a classicist, and a mathematician in translating EULER's monograph may have reduced somewhat the number of errors which any one of the three unaided might easily have made.

Urbana, Illinois, April 27, 1932.

W. A. OLDFATHER.

## ADDITIONAL NOTE BY DONALD M. BROWN

Such factual errors and mistakes as were made by EULER, and have been pointed out by others, have been corrected in the body of the text, the errors themselves being indicated in the notes. An exception to this is the error pointed out in note 31, where the correction would involve the incorporation of several sections of the body of the text into the notes. In this case, since the error involved was not great enough to make any essential difference in the calculations, the text was included as translated, and the correct values have been inserted within square brackets immediately following the incorrect values at all places where errors had been made. With the exception of the error indicated in note 23, all the errors were pointed out by H. LINSENBARTH in his German translation of the text in "Ostwald's Klassiker der exakten Wissenschaften," vol. 175 (Leipzig, 1910). In fact, all the notes, correctional, explanatory, and introductory, have been incorporated as translated, but several of the correctional notes have been modified to conform to the plan mentioned above of correcting the text, and indicating the errors themselves in the notes.

LINSENBARTH's admirable translation was used throughout as a check, and since this work includes numerous cross references with the text which are not found in the original, such references have been inserted within square brackets.

The facsimiles of the title page and of the figures are from the Harvard Library copy; they are reduced to about two-thirds of the original size; Figs. 26, 27, and 28 belong to Additamentum II, *De motu projectorum in medio non resistente.*

Nothing has been omitted in this translation.

D. M. BROWN.

## INTRODUCTION

"It is of the utmost importance," writes Professor G. A. MILLER (1), "that those students who desire a deep mathematical insight should accustom themselves early to go directly to the original developments,—at least in those cases where the original developments are direct, and are found in a language which does not impose too great difficulties." The statement might be equally applicable to the same kind of student of engineering, and it is for such students of mathematics and of engineering that the present translation from the original Latin into English is intended.

LEONHARD EULER (1707-1783), probably the most versatile, certainly quite the most prolific mathematician of all time (2), needs no commendation from us. CONDORCET, in his celebrated *Éloge* (3), after remarking that "all the celebrated mathematicians of today are his pupils," quite justly lists him as "one of the

(1) *Historical Introduction to Mathematical Literature* (New York, 1916) 74.

(2) By early in 1783, 530 studies of his had been published; by 1826 the number had increased to 771. In 1844 a great-grandson discovered still further material in manuscript. (G. DU PASQUIER : *Léonard Euler et ses amis* (Paris, 1927) 116). His complete bibliography, by G. ENESTRÖM : *Verzeichnis der Schriften Leonhard Eulers* (Leipzig, 1910 and 1913), lists 866 separate items, together with a voluminous correspondence. The *Leonhard Euler-Gesellschaft*, a society organized for the sole purpose of publishing his works in proper modern form, produced its first volume in 1911, and down to date has brought out some 20 volumes. The completed undertaking will require 69 quarto volumes, of which 55 are assigned to Mathematics, Mechanics, and Astronomy, the remainder to Physics and Varia.

(3) Published in *Les Lettres de L. Euler à une Princesse d'Allemagne* (Paris, 1842), xlviii and 1.

very greatest and most extraordinary men that nature has ever produced."

The treatise also to which the present study is merely an appendix *(Methodus Inveniendi Lineas Curvas)* is called by DU PASQUIER " one of the finest monuments of the genius of EULER," who, he continues, " founded the calculus of variations which has become, in the twentieth century, one of the most efficient of the means of investigation employed by mathematicians and physicists. The recent theories of EINSTEIN and the applications of the principle of relativity have greatly increased the importance of the calculus of variations which EULER created" (*op. cit.*, 50-51).

The special interest for engineering in the present little paper, lies in the fact that it is the first systematic treatment of elastic curves, laying the foundation for subsequent studies, and of course most immediately for the celebrated 'Euler formula,' " which expresses the critical load at which a slender column buckles." (4)

Referring to the *Additamentum I*, TODHUNTER and PEARSON say, " Euler distinguishes the various species of curves included under the general differential equation...

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}$$ [p. 82 of this translation.]

... The whole discussion is worthy of this great master of analysis;" (5) Again, " From page 282 [p. 121 of this translation] to the end EULER devotes his attention to the *oscillations* of an elastic lamina; the investigation is somewhat obscure for the science of dynamics had not yet been placed on the firm foundation of *D'Alembert's Principle:* Nevertheless, the results obtained by EULER will be found in substantial agreement with those in POISSON's *Traité de Mécanique*, Vol. II, pages 368-392. The important equations (a) and (a') on POISSON's pages 377 and 387 respectively agree with corresponding equations on EULER's " [pages 297 and 287 = pages 135 and 125 respectively of this translation] (6).

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(4) H. M. WESTERGARD : *One Hundred and Fifty Years Advance in Structural Analysis. Transactions American Society of Civil Engineers* 94 (1930) 228. Compare also remarks by S. TIMOSHENKO (*ibid.* 241).

(5) TODHUNTER and PEARSON : *A History of the Theory of Elasticity and of the Strength of Materials.* Cambridge, University Press, 1886, p. 36.

(6) *Op. cit.*, p. 38.

Referring to EULER's *Sur la force des colonnes*, *Mémoires de l'Académie de Berlin*, Tom. XIII, 1759, pages 252-282, TODHUNTER and PEARSON (7) say, "This is one of EULER's most important contributions to the theory of elasticity.

The problem with which this memoir is concerned, is the discovery of the least force which will suffice to give the least curvature to a column, when applied at one extremity parallel to its axis, the other extremity being fixed. EULER finds that the force must be at least $\pi^2 \frac{Ek^2}{a^2}$, where $a$ is the length of the column and $Ek^2$ is the 'moment of the spring,' or the 'moment of stiffness of the column'..." The volume from which this is quoted is intended to be a chronological development of the theory of elasticity, but the authors obviously overlooked the fact that the problem stated is precisely the one considered in section 37, pages 267-268 of the original *Additamentum I* [pages 102-103 of this translation]. Hence the present work is the first known source of the famous 'EULER formula' mentioned above.

It might, in conclusion, be observed that the problem of elastic curves and the buckling of long thin struts under thrust was first worked out by EULER (8).

W. A. OLDFATHER and D. M. BROWN.

## CONCERNING ELASTIC CURVES

### L. EULER

1. All the greatest mathematicians have long since recognized that the method presented in this book is not only extremely useful in analysis, but that it also contributes greatly to the solution of physical problems. For since the fabric of the universe is most perfect, and is the work of a most wise Creator, nothing whatsoever takes place in the universe in which some relation of maximum and minimum does not appear. Wherefore there

(7) *Op. cit.*, p. 39.

(8) LOVE: *The Mathematical Theory of Elasticity*. Cambridge, University Press, Third Edition, 1920. Footnotes p. 407 and p. 411. See also Introduction, p. 3.

is absolutely no doubt that every effect in the universe can be explained as satisfactorily from final causes, by the aid of the method of maxima and minima, as it can from the effective causes themselves. Now there exist on every hand such notable instances of this fact, that, in order to prove its truth, we have no need at all of a number of examples; nay rather one's task should be this, namely, in any field of Natural Science whatsoever to study that quantity which takes on a maximum or a minimum value, an occupation that seems to belong to philosophy rather than to mathematics. Since, therefore, two methods of studying effects in Nature lie open to us, one by means of effective causes, which is commonly called the direct method, the other by means of final causes, the mathematician uses each with equal success. Of course, when the effective causes are too obscure, but the final causes are more readily ascertained, the problem is commonly solved by the indirect method; on the contrary, however, the direct method is employed whenever it is possible to determine the effect from the effective causes. But one ought to make a special effort to see that both ways of approach to the solution of the problem be laid open; for thus not only is one solution greatly strengthened by the other, but, more than that, from the agreement between the two solutions we secure the very highest satisfaction. Thus the curvature of a rope or of a chain in suspension has been discovered by both methods; first, *a priori*, from the attractions of gravity; and second, by the method of maxima and minima, since it was recognized that a rope of that kind ought to assume a curvature whose center of gravity was at the lowest point. Similarly, the curvature of rays passing through a transparent medium of varying density has been determined both *a priori*, and also from the principle that they ought to arrive at a given point in the shortest time.

Other similar examples have been brought forward in large numbers by the most eminent BERNOULLIS and others, who have made very great contributions both to the method of *a priori* solution, and to the knowledge of effective causes. Although, therefore, thanks to these so numerous and striking instances, there can be no doubt that in the case of all curved lines which appear in the solution of physical-mathematical problems, there enters in the character of some maximum or minimum; still this very maximum or minimum is frequently very hard to recognize, although one might have reached a solution *a priori*. Thus, although the figure which a curved elastic ribbon assumes has long since been known, nevertheless no one has observed as yet how this curve can be studied by the method of maxima and minima, that is to say, by means of final causes. Wherefore, seeing that the most illustrious and, in this sublime fashion of studying nature, most perspicacious man, DANIEL BERNOULLI, had pointed out to me that he could express in a single formula, which he calls the potential force, the whole force which inheres in a curved elastic ribbon, and that this expression must be a minimum in the elastic curve [1]), and since by this discovery my method of maxima and minima as set forth in this book has had new light cast upon it in a marvelous fashion, and its most extensive application is thoroughly established, I cannot let pass this most desired opportunity without making clearer the application of my method at the same time that I publish this remarkable characteristic of the elastic curve discovered by the celebrated BERNOULLI. For that characteristic contains within itself differentials of the second order in such a fashion that the methods hitherto published of solving the isoperimetric problem are not capable of disclosing it.

## ON THE CURVATURE OF UNIFORM ELASTIC RIBBONS

2. Let the elastic ribbon $AB$ (FIG. 1) be curved in any direction whatsoever; let the arc $AM = s$, and the radius of curvature $MR = R$; furthermore, according to BERNOULLI, let the potential force contained in the portion $AM$ of the ribbon be designated by the expression $\int \frac{ds}{R^2}$. If the ribbon be of uniform cross section and elasticity, and if it be straight when in its natural position, [2]) the character of the curve $AM$ will be such that in this case the expression $\int \frac{ds}{R^2}$ is an absolute minimum. But since the differentials of the second order appear in the radius of curvature $R$, in order to determine a curve of this character we shall need four conditions, and this is precisely the

subject of our investigation. For since through the given ends *A* and *B*, an infinite number of elastic ribbons of the same length can be bent, the problem will not have been solved unless, in addition to the two points *A* and *B*, two other points, or what amounts to the same thing, the position of the tangents at the points *A* and *B* be given at the same time. For given an elastic ribbon which is longer than this distance between *A* and *B*, it can be curved not only in such a way that it is contained between the ends *A* and *B*, but also in such a way that its tangents have a given direction through these points. Wherefore the problem of finding the curvature of an elastic ribbon, which is to be solved by this method, must be expressed thus:

That among all curves of the same length, which not only pass through the points *A* and *B*, but also are tangent to given straight lines at these points, that curve be determined in which the value of $\int \frac{ds}{R^2}$ is a minimum.

3. Because the solution is to be referred to rectangular coordinates, let any straight line *AD* be taken as an axis, the abscissa $AP = x$, (FIG. 2) and the ordinate $PM = y$; then, according to this method, letting $dy = pdx$, and $dp = qdx$, the element of the curve *Mm* will be $ds = dx\sqrt{1 + p^2}$. Since the curves from among which the curve sought has to be discovered are to be isoperimetric, in the first place, the expression $dx\sqrt{1 + p^2}$ will have to be considered; this, compared with the general expression $\int Zdx$, gives the differential value $\frac{d}{dx}\frac{p}{\sqrt{1 + p^2}}$. Secondly, since the radius of curvature is $\frac{dx\,(1 + p^2)}{dp} = \frac{(1 + p_2)^{\frac{3}{2}}}{q} = R$, the expression $\int \frac{ds}{R^2}$ which must be a minimum, is transformed into $\int \frac{q^2\,dx}{(1 + p_2)^{\frac{5}{2}}}$. Let this be compared with the general expression $\int Zdx$, and this gives $Z = \frac{q^2}{(1 + p_2)^{\frac{5}{2}}}$, and letting $dZ = Mdx + Ndy + Pdp + Qdq$, then $M = o$,

$N = o$, $P = \frac{-5pq^2}{(1+p^2)^{\frac{7}{2}}}$, and $Q = \frac{2q}{(1+p^2)^{\frac{5}{2}}}$. Therefore the differential value to be derived from the expression $\int \frac{q^2\, dx}{(1+p^2)^{\frac{5}{2}}}$ will be $-\frac{dP}{dx} + \frac{d^2Q}{dx^2}$. And so for the curve sought we shall have the equation :

$$a \frac{d}{dx} \frac{p}{\sqrt{1+p^2}} = \frac{dP}{dx} - \frac{d^2Q}{dx^2}$$

which, multiplied by $dx$ and integrated, gives

$$\frac{ap}{\sqrt{1+p^2}} + \beta = P - \frac{dQ}{dx}$$

Let this equation be multiplied by $qdx = dp$.

$$\frac{apdp}{\sqrt{1+p^2}} + \beta dp = Pdp - qdQ$$

Since $M = o$, and $N = o$, then $dZ = Pdp + Qdq$, or $Pdp = dZ - Qdq$.
Substituting this value for $Pdp$ gives

$$\frac{apdp}{\sqrt{1+p^2}} + \beta dp = dZ - Qdq - qdQ$$

Integrating :

$$a\sqrt{1+p^2} + \beta p + \gamma = Z - Qq$$

Now since

$$Z = \frac{q^2}{(1+p^2)^{\frac{5}{2}}} \text{ and } Q = \frac{2q}{(1+p^2)^{\frac{5}{2}}}$$

$$a \sqrt{1+p^2} + \beta p + \gamma = \frac{-q^2}{(1+p^2)^{\frac{5}{2}}}$$

Taking the arbitrary constants negatively,

$$q = (1+p^2)^{\frac{5}{4}} \sqrt{a \sqrt{1+p^2} + \beta p + \gamma} = \frac{dp}{dx}$$

whence

$$dx = \frac{dp}{(1+p^2)^{\frac{5}{4}} \sqrt{a \sqrt{1+p^2} + \beta p + \gamma}}$$

Then, since $dy = pdx$,

$$dy = \frac{pdp}{(1+p^2)^{\frac{5}{4}} \sqrt{a\sqrt{1+p^2} + \beta p + \gamma}}$$

These two equations would be sufficient for constructing a curve by means of quadratures.

4. Neither of these equations regarded thus in general can be integrated, but they can be combined in a certain fashion so that the sum can be integrated. For, since

$$d2\frac{\sqrt{a\sqrt{1+p^2} + \beta p + \gamma}}{\sqrt[4]{1+p^2}} = \frac{dp\ (\beta - \gamma p)}{(1+p^2)^{\frac{5}{4}} \sqrt{a\sqrt{1+p^2}+\beta p+\gamma}},$$

then $$\frac{2\sqrt{a\sqrt{1+p^2}+\beta p+\gamma}}{(1+p^2)^{\frac{1}{4}}} = \beta x - \gamma y + \delta.$$

Since the position of the axis is arbitrary, the constant $\delta$ can be left out without any loss in generality. Moreover, the axis can be so changed that the abscissa will become $X = \frac{\beta x - \gamma y}{\sqrt{\beta^2 + \gamma^2}}$, and the ordinate will become $Y = \frac{\gamma x + \beta y}{\sqrt{\beta^2 + \gamma^2}}$. [3]) Also $\gamma$ can be safely made equal to zero, because nothing prevents the new abscissa from being expressed by $x$. For this reason we will get the following equation for the elastic curve:

$$2\sqrt{a\sqrt{1+p^2} + \beta p} = \beta x\ (1+p^2)^{\frac{1}{4}},$$

which, after squaring becomes $4a\sqrt{1+p^2} + 4\beta p = \beta^2 x^2 \sqrt{1+p^2}$.

To introduce homogeneity, let $a = \frac{4m}{a^2}$, and $\beta = \frac{4n}{a^2}$;

then $na^2p = (n^2x^2 - ma^2)\sqrt{1+p^2}$,
whence $n^2a^4p^2 = (n^2x^2 - ma^2)^2\ (1+p^2)$,
and therefore

$$p = \frac{n^2 x^2 - ma^2}{\sqrt{n^2 a^4 - (n^2 x^2 - ma^2)^2}}.$$

By changing the constants, and either by increasing or diminishing the abscissa $x$ by a given constant, 4) the following general equation for the elastic curve will be secured :

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

from which

$$ds = \frac{a^2 dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

From these equations, the agreement of this discovered curve with the elastic curve already determined is perfectly clear.

5. In order that this agreement be placed more clearly before the reader, I shall investigate also *a priori* the nature of the elastic curve. Although this has been done already in a most excellent fashion by that very great man, JACOB BERNOULLI, nevertheless, since this favorable occasion has been offered, I shall add certain things about the character of elastic curves and their different kinds and figures which I see have been either neglected by other men, or else have been but lightly touched upon.

Let the elastic ribbon $AB$ (FIG. 3) be fixed in a wall or solid pavement at $B$ in such a fashion that the extremity $B$ is not only held firmly, but also the position of the tangent at $B$ is fixed. Now at $A$ let the ribbon have fastened to it the rigid rod $AC$, to which let there be applied normally the force $CD = P$, whereby the ribbon is brought into the curved position $BMA$. Let this straight line $AC$ be considered as produced for an axis, and, having assumed that $AC = c$, let the abscissa $AP = x$, and the ordinate $PM = y$. If now the ribbon at $M$ should suddenly lose all elasticity and become perfectly flexible, it would assuredly be turned by the force $P$, the inflexion being caused by the moment of the force $P = P\,(c + x)$. The reason why this inflexion does not actually follow, therefore, is that the elasticity depends, in the first place, upon the character of the material of which the ribbon is composed and which I assume always to be the same; but in the second place the elasticity depends, at the same time, upon the curvature of the ribbon at the point $M$, in such a way that it is inversely proportional to the radius of curvature at $M$. Therefore let the radius of curvature at $M$ be $R = \frac{(ds)^3}{-dxd^2y}$; here $ds = \sqrt{dx^2 + dy^2}$, and $dx$ remains constant; and let $\frac{Ek^2}{R}$ express the elastic force of the ribbon at $M$, which stands in equilibrium with the moment of the external force $P(c + x)$, in such a manner that

$$P\ (c + x) = \frac{Ek^2}{R} = -\frac{Ek^2dxd^2y}{(ds)^3}.$$

This equation, multiplied by $dx$, becomes integrable, and the integral will be

$$P\ (\tfrac{1}{2}x^2 + cx + f) = \frac{-Ek^2dy}{\sqrt{dx^2 + dy^2}},$$

whence

$$dy = \frac{-Pdx\ (\frac{1}{2}x^2 + cx + f)}{\sqrt{E^2k^4 - P^2\ (\frac{1}{2}x^2 + cx + f)^2}}.$$

This equation agrees absolutely with that which I have just secured through the method of maxima and minima from Bernoulli's principle.

6. From the comparison of this equation with the one found before, it will be possible to determine the force which is required to produce the given curvature of the ribbon, since the curvature is contained in the discovered general equation. In other words, let the elastic ribbon have the shape $AMB$, the nature of which is expressed by the equation

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\ dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

and let $Ek^2$ express the absolute elasticity of this ribbon in such a manner, indeed, that $Ek^2$ at any point you please, divided by the radius of curvature, represents the true elastic force. To institute a comparison, let the numerator and denominator be each multiplied by $\frac{Ek^2}{a^2}$, so that we have

$$dy = \frac{\dfrac{Ek^2\ dx\ (\alpha + \beta x + \gamma x^2)}{a^2}}{\sqrt{E^2k^4 - \dfrac{E^2k^4}{a^4}\ (\alpha + \beta x + \gamma x^2)^2}}$$

Therefore

$$-\tfrac{1}{2}P = \frac{Ek^2\gamma}{a^2};\ -PC = \frac{Ek^2\beta}{a^2};\ \text{and}\ -Pf = \frac{Ek^2\alpha}{a^2};$$

and hence the external force $CD = \frac{-2Ek^2\gamma}{a^2}$, the length $AC = c = \frac{\beta}{2\gamma}$, and the constant $f = \frac{\alpha}{2\gamma}$.

7. In order, therefore, that the elastic ribbon $AB$, fixed in the wall at one end $B$, be curved into the shape $AMB$, the character of which is expressed by the following equation:

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

it is necessary that the ribbon be drawn in the direction $CD$ normal to the axis $AP$ (assuming that the distance $AC = \frac{\beta}{2\gamma}$), by the force $CD = \frac{-2Ek^2\gamma}{a^2}$. This force will act, of course, in an opposite direction from that shown in the figure, if $\gamma$ be a positive quantity. Because $\frac{Ek^2}{R}$ is equivalent to the moment of the external force, the expression $\frac{Ek^2}{a^2}$ will be equivalent to the weight, or to a pure force, which force, $\frac{Ek^2}{a^2}$, for that reason, will be determined by the elasticity of the ribbon. Let this force be $F$; then the deflecting force $CD$ will be to this force $F$ as $-2\gamma$ is to 1, for $\gamma$ will be an abstract number.

8. Now in addition, the force required to keep the portion $BM$ of the ribbon in its position, if the portion $AM$ should be entirely cut off, can be determined from this. When this portion $AM$ is cut off, the elastic ribbon becomes a rigid rod $MT$ [Fig. 3] without any flexure at all, and this is so connected with the ribbon that it always makes a tangent at the point $M$, no matter how the ribbon may be bent. If we assume this, it is clear from what precedes that to preserve the curvature $BM$, it is necessary

that the rod $MT$ at the point $N$ be drawn in the direction $ND$ by a force which equals $-\frac{2Ek^2\gamma}{a^2}$; the direction $ND$ will be normal to the axis $AP$, and likewise the interval $AC$ will be equal to $\frac{\beta}{2\gamma}$. And thus the distance $MN$ will be

$$\frac{ds}{dx}.CP = \frac{ds}{dx}.\frac{\beta+2\gamma x}{2\gamma} = \frac{(\beta+2\gamma x)\,ds}{2\gamma\,dx}.$$

Then

$$\frac{ds}{dx} = \frac{a^2}{\sqrt{a^4-(\alpha+\beta x+\gamma x^2)^2}}$$

But if this force $ND$, which equals $-\frac{2Ek^2\gamma}{a^2}$, be resolved into two components, $NQ$ normal to the tangent $MT$, and $NT$ along $MT$, the normal force $NQ = \frac{-2Ek^2\gamma}{a^2}.\frac{dx}{ds}$, and the tangential force $NT = \frac{-2Ek^2\gamma}{a^2}.\frac{dy}{ds}$.

9. But now if the part $BM$ be cut off, leaving the part $AM$, which is drawn as before in the direction $CD$ by the force $-\frac{2Ek^2\gamma}{a^2}$, in order to preserve the curvature $AM$, the extremity $M$, which is understood to be connected with the rigid tangent rod $MN$, will have to be drawn, at the point $N$, by a force also equal to $\frac{-2Ek^2\gamma}{a^2}$, but in the direction opposite to that which we have discovered in the preceding case. For the forces which will have to be applied to both extremeties of the curved ribbon constantly oppose each other, and consequently must be equal and opposite. For otherwise the whole ribbon would be moved, and to restrain this movement, a force would be necessary to cause equilibrium between these forces. Hence the forces to be applied at any portion of the part which has been cut off can be determined; these forces will preserve the curvature already induced.

10. Let $AM$ (FIG. 4) be an elastic ribbon, to which, at $A$ and $M$ are attached rigid rods $AD$ and $MN$, and to which in opposite directions $DE$ and $NR$ there have been applied equal forces $DE$ and $NR$, which, being in equilibrium, induce the curvature $AM$ in the ribbon. For this curvature an equation will be sought. First, therefore, let there be taken as an axis the straight line $AP$ passing through the point $A$, and normal to the direction $ER$ of the external force. Let the absolute elasticity of the ribbon be $Ek^2$; and let the sine of the angle $CAD$ which the tangent forms with the axis at $A$, and which has been given, equal $m$, and the cosine equal $n$, so that $m^2 + n^2 = 1$. Furthermore, let the distance $AC = c$, and the bending force $DE = NR = P$. Letting the abscissa $AP = x$, and the ordinate $PM = y$, the character of the curve will be expressed by the following equation:

$$dy = \frac{- Pdx\ (\frac{1}{2}x^2 + cx + f)}{\sqrt{E^2k^4 - P^2\ (\frac{1}{2}x^2 + cx + f)^2}}$$

But since the direction of the tangent at $A$ is given, when $x = o$, $\frac{dy}{dx} = \frac{m}{n}$ whence $\frac{m}{n} = \frac{-Pf}{\sqrt{E^2k^4 - P^2f^2}} = \frac{m}{\sqrt{1-m^2}}$, and $m = \frac{-Pf}{Ek^2}$.

Therefore the constant $f$ is determined, so that $f = \frac{-mEk^2}{P}$, and hence the whole curve is thereby determined.

11. To produce, therefore, the curvature of the ribbon $AM$, expressed by the foregoing equation, the force $DE = P$ must be applied to the tangent $AD$, at the point $D$, in such a way that $AD = \frac{c}{n}$, and in a direction parallel to the ordinate $PM$. Let this force $DE$ be resolved into two rectangular components $Dd$ and $Df$ (FIG. 5), normal to one another, the force $Dd = Pn$, and the force $Df = Pm$. Now in order that the consideration of the straight line $AD$ may be eliminated from the computation, in place of the force $Dd$ at the given points $A$ and $B$ (assuming that $AB = h$), two forces $Aa = p$ and $Bb = q$ can be substituted, likewise normal to the rod $AB$, if we let $Ph = Pn.BD = nP(\frac{c}{n} - h)$, and $q = p + nP$. In the next place, because it makes no differ-

ence at which point of the rod $AB$ the tangential force $Df = mP$ be applied, let it be applied at the point $A$, where $AF = mP$. Now let the force $AF = r$ in such a way that the ribbon $MA$ is acted upon by the three forces $Aa = p$, $Bb = q$, and $AF = r$. We shall investigate how the curvature is affected by them.

12. First, since $mP = r$, $P = \frac{r}{m}$, which value, substituted in the former equations, will give $ph = \frac{cr}{m} - \frac{nhr}{m}$, and $q = p + \frac{nr}{m}$; hence $\frac{n}{m} = \frac{q-p}{r}$ from which equation first the position of the axis $AP$ becomes known; for $\tan CAD = \frac{r}{q-p}$.

Hence

$$m = \frac{r}{\sqrt{r^2 + (q-p)^2}}, \text{ and } n = \frac{q-p}{\sqrt{r^2 + (q-p)^2}}.$$

Secondly, from the equation $hp = \frac{cr}{m} - \frac{nhr}{m} = \frac{cr}{m} - hq + hp$, it follows that $c = \frac{mhq}{r}$, or $c = \frac{hq}{\sqrt{r^2 + (q-p)^2}}$, and

$$P = \sqrt{r^2 + (q-p)^2}.$$

Now since

$$f = \frac{-mEk^2}{P} = \frac{-Ek^2r}{r^2 + (q-p)^2},$$

then

$$\frac{x^2}{2} + cx + f = \frac{x^2}{2} + \frac{hqx}{r^2 + (q-p)^2} - \frac{Ek^2r}{r^2 + (q-p)^2},$$

from which the following equation of the curve sought will be obtained:

$$dy = \frac{dx\left[\frac{Ek^2r}{\sqrt{r^2+(q-p)^2}} - hqx - \frac{1}{2}x^2\sqrt{r^2+(q-p)^2}\right]}{\sqrt{E^2k^4 - \left[\frac{Ek^2r}{\sqrt{r^2+(q-p)^2}} - hqx - \frac{1}{2}x^2\sqrt{r^2+(q-p)^2}\right]^2}}$$

Now this equation is very convenient for the most common method of bending ribbons when they are held either by forceps or two fingers, one of which presses in a direction *Aa*, the other in the direction *Bb*, while, at the same time, the ribbon can be stretched in the direction *AF*.

13. If the tangential force $AF = r$ should disappear, the axis *AP* will fall upon the tangent *AF* produced, and

$$dy = \frac{-\,dx\,[hqx + \frac{1}{2}\,(q-p)\,x^2]}{\sqrt{E^2k^4 - [hqx + \frac{1}{2}\,(q-p)\,x^2]^2}}.$$

But if the normal forces $p$ and $q$ should be equal, the axis *AP* will be normal to the tangent *AF*, because $n = o$, and we shall have the following equation for the curve:

$$dy = \frac{dx\,(Ek^2 - hqx - \frac{1}{2}\,rx^2)}{\sqrt{2\,Ek^2\,(hqx + \frac{1}{2}\,rx^2) - (hqx + \frac{1}{2}\,rx^2)^2}}.$$

Hence if also $r = o$ in such a way that the ribbon at the points $A$ and $B$ be subjected to equal and opposite forces *Aa* and *Bb*, the character of the curve will be expressed by

$$dy = \frac{dx\,(Ek^2 - hqx)}{\sqrt{hq\,(2Ek^2x - hqx^2)}},$$

which, when integrated, gives

$$y = \sqrt{\frac{2Ek^2x - hqx^2}{hq}}.$$

This is the equation of a circle, and therefore, in this case, the ribbon is bent into the arc of a circle, the radius of which will be $\frac{Ek^2}{hq}$.

## The Enumeration of Elastic Curves

14. Since therefore we observe that not only is the circle included in the class of elastic curves, but more than that, there is an infinite variety of these elastic curves, it will be worth while to enumerate all the different kinds included in this class of curves. For in this way not only will the character of these curves be more profoundly perceived, but also, in any case whatsoever

offered, it will be possible to decide from the mere figure into what class the curve formed ought to be put. We shall also list here the different kinds of curves in the same way in which the kinds of algebraic curves included in a given order are commonly enumerated [5]).

15. The general equation for elastic curves is

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

which, if the origin of the abscissas be moved on the axis through the distance $\frac{\beta}{2\gamma}$, and if $a^2$ be written for $\frac{a^2}{\gamma}$ (or making $\gamma = 1$), takes the simpler form

$$dy = \frac{(\alpha + x^2)\, dx}{\sqrt{a^4 - (\alpha + x^2)^2}}.$$

But because $a^4 - (\alpha + x^2)^2 = (a^2 - \alpha - x^2)(a^2 + \alpha + x^2)$, let $a^2 - \alpha = c^2$, so that $\alpha = a^2 - c^2$, and the equation will be transformed into

$$dy = \frac{(a^2 - c^2 + x^2)\, dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

Let the character of the curve *AMC* (FIG. 6) be expressed by this equation, and the abscissa $AP = x$, and the ordinate $PM = y$. Therefore, since $\beta = o$ [see FIG. 3, Sec. 6], the direction of the force which bends the elastic ribbon will be normal to the axis *AP* at the point *A*, and therefore *AD* will represent the direction of the acting force. This force will equal $\frac{2Ek^2}{a^2}$ where $Ek^2$ expresses the absolute elasticity.

16. If x = o, then $\frac{dy}{dx} = \frac{a^2 - c^2}{c\sqrt{2a^2 - c^2}}$. This expression gives the tangent of the angle which the curve *AM* makes with the axis *AP* at *A*, the sine of which angle will equal $\frac{a^2 - c^2}{a^2}$. Wherefore, if $a^2 = \infty$, the ribbon will be normal to the axis *AP* at the point *A*, and will have no curvature, because the curving

force $\frac{2Ek^2}{a^2}$ disappears. Therefore, in the case where $a^2 = \infty$, the natural shape of the ribbon appears, that is, a straight line. This, then, constitutes the first class of elastic curves, which the straight line $AB$ produced in both directions to infinity will represent.

17. Before enumerating the remaining classes, it will be convenient to make certain observations in general about the figure of the elastic curve. Now it is understood that the angle $PAM$ [FIG. 6] which the curve makes with the axis $AP$ at $A$, decreases as the quantity $a^2$ becomes smaller, that is, the more the curving force $\frac{2Ek^2}{a^2}$ is applied. And if $a^2$ should become equal to $c^2$, then the axis $AP$ will be tangent to the curve at $A$; but if $a^2 < c^2$, then the curve $AM$, which hitherto ran downwards [as in FIG. 6], will now turn upwards until [as in FIG. 7] the point is reached where $a^2 = \frac{c^2}{2}$, in which case the tangent of the curve will fall upon the straight line $Ab$. But if $a^2 < \frac{1}{2}c^2$, then the angle $PAM$ will be absolutely imaginary, and therefore no portion of the curve will exist at $A$. These different cases will constitute a variety of classes.

18. Furthermore from the equation it is understood (because if $x$ and $y$ are both made negative, the form of the equation is not changed) that the curve on both sides of $A$ has similar and equal branches $AMC$ and $Amc$ alternately disposed, in such a way that $A$ is the point of contraflexure; whence, the portion $AMC$ of the curve being known at the same time, its continuation $Amc$ beyond $A$ will be known, in as much as the latter is similar and equal to the former. Thus, letting $Ap = AP$, $pm$ will also equal $PM$. Now in receding from $A$, the curve on both sides is bent back further from the axis, until the abscissa $AE = c$, the ordinate $EC$ will be tangent to the curve; for if $x = c$, then $\frac{dy}{dx} = \infty$. It is clear that the abscissa $x$ cannot increase beyond $AE = c$, for otherwise $\frac{dy}{dx}$ would become imaginary. Hence

the whole curve will be contained between the extreme ordinates $EC$ and $ec$, beyond which limits it cannot pass. Now therefore we have, so far, the two branches $AC$ and $Ac$ of the curve extending on both sides from $A$ to the limits.

19. Let us see, then, under what conditions the curve may pass beyond $C$ and $c$. To this end let us take the straight line $CD$ parallel to $AE$ as an axis, and let these new co-ordinates $CQ = t$ and $QM = u$; then $x + t = AE = CD = c$, and $y + u = CE = AD = b$, whence $x = c - t$, and $y = b - u$, or $dy = - du$, and $dx = - dt$. Substituting these values, there will arise an equation for the curve in terms of the new co-ordinates $CQ = t$ and $QM = u$; this equation will be

$$du = \frac{(a^2 - 2ct + t^2)\, dt}{\sqrt{t\,(2c - t)\,(2a^2 - 2ct + t^2)}}$$

Here it is clear, in the first place, that if $t$ be taken as infinitely small, then $du = \frac{a^2 dt}{2a\sqrt{ct}}$, and $u = a\sqrt{\frac{t}{c}}$. The latter equation indicates that the curve beyond $C$ begins to advance towards $N$ in a way similar to that in which it extends from $C$ to $M$ [6]). Now the ambiguity of the radical sign in the denominator of the equation shows admirably that the ordinate $u$ can be taken negatively as well as positively; whence it is manifest that the straight line $CD$ is a diameter of the curve, and moreover, that the arc $CNB$ will be similar and equal to the arc $CMA$.

20. Now in a similar way the straight line $cd$ produced through $c$ on the other side of and parallel to the axis $AE$ will be a diameter of the curve; because the branch $Acb$ is similar and equal to the branch $ACB$. Therefore at the points $B$ and $b$ there will also be points of contraflexure as at $A$; whence the curve will extend further in a similar fashion. Therefore the curve will have an infinite number of diameters $CD$, $cd$, etc., mutually distant from one another by the same interval $Dd$, and parallel to one another; and because of this, the curve will consist of an infinite number of parts similar and equal to one another; and therefore the whole curve will be known if only a single portion $AMC$ be known.

21. Because the point of contraflexure is at $A$, the radius of curvature will be infinitely great at that point, which is clear from the nature of the curve. For since the curve at $A$ is drawn by the force $\frac{2Ek^2}{a^2}$ in the direction $AD$, at any point $M$, if the radius of curvature be set equal to $R$, because of the nature of the elasticity, the force will be $\frac{2Ek^2x}{a^2} = \frac{Ek^2}{R}$, whence $R = \frac{a^2}{2x}$. Therefore at the point $A$ $(x = o)$ the radius of curvature is infinite; but because $AE = Ae = c$ at the points $C$ and $c$, the radius of curvature will equal $\frac{a^2}{2c}$; in other words, at these places, the farthest distant from the straight line $BAb$, the curvature is greatest [7]).

22. Now although for the point $C$ it is known that the abscissa is $AE = c$, nevertheless the distance $EC$ cannot be determined except by the integration of the equation

$$dy = \frac{(a^2 - c^2 + x^2)\, dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

For if after the integration, $x$ be made equal to $c$, the value of $y$ will give the distance $CE$, which taken twice will give the distance $AB$, or the interval $Dd$ lying between the diameters. Similarly, integration will be necessary to determine the length of the curved ribbon $AC$. For since, if the arc $AM = s$,

$$ds = \frac{a^2\, dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}, \text{ and}$$

its integral, evaluated at $x = c$, will give the length of the curve $AC$ [8]).

23. Now since these formulas do not admit of integration, let us try to express conveniently by approximation the values of the interval $AD$ and of the arc $AC$. To this end, let $\sqrt{c^2 - x^2} = z$, whence

$$PM. = y = \int \frac{(a^2 - z^2)\, dx}{z\sqrt{2a^2 - z^2}}, \text{ and } AM = s = \int \frac{a^2\, dx}{z\sqrt{2a^2 - z^2}}$$

Expressed as a series,

$$\frac{1}{\sqrt{2a^2-z^2}} = \frac{1}{a\sqrt{2}}\left(1 + \frac{1}{4}\cdot\frac{z^2}{a^2} + \frac{1.3}{4.8}\cdot\frac{z^4}{a^4} + \frac{1.3.5}{4.8.12}\cdot\frac{z^6}{a^6} + \ldots\right),$$

whence

$$s = \frac{1}{\sqrt{2}}\int\left(\frac{a}{z} + \frac{1}{4}\cdot\frac{z}{a} + \frac{1.3}{4.8}\cdot\frac{z^3}{a^3} + \frac{1.3.5}{4.8.12}\cdot\frac{z^5}{a^5} + \ldots\right)dx,$$

and

$$s-y = \frac{1}{\sqrt{2}}\int\left(\frac{z}{a} + \frac{1}{4}\cdot\frac{z^3}{a^3} + \frac{1.3}{4.8}\cdot\frac{z^5}{a^5} + \frac{1.3.5}{4.8.12}\cdot\frac{z^7}{a^7} + \ldots\right)dx.$$

24. Since we desire these integrals only for the case $x = c$, in which $z = 0$, they can be expressed conveniently by the aid of the circumference of the circle. For assuming that the ratio of the diameter to the circumference is as 1 is to $\pi$,

$$\int\frac{dx}{z} = \int_0^c\frac{dx}{\sqrt{c^2-x^2}} = \frac{\pi}{2}.$$

Now in the same way the following integrals will be determined [9])

$$\int_0^c z\,dx = \frac{1}{2}\cdot\frac{\pi c^2}{2},\quad \int_0^c z^3dx = \frac{1.3}{2.4}\cdot\frac{\pi}{2}c^4$$

$$\int_0^c z^5dx = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}c^6, \text{ and } \int_0^c z^7dx = \frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}c^8.$$

By the aid of these integrals we have

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2}\cdot\frac{c^2}{2a^2} + \frac{1^2.3^2}{2^2.4^2}\cdot\frac{c^4}{4a^4} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\frac{c^6}{8a^6} + \ldots\right)$$

and

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2}\cdot\frac{3}{1}\cdot\frac{c^2}{2a^2} - \frac{1^2.3^2}{2^2.4^2}\cdot\frac{5}{3}\frac{c^4}{4a^4} - \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\cdot\frac{7}{5}\cdot\frac{c^6}{8a^6} - \ldots\right)$$

If accordingly $AE = c$ and $AD = b$ be given, from these equations the constant $a$ and the length of the curve $AC$ will be determined. Conversely also, from the given length of this curve $AC$, and from the constant $a$ by which the external force is determined, it will be possible to find the distances $AD$ and $CD$.

## First Class

25. Since we have so determined the first class that, in the general equation

$$dy = \frac{(a^2 - c^2 + x^2)\, dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}},$$

$c = o$, or $\frac{a}{c} = \infty$, a straight line represents the natural condition of the ribbon; and to this same first class let us refer also those cases for which $c$ is an infinitely small quantity, in such a way that in comparison with $a$ it can be regarded as on the point of vanishing. However, because $x$ cannot be greater than $c$, likewise $x$ in comparison with $a$ will be on the point of vanishing, and therefore the following equation will result:

$$dy = \frac{a\, dx}{\sqrt{2\,(c^2 - x^2)}}.$$

The integral of this is $y = \frac{a}{\sqrt{2}} \text{ arc sin } \frac{x}{c}$, which is the equation for a trochoid curve infinitely elongated [10]). Now $AD$ will become equal to $\frac{\pi a}{2\sqrt{2}}$, from which the length of the curve differs only infinitesimally, because the angle $DAM$ is infinitely small. Let the length of the ribbon $ACB = 2f$, and its absolute elasticity be $Ek^2$. Because $f = \frac{\pi a}{2\sqrt{2}}$, the force requisite to produce this infinitely small curvature of the ribbon will be of a finite magnitude, and will equal $\frac{Ek^2}{f^2} \cdot \frac{\pi^2}{4}$; that is to say, if the extremeties $A$ and $B$ be fastened together with a string $AB$, the string will necessarily be stretched by the force $\frac{Ek^2}{f^2} \frac{\pi^2}{4}$.

## Second Class

26. Let the case in which $c$ is contained between the limits $o$ and $a$, constitute the second class. In these cases the angle

*DAM* will be less than a right angle; for the sine of the angle *PAM*, or the cosine of the angle $DAM = \frac{a^2 - c^2}{a^2}$. Therefore in this case the form of the curve will be similar to that which FIG. 6 represents. Since $c < a$, therefore $\frac{c^2}{2a^2} < \frac{1}{2}$; but since $\frac{c^2}{2a^2} > o$, assuredly $AC = f > \frac{\pi a}{2\sqrt{2}}$, whence $a^2 < \frac{8f^2}{\pi^2}$; wherefore the force whereby the extremeties *A* and *B* of the ribbon are drawn together, by the aid of the string *AB*, will be greater than in the preceding case, that is the force will be greater than $\frac{Ek^2}{f^2} \cdot \frac{\pi^2}{4}$.

## THIRD CLASS

27. In the third class I consider the unique case in which $c = a$, because in this case the axis *AP* is tangent to the curve at the point *A*. This class has the special name of the rectangular elastic curve. In this case

$$dy = \frac{x^2\,dx}{\sqrt{a^4 - x^4}}, \text{ and } ds = \frac{a^2 dx}{\sqrt{a^4 - x^4}},$$

and hence

$$AC = f = \frac{\pi\, a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} \cdot \frac{1}{2} + \frac{1^2 . 3^2}{2^2 . 4^2} \cdot \frac{1}{4} + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} \cdot \frac{1}{8} + \ldots\right),$$

and

$$AD = b = \frac{\pi\, a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2} \cdot \frac{3}{1.2} - \frac{1^2 . 3^2}{2^2 . 4^2} \cdot \frac{5}{3.4} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} \cdot \frac{7}{5.8} - \ldots\right).$$

Now although from this neither *b* nor *f* can be accurately expressed in terms of *a*, yet I have elsewhere pointed out that there is a remarkable relation between these two quantities [11]). In other words I have shown that $4bf = \pi a^2$, or the rectangle formed by *AD* and *AC* will equal the area of a circle the diameter of which is *AE*. Now it will be found by introducing calculus that $f = \frac{5a}{6} \cdot \frac{\pi}{2}$ approximately, so that $a = \frac{12f}{5\pi}$; hence the force by which the extremities $A$ and $B$ must be drawn toward each other will equal $\frac{Ek^2}{f^2} \cdot \frac{25}{72} \pi^2$. A closer approximation gives

$$f = 1.1803206 \frac{a\pi}{2\sqrt{2}}, \text{ hence}$$

$$b = \frac{\pi a^2}{4f} = \frac{a}{1.1803206\sqrt{2}}, \text{ whence } \frac{f}{a} = 1.311006, \text{ and}$$

$$\frac{b}{a} = 0.59896 \text{ }^{12)}.$$

## Fourth Class

28. If $c > a$, the fourth class will arise (Fig. 7), the ribbon opening out horizontally until $AD = b > o$. This second limit of $c$ will be defined by the equation

$$1 = \frac{1^2}{2^2} \cdot \frac{3}{1} \cdot \frac{c^2}{2a^2} + \frac{1^2 \cdot 3^2 \cdot}{2^2\, 4^2} \cdot \frac{5}{3} \cdot \frac{c^4}{4a^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{7}{5} \cdot \frac{c^6}{8a^6} + \ldots .$$

In this class therefore, since $c > a$, the curve at $A$ will rise above the axis $AE$, and will form the angle $PAM$, the sine of which will equal $\frac{c^2 - a^2}{a^2}$. Now we shall soon see that this angle $PAM$ is less than $40^0$ -41′; since if it reaches this value, the interval $AD$ disappears, a case which I refer to the fifth class. Hence in the fourth class are included the curves in which the value $\frac{c^2}{a^2}$ is contained between the limits 1 and 1.651868. Now the form of these curves is understood from the figure, provided only that it be observed that the closer $\frac{c^2}{a^2}$ approaches the latter limit 1.651868, the shorter the interval $AD$ will become, and the closer the end points $A$ and $B$ will be brought to each other. Therefore it can happen that the humps of the ribbon $m$ and $R$ and likewise $M$ and $r$ are not merely mutually tangent, but even intersect, and intersections of this kind will be repeated indefinitely until all the diameters $DC$ and $dc$ coincide and merge with the axis $AE$.

## Fifth Class

29. If this happens, the fifth class (Fig. 8) will arise, the character of which will be expressed by the equation between the co-ordinates $AP = x$ and $PM = y$,

$$dy = \frac{(a^2 - c^2 + x^2)\, dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}},$$

the following relation existing between $a$ and $c$, viz., that the interval $AD = b = o$. Let $\frac{c^2}{2a^2} = v$, and then $v$ must be defined by the following equation:

$$1 = \frac{1.3}{2.4} v + \frac{1.1.3.5}{2.2.4.4} v^2 + \frac{1.1.3.3.5.7}{2.2.4.4.6.6} v^3 + \ldots.$$

Let there be sought first, by methods familiar to everyone, or else by mere trial, the limits between which the true value of $v$ is contained, and these limits will be found to be $v = 0.824$, and $v = 0.828$. But if now both of these values be substituted in the equation, from the two errors which are certain to arise, it will finally be concluded that $v = 0.825934 = \frac{c^2}{2a^2}$, whence $\frac{c^2}{a^2} = 1.651868$, and $\frac{c^2 - a^2}{a^2} = 0.651868$; and since this expression equals the sine of the angle $PAM$, it will be found from the tables that the angle $PAM = 40^\circ\, \text{-}41'$; and therefore twice this, or the angle $MAN$, will equal $81^\circ\, \text{-}22'$. Wherefore, if the extremities of the elastic ribbon be brought toward each other until they touch, they will form the curve $AMCNA$ [13]), [Fig. 8] and the two extremities will form at $A$ an angle $= 81^\circ\, \text{-}22'$.

## Sixth Class

30. If the two extremities $A$ and $B$ of the ribbon, after they have been brought together, should be drawn apart in opposite directions by an increased force, there will arise the curve of the shape $AMCNB$ (Fig. 9) which constitutes the sixth class.

Therefore in the curves belonging to this class, $\frac{c^2}{2a^2} > 0.825934$; but $\frac{c^2}{2a^2} < 1$. For if $c^2 = 2a^2$, there will arise the seventh class, to be explained in a moment. In these curves the angle *PAM* which the curve makes with the axis at *A* is greater than $40^\circ$ -$41'$, but less than a right angle; for since its sine is $\frac{c^2 - a^2}{c^2}$, because $c^2 < 2a^2$, the sine is necessarily less than 1, and hence the angle *PAM* cannot become a right angle unless $c^2 = 2a^2$.

## Seventh Class

31. Now let $c^2 = 2a^2$, in which case the seventh class is constituted, and the character of the curve will be expressed by the equation

$$dy = \frac{(a^2 - x^2)\,dx}{x\sqrt{2\,a^2 - x^2}},$$

from which it is gathered that the branches *A* and *B* of the curve (Fig. 10) are extended indefinitely, in such a way that the straight line *AB* becomes the asymptote of the curve. Therefore each branch *AMC* and *BNC* will become infinite, as is understood from the series discovered above for the arc *AC*; for

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} + \frac{1^2.\,3^2}{2^2.\,4^2} + \frac{1^2.\,3^2.\,5^2}{2^2.\,4^2.\,6^2} + \ldots\right),$$

the sum of which series is infinite. If therefore the length of the ribbon *AC* be finite and equal to *f*, it is necessary that $a = o$, and hence also $CD = c = o$; therefore the ribbon after it has been curved to a knot, in this case will be extended again in a straight line, for which an infinite force will be needed. But if the ribbon be infinitely long, it will form a knotted curve converging to the asymptote *AB*, *CD* being equal to *c*. Now the equation for this curve can be integrated by the aid of logarithms, for

$$y = \sqrt{c^2 - x^2} - \frac{c}{2}\,log\,\frac{c + \sqrt{c^2 - x^2}}{x},$$

taking the abscissa *x* on the diameter *DC* in such a way that

$DQ = x$ and $QM = y$; for the ordinate $y$ disappears when $x = CD = c$; also at the knot $O$ the ordinate disappears. To find this point let us put

$$\frac{2\sqrt{c^2 - x^2}}{c} = log\ \frac{c + \sqrt{c^2 - x^2}}{x}.$$

Let $\phi$ be the angle whose cosine is $\frac{x}{c}$ and whose sine is $\frac{\sqrt{c^2 - x^2}}{c}$,

then $2 \sin \phi = \log \tan (45^\circ + \frac{\phi}{2})$.

The logarithm must be taken from a table of natural logarithms. If a table of this kind be lacking, let there be taken from a table of common logarithms the logarithm of the tangent of the angle $(45^\circ + \frac{\phi}{2})$, from the characteristic of which let 10 be subtracted, and let the remainder be $\omega$; by so doing

$2 \sin \phi = 2.30258509\ \omega$ [14]).

Taking common logarithms once more, we have

$\log 2 + \log \sin \phi = \log \omega + 0.3622156886$,

or $\log \sin \phi = \log \omega + 0.0611856930$.

By trying this artifice, a value of the angle $\phi$ very close to the true value will soon be secured; whence by the rule of the false value the true value of the angle $\phi$ will be determined, and from it the abscissa $x = DO$. Now in this way the angle $\phi$ is found to be $73^\circ$ -14′ -12″, whence it results that $\frac{x}{c} = 0.2884191$, and $\frac{\sqrt{c^2 - x^2}}{c} = 0.9575042$. But the angle $QOM$ is $2\phi - 90^\circ = 56^\circ$ -28′ -24″, and therefore the angle $MON = 112^\circ$ -56′ -48″. Therefore, since in the fifth class the angle at the knot was $81^\circ$ -22′, in the sixth class the angle $MON$ at the knot will be contained between the limits $81^\circ$ -22′ and $112^\circ$ -56′ -48″. In the fourth class, if the knot appears, its angle will be less than $81^\circ$ -22′.

## Eighth Class

32. Now let $c^2 > 2a^2$, and $c^2 = 2a^2 + g^2$; since $a^2 = \frac{c^2 - g^2}{2}$,

the equation of the curve will be

$$dy = \frac{(x^2 - \frac{1}{2}c^2 - \frac{1}{2}g^2)\,dx}{\sqrt{(c^2 - x^2)(x^2 - g^2)}}.$$

By this equation the eighth class is expressed (FIG. 11), and if the straight line *dDd* represents the direction of the external force, then $x = DQ$, and $y = QM$. First therefore, it is clear that the ordinate $y$ cannot be real unless $x > g$; but $x$ cannot exceed the straight line $DC = c$, whence, putting $DF = g$, the whole curve will be contained between the straight lines parallel to *dd* drawn through the points *C* and *F*, and these lines will be tangent to the curve. Now it is indifferent which one of the straight lines $c$ and $g$ be the greater, provided only that they be unequal; for the equation is not changed if the straight lines $c$ and $g$ be interchanged. Furthermore this curve will also have an infinite number of diameters parallel to one another *DC*, *dc*, *dc*, etc., and also straight lines which are drawn through the individual points *G* and *H*, likewise normal to *dDd* [15]). But nowhere along the whole curve will there be a point of contraflexure, and therefore the continual curvature will progress in both directions indefinitely, as the figure shows; and the angles *MON*, *mon*, etc., which are made at the knots will be greater than 112° -56′ -48″.

## NINTH CLASS

33. Since in the eighth class are contained not only the cases in which $g^2 < c^2$, but also those in which $g^2 > c^2$, there remains but the one case in which $c = g$; in which case, because $CF = o$, the curve is reduced, vanishing into space. But on the other hand, if we take $c$ and $g$ both as infinite, but in such a way that their difference is finite, the curve will occupy a finite space. Therefore, to find this curve, let $g = c - 2h$, and $x = c - h - t$, and, because $c = \infty$, but the quantities $h$ and $t$ are finite,

$$\frac{c^2}{2} + \frac{g^2}{2} = c^2 - 2\,ch, \text{ and } x^2 - \frac{c^2}{2} - \frac{g^2}{2} = -2\,ct;$$

Then $c^2 - x^2 = 2c\,(h + t)$, and $x^2 - g^2 = 2c\,(h - t)$, from which the following equation

$$dy = \frac{tdt}{\sqrt{h^2 - t^2}}$$

for the circle will result. Therefore the elastic band in this case will be curved into a circle, as we have already observed above. Wherefore the circle will constitute the ninth and last class.

34. Now that we have enumerated the classes, it will be easy, in any given case, to determine to which class the curve belongs. Let the elastic ribbon be fixed in the wall at *G* (FIG. 12), and from the end *A* let there be hung the weight *P*, by which the ribbon is curved into the shape *GA*. Let the tangent *AT* be drawn, and the whole decision will have to be sought from the angle *TAP*. For if this angle be acute, the curve will belong to the second class; but if it be a right angle, the curve will belong to the third class, and it will be an elastic rectangular curve. But if the angle *TAP* be obtuse, yet less than 130° -41′, the curve will belong to the fifth class; if, however, the angle *TAP* be greater than 130° -41′, the curve will belong to the sixth class. Now it would belong to the seventh class if the angle should be equal to two right angles, but that cannot happen. This class, therefore, together with the following classes, cannot be produced by hanging a weight directly to the ribbon.

35. Now in order that it may appear how the remaining classes can be produced by curving the ribbon, let a rigid rod *AC* be firmly fastened at *A*, the end of the ribbon fixed at *B* (FIG 3), and let the weight *P*, which draws in the direction *CD*, be appended at *C*. Let the interval *AC* be *h*, the absolute elasticity of the ribbon be $Ek^2$, and the sine of the angle *MAP* which the ribbon makes with the horizontal at *A* be *m*. All this being stipulated, if we let the abscissa $AP = t$, and the ordinate $PM = y$, the following equation will be found for the curve

$$\text{I. } dy = \frac{dt\,(mEk^2 - Pht - \frac{1}{2}Pt^2)}{\sqrt{E^2k^4 - (mEk^2 - Pht - \frac{1}{2}Pt^2)^2}}.$$

Now let $CP = x = h + t$, whereby the equation is reduced to the form which we have used in the division of the classes, viz.,

$$\text{II. } dy = \frac{dx\,(mEk^2 + \frac{1}{2}Ph^2 - \frac{1}{2}Px^2)}{\sqrt{E^2k^4 - (mEk^2 + \frac{1}{2}Ph^2 - \frac{1}{2}Px^2)^2}}$$

which, compared with the form

$$dy = \frac{dx\,(a^2 - c^2 + x^2)}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}},$$

or

$$\text{III. } dy = \frac{dx\,(a^2 - c^2 + x^2)}{\sqrt{a^4 - (a^2 - c^2 + x^2)^2}},\ ^{16)}$$

will give $\frac{1}{2}Pa^2 = Ek^2$, or $a^2 = \frac{2Ek^2}{P}$; and $\frac{1}{2}Pc^2 - \frac{1}{2}Pa^2 = m\ Ek^2 + \frac{1}{2}Ph^2$; therefore

$$c^2 = \frac{2\,(1 + m)\,E\,k^2}{P} + h^2$$

36. Therefore the curve will belong to the second class if $\frac{2\,m\,E\,k^2}{P} + h^2 < o$, *or* $P < \frac{-\,2mEk^2}{h^2}$, Hence unless the angle *PAM* be negative, the force *P* will have to be negative, and the rod at *C* will have to be drawn upwards. The curvature will belong to the third class if $P = \frac{-\,2mEk^2}{h^2}$. The fourth class will appear if $2mEk^2 + Ph^2 > o$, but at the same time $2mEk^2 + Ph^2 < 2\alpha Ek^2$, $\alpha$ being equal to 0.651868. But if $P = \frac{2(\alpha - m)Ek^2}{h^2}$, then the curve will belong to the fifth class. If, however, $Ph^2 > 2(\alpha - m)Ek^2$, but at the same time $Ph^2 < 2(1 - m)Ek^2$, the curve is to be referred to the sixth class. The seventh class will arise if $Ph^2 = 2(1 - m)Ek^2$. The eigth class will be obtained if $Ph^2 > 2(1 - m)Ek^2$. Wherefore if the angle *PAM* be a right angle, because $1 - m = o$, the curve will always belong to the eighth class. Finally, the ninth class will arise if $h = \infty$ , as I have already observed above.

## Strength of Columns

37. That which has been observed above about the first class can help us judge of the strength of columns. For let the column *AB* (Fig. 13), sustaining the load *P*, be placed vertically upon the base *A*. If the column be so constituted

that it cannot slip, nothing else need be feared from the weight $P$, if it be not excessively great, except the bending of the column; therefore in this case the column can be considered as elastic. Let the absolute elasticity of the column be $Ek^2$, and let its height $AB = 2f = a$. As we have seen above in section 25, the force necessary to bend this column even in the least degree is

$$\frac{\pi^2 Ek^2}{4f^2} = \frac{\pi^2}{a^2} \cdot Ek^2.$$

Therefore, unless the load $P$ to be borne be greater than $\frac{E\pi^2 k^2}{a^2}$, there will be absolutely no fear of bending; on the other hand, if the weight $P$ be greater, the column will be unable to resist bending. Now when the elasticity of the column and likewise its thickness remain the same, the weight $P$ which it can carry without danger will be inversely proportional to the square root of the height of the column; and a column twice as high will be able to bear only one-fourth of the load. This principle can, therefore, be applied in the case of wooden columns, since they are subject to bending.

## Determination of the Absolute Elasticity by Experiment

38. Now in order that the force and the bending of any elastic ribbon whatsoever may be determined *a priori*, it is necessary that the absolute elasticity which we have hitherto expressed by $Ek^2$, be known. This can be conveniently determined by a single experiment. Let the uniform elastic ribbon, the absolute elasticity of which is to be investigated, be fixed at one end F, in a solid wall $GK$ (Fig. 14), in such a way that it is held horizontally, for here we may neglect the weight of the ribbon. To the other end $H$ let there be hung any weight $P$ by which the ribbon is curved to the position $AF$. Let the length of the ribbon $AF = HF = f$, the horizontal distance $AG = g$, and the vertical distance $GF = h$, all of which values will be known by experiment. Now let this curve be compared with the general equation

$$dy = \frac{(c^2 - a^2 - x^2)\, dx}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}},$$

in which, if $a$ and $c$ have been determined by $f$, $g$, and $h$, the curving force $P$ will equal $\frac{2Ek^2}{a^2}$. Therefore the absolute elasticity $Ek^2 = \frac{1}{2} Pa^2$.

39. Because now the tangent at $F$ is horizontal, $\frac{dy}{dx} = o$, and therefore $x = \sqrt{c^2 - a^2}$. Hence $AG = g = \sqrt{c^2 - a^2}$, and $a^2 = c^2 - g^2$; and therefore

$$dy = \frac{(g^2 - x^2)\, dx}{\sqrt{(c^2 - x^2)\,(c^2 - 2g^2 + x^2)}},$$

and if we let $x = g$ here, $y$ will have to be equal to $GF = h$, or $s = AF = f$; whence

$$ds = \frac{(c^2 - g^2)\, dx}{\sqrt{(c^2 - x^2)\,(c^2 - 2g^2 + x^2)}}$$

Now if the weight $P$ be taken as very small, so that the ribbon be depressed only a very little, then $c$ will become a very great quantity, and therefore

$$\frac{1}{\sqrt{(c^2 - \chi^2)\,(c^2 - 2g^2 + \chi^2)}} = (c^4 - 2c^2g^2 + 2g^2 - x^4)^{-\frac{1}{2}}$$

$$= \frac{1}{c^2} + \frac{g^2}{c^4} + \frac{3g^4}{2c^6} - \frac{g^2x^2}{c^6} - \frac{3g^4x^2}{8} + \frac{x^4}{2c^6} + \frac{3g^2x^4}{2c^8} + \ldots.,$$

and therefore by integration we have approximately

$$s = \frac{(c^2 - g^2)x}{c^2} + \frac{(c^2 - g^2)g^2x}{c^4} + \frac{3(c^2 - g^2)g^4x}{2c^6} - \frac{(c^2 - g^2)g^2x^3}{3c^6} - \frac{(c^2 - g^2)g^4x^3}{c^8} + \frac{(c^2 - g^2)x^5}{10c^6} + \frac{3(c^2 - g^2)g^2x^5}{10c^8} + \ldots.,$$

and

$$y = \frac{g^2x}{c^2} + \frac{g^4x}{c^4} + \frac{3g^6x}{2c^6} - \frac{g^4x^3}{3c^6} - \frac{g^6x^3}{c^8} + \frac{g^2x^5}{10c^6} + \frac{3g^4x^5}{10c^8} - \frac{x^3}{3c^2} - \frac{g^2x^3}{3c^4} - \frac{g^4x^3}{2c^6} + \frac{g^2x^5}{5c^6} + \frac{3g^4x^5}{5c^8} - \frac{x^7}{14c^6} - \frac{3g^2x^7}{14c^8} + \ldots$$

Now letting $x = g$, we have

$$f = g + \frac{4g^5}{15c^4},$$ [17]) and $$h = \frac{2g^3}{3c^2} + \frac{2g^5}{3c^4}.$$

Therefore if the straight line $FG = h$ be called into use

$$c^2 = \frac{2g^3}{3h}$$

and $a^2 = \dfrac{g\,(2g^2 - 3gh)}{3h}$

Whence the absolute elasticity is secured as

$$Ek^2 = \frac{Pg^2\ (2g - 3h)}{6h}$$

This value will differ hardly at all from the true value, provided that a not too great curvature of the ribbon be caused.

40. Now this absolute elasticity $Ek^2$ will depend in the first place upon the character of the material out of which the ribbon has been made; whence one material is said to have more elasticity than another. Also, in the second place, it so depends upon the breadth of the ribbon that the expression $Ek^2$ ought everywhere to be proportional to the breadth, other things being equal. But thirdly, the thickness of the ribbon contributes a great deal to determining the value of $Ek^2$, which seems to be composed in such a way that, other things being equal, $Ek^2$ is proportional to the square of the thickness. Therefore, taken all together, the expression $Ek^2$ will have a total ratio composed of the ratio of the elasticity of the material, the breadth of the ribbon, and the square of the thickness. Hence by experiments in which it is possible to measure the width and thickness, the elasticity of all materials can be compared one with another and determined.

## Curvature of Elastic Ribbons of Variable Cross Section

41. As hitherto I have taken the absolute elasticity $Ek^2$ as constant throughout the whole length of the ribbon the curvature of which I have determined, so the solution can also be made by the same method if the quantity $Ek^2$ be taken as a variable in any manner. In other words, if the absolute elasticity of the ribbon $AM$ (Fig. 2) be any function $S$ of the arc $AM = s$, and if the radius of curvature at $M$ be $R$, the curve $AM$, which

the ribbon takes on, will be so constituted that among all other curves of the same length, $\int \frac{Sds}{R^2}$ is a minimum. This case will therefore be solved by the second general formula [18]).

Let $dy = pdx$, $dp = qdx$, and $dS = Tds$. Among all curves in which $\int dx\sqrt{1 + p^2}$ is of the same magnitude, that curve is to be determined in which $\int \frac{Sq^2dx}{(1 + p_2)^{\frac{5}{2}}}$ is a minimum. The first formula, $\int dx\sqrt{1 + p^2}$, gives for differential value $\frac{d}{dx}\ \frac{p}{\sqrt{1 + p^2}}$.

The second formula $\int \frac{Sq^2dx}{(1 + p^2)^{\frac{5}{2}}}$, compared with $\int Zdx$, will give

$$Z = \frac{Sq^2}{(1 + p^2)^{\frac{5}{2}}}.$$

Now if we let $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq$, where $\Pi = \int [Z]\overline{dx}$, then $d[Z] = [M]\overline{dx} + [N]\overline{dy} + [P]\overline{dp}$, and $Ld\,\overline{\Pi} = \frac{q^2T\overline{ds}}{(1 + p^2)^{\frac{5}{2}}}$, whence $L = \frac{q^2T}{(1 + p^2)^{\frac{5}{2}}}$, and $d\overline{\Pi} = \overline{ds} = \overline{dx}\ \sqrt{1 + p^2}$.

Therefore $[Z] = \sqrt{1 + p^2}$, $[M] = 0$, $[N] = 0$, and $[P] = \frac{p}{\sqrt{1+p^2}}$.

But then $M = 0$, $N = 0$, and also $P = -\frac{5\ Sq^2p}{(1 + p^2)^{\frac{7}{2}}}$, and $Q = \frac{2Sq}{(1 + p^2)^{\frac{5}{2}}}$, so that $dZ = \frac{q^2dS}{(1 + p^2)^{\frac{5}{2}}} + Pdp + Qdq$.

42. Now let $\int Ldx = \int \frac{q^2Tdx}{(1 + p^2)^{\frac{5}{2}}} = \int \frac{q^2dS}{(1 + p^2)^3}$, and let $H$ be the value of this integral when $x = a$. The consideration of the constant $a$ will soon disappear again from the calculation. Therefore $V = H - \int \frac{q^2\ dS}{(1 + p^2)^3}$, whence

the differential expression will become $-\frac{dP}{dx} - \frac{d}{dx}[P]V + \frac{d^2Q}{dx^2}$.

Wherefore from these two differential values, the following equation for the curve sought will arise :

$$\alpha \frac{d}{dx} \frac{p}{\sqrt{1+p^2}} = + \frac{dP}{dx} + \frac{d}{dx}[P]V - \frac{d^2Q}{dx^2}.$$

This, being integrated, gives

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = P + [P]V - \frac{dQ}{dx}.$$

or

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = \frac{Hp}{\sqrt{1+p^2}} - \frac{p}{\sqrt{1+p^2}} \int \frac{q^2\,dS}{(1+p^2)^3} + P - \frac{dQ}{dx}.$$

The constant $H$ can be absorbed into the arbitrary constant $\alpha$, whereby the constant $a$ disappears from the calculation. On this account, the following equation will result :

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = P - \frac{dQ}{dx} - \frac{p}{\sqrt{1+p^2}} \int \frac{q^2\,dS}{(1+p^2)^3}.$$

43. Let this equation be multiplied by $dp = qdx$, and there will result

$$\frac{\alpha\,pdp}{\sqrt{1+p^2}} + \beta\,dp = Pdp - qdQ - \frac{pdp}{\sqrt{1+p2}} \int \frac{q^2 dS}{(1+p^2)^3}.$$

Now since $dZ = \frac{q^2\,dS}{(1+p^2)^{\frac{5}{2}}} + Pdp + Qdq,$

then $Pdp = dZ - Qdq - \frac{q^2\,dS}{(1+p^2)^{\frac{5}{2}}}$.

If this value is substituted, the following integrable equation will result :

$$\frac{\alpha\,pdp}{\sqrt{1+p^2}} + \beta\,dp = dZ - qdQ - Qdq - \frac{q^2\,dS}{(1+p^2)^{\frac{5}{2}}}$$
$$- \frac{pdp}{\sqrt{1+p^2}} \int \frac{q^2\,dS}{(1+p^2)^3},$$

the integral of which is

$$\alpha\sqrt{1+p^2} + \beta p + \gamma = Z - Qq - \sqrt{1+p^2} \int \frac{q^2\,dS}{(1+p^2)^3},$$

$$or\ \alpha\sqrt{1+p^2}+\beta p+\gamma=-\frac{Sq^2}{(1+p^2)^{\frac{5}{2}}}-\sqrt{1+p^2}\int\frac{q^2dS}{(1+p^2)^3}.$$

In order to eliminate the integral sign, divide the equation by $\sqrt{1+p^2}$ and differentiate again, obtaining

$$\frac{\beta\, dp}{(1+p^2)^{\frac{3}{2}}}-\frac{\gamma\, pdp}{(1+p^2)^{\frac{3}{2}}}+\frac{2\,q^2dS}{(1+p^2)^3}+\frac{2\,Sqdq}{(1+p^2)^3}-\frac{6\,Spq^2dp}{(1+p^2)^4}=0,$$

which, multiplied by $\frac{(1+p^2)^{\frac{3}{2}}}{2q}$, gives

$$\frac{\beta\, dp}{2q}-\frac{\gamma\, pdp}{2q}+\frac{qdS+Sdq}{(1+p^2)^{\frac{3}{2}}}-\frac{3\,Spqdp}{(1+p^2)^{\frac{5}{2}}}=0.$$

The integral of this, because $dp = qdx$, and $dy = pdx$, will be

$$\alpha+\tfrac{1}{2}\,\beta x-\tfrac{1}{2}\,\gamma y+\frac{Sq}{(1+p^2)^{\frac{3}{2}}}=0.$$

But $-\frac{(1+p^2)^{\frac{3}{2}}}{q}$ is the radius of curvature $R$; whence, by doubling the constants $\beta$ and $\gamma$, the following equation will arise:

$$\frac{S}{R}=\alpha+\beta x-\gamma y.$$

This equation agrees admirably with that which the second or direct method supplies. For let $\alpha + \beta x - \gamma y$ express the moment of the bending power, taking any line you please as an axis [19]), to which moment the absolute elasticity $S$, divided by the radius of curvature $R$ must be absolutely equal. Thus, therefore, not only has the character of the elastic curve observed by the celebrated BERNOULLI been most abundantly demonstrated, but also the very great utility of my somewhat difficult formulas has been established in this example.

44. If, therefore, the curve be given which a variable elastic ribbon, acted upon by the force $CD = P$ (FIG. 3) forms, the absolute elasticity of the ribbon at any point can be found. For, taking the straight line $CP$, which is normal to the direction of the force, as an axis, and putting $CP = x$, $PM = y$, the arc of the curve $AM = s$, and the radius of curvature at $M$ equal to $R$, because the moment of the force with reference to the

point $M$ is $Px$, then $\frac{S}{R} = Px$, and therefore the absolute elasticity $S$, at the point $M$, is $PRx$. Hence, since the radius of curvature $R$ is known at every point when the curve is given, the absolute elasticity at any point becomes known. Therefore, if the material of the ribbon together with its thickness be everywhere the same, but the width is variable, because the absolute elasticity is proportional to the width, the width of the ribbon at every point is learned from the form of the curve.

45. Let the triangular tonguelet $fAf$ (FIG. 15) be cut out of an elastic ribbon of uniform thickness. Since the width $mm$ at any section $M$ is proportional to the length $AM$, if we let $AM = s$, the absolute elasticity at $M$ will be proportional to $s$. Let the absolute elasticity be $Eks$, and to the ribbon fastened at the end $ff$ horizontally in a wall let there be hung, at the point $A$, the weight $P$, by which the median straight line $AF$ is bent into the curve $FmA$ (FIG. 14), the character of which curve is sought. Now on the horizontal axis let the abscissa $Ap = x$, the ordinate $pm = y$, and the arc $Am = s$; then $Px = \frac{Eks}{R}$, in which $R$ denotes the radius of curvature at $m$. Let this equation be multiplied by $dx$, and because $R = -\frac{ds^3}{dxd^2y}$, assuming $dx$ as constant, we have $Px.dx = \frac{-Eksdx^2\,d^2y}{ds^3}$

or
$$\frac{Pxdx}{Ek} + \frac{sdx^2.d^2y}{ds^3} = 0$$

But since $\frac{d}{ds}\,sdy = \frac{sd^2y}{ds} - \frac{sdyd^2s}{ds^3} + dy = \frac{sdx^2.d^2y}{ds^3} + dy$, and because $d^2s = \frac{dyd^2y}{ds}$, then
$$\int\frac{sdx^2d^2y}{ds^3} = \frac{sdy}{ds} - y.$$
Whence by integration,
$$\frac{Px^2}{2Ek} + a = \frac{-sdy}{ds} + y.$$

46. Let $dy = pdx$, so that $ds = dx\sqrt{1+p^2}$, and $\frac{2\,Ek}{P} = c$;

then $a + \frac{x^2}{c} = y - \frac{sp}{\sqrt{1+p^2}}$, and therefore

$$\frac{a\sqrt{1+p^2}}{p} + \frac{x^2\sqrt{1+p^2}}{cp} = \frac{y\sqrt{1+p^2}}{p} - s.$$

This, differentiated, gives

$$\frac{-\,adp}{p^2\sqrt{1+p^2}} + \frac{2\,x\,dx\sqrt{1+p^2}}{cp} - \frac{x^2\,dp}{cp^2\sqrt{1+p^2}} = \frac{dy\sqrt{1+p^2}}{p}$$

$$-\,y\,\frac{dp}{p^2\sqrt{1+p^2}} - dx\sqrt{1+p^2} = \frac{-\,y\,dp}{p^2\sqrt{1+p^2}}.$$

Hence $a - y = \frac{2pxdx\,(1+p^2)}{cdp} - \frac{x^2}{c}$ results.

Let $dp$ be taken as constant and differentiate; then

$$-\,pdx = \frac{2\,pxd^2x\,(1+p^2)}{cdp} + \frac{2\,pdx^2\,(1+p^2)}{cdp} + \frac{2\,xdx\,(1+3p^2)}{c} - \frac{2xdx}{c},$$

or

$$cdxdp + 2xd^2x\,(1+p^2) + 2dx^2\,(1+p^2) + 6pxdx = 0.$$

A further solution of this equation is impossible. The most simple equation for the curve is the following:

$$\frac{yds - sdy}{ds} = \frac{Px^2}{2Ek},$$

for when $x = 0$, both $y$ and $s$ must vanish, and the constant $a = 0$.

## The Curvature of Elastic Ribbons which in Their Natural State are Not Straight

47. In the previous discussion the curvature of a ribbon, whether uniformly elastic or not, is determined if it be subjected to a single force, and, which is especially to be noted, if the ribbon be naturally straight. But if the ribbon in its natural state be already curved, then it will certainly take on a different curvature due to the acting force. To find this, one must know its natural shape in addition to its elasticity and the acting force. Let,

therefore, the elastic ribbon *Bma* (FIG. 16) be naturally curved; let the elasticity of it be everywhere the same, viz. $Ek^2$, and let it be curved by the force $P$ into the shape $BMA$. Through the point $A$ let there be drawn the straight line $CAP$ normal to the direction of the acting force, and let this line be taken as the axis; let also the distance $AC = c$, the abscissa $AP = x$, the ordinate $PM = y$; and the moment of the acting force at the point $M$ be equal to $P(c + x)$.

48. Furthermore let the radius of curvature of the curve sought be equal to $R$ at the point $M$; let the arc *am* in the natural state be $AM = s$, and let the radius of curvature at the point $m$ be $r$; this radius, because the curve *amB* is known, will be given by the arc $s$. At $M$, therefore, because the curvature is greater, the radius of curvature $R$ is less than $r$, and the excess of the elementary angle over the angle in the natural state will be $\frac{ds}{R} - \frac{ds}{r}$, which excess will be the effect produced by the acting force. Wherefore $p(c + x) = Ek^2 \left(\frac{1}{R} - \frac{1}{r}\right)$, which, since $r$ is given by $s$, will be the equation of the curve sought; and this considered thus cannot be reduced to one of the previously described classes.

49. Therefore let us assume that the ribbon has a circular shape *amB* in its natural state; $r$ will be the radius $a$ of that circle, whence $P(c + x) = Ek^2 \left(\frac{1}{R} - \frac{1}{a}\right)$. Let this equation be multiplied by $dx$ and integrated; then [see sec. 5 towards the end]

$$\frac{P}{Ek^2}\left(\frac{x^2}{2} + cx + f\right) = \frac{-dy}{ds} - \frac{x}{a},$$

will arise, which, if $c - \frac{Ek^2}{Pa}$ be written for $c$, will go over into

$$\frac{P}{Ek^2}\left(\frac{x^2}{2} + cx + f\right) = -\frac{dy}{ds}.$$

This is the same equation that we discovered above for the ribbon which was straight in its natural condition [sec. 5]. Let, therefore, the ribbon which is circular in its natural condition be curved into the same curves which are produced for the ribbon that is straight in its natural state; of course, the place of the application of the force, or the distance $AC = c$ will have to vary for each case according to the given law. Therefore the same nine classes of curves will appear for the figures which the ribbon that is circular in its natural state can produce, and these we have enumerated above. For the circular ribbon, if the distance $AC$ be taken as infinite, can be drawn first into a straight line [see class 9]; then any force whatever applied in addition will produce the same effect as if it were applied alone to the elastic ribbon which was straight in its natural state.

50. Now let us assume that, whatever be the natural shape of the ribbon, the point $C$ is infinitely distant, in such a way that the moment of the acting force be everywhere the same, and let the moment, when divided by $Ek^2$ be taken as $\frac{1}{b}$, then

$$\frac{1}{b} = \frac{1}{R} - \frac{1}{r} \; and \; \frac{1}{R} = \frac{1}{b} + \frac{1}{r}.$$

Hence $\int \frac{ds}{R} = \frac{s}{b} + \int \frac{ds}{r}$

is the amplitude of the arc $AM$, just as $\int \frac{ds}{r}$ expresses the amplitude of the arc *am*, precisely as the celebrated JOHN BERNOULLI is accustomed to use the term amplitude in his superb treatise *De motu reptorio* [20].) Let therefore $\frac{s}{b} + \int \frac{ds}{r}$ be the arc in the circle whose radius equals 1, which, because $r$ is given by $s$, will also be a known function of $s$. Hence the rectangular co-ordinates $x$ and $y$ will be found in such a way that

$$x = \int ds \sin\left(\frac{s}{b} + \int \frac{ds}{r}\right), \text{ and } y = \int ds \cos\left(\frac{s}{b} + \int \frac{ds}{r}\right),$$

whence the curve sought can be constructed by quadratures.

51. Hence the figure *amB* (FIG. 17), which the ribbon must

have in its natural state, can be determined, so that by the force $P$, acting in the direction $AP$, it can be unfolded into the straight line $AMB$. For letting $AM = s$, the moment of the force acting at the point $M$ will equal $Ps$, and the radius of curvature at $M$ will be infinite by hypothesis, or $\frac{1}{R} = 0$. Now the arc $am$ in its natural state being equal to $s$, and the radius of curvature at $m$ being taken as $r$, because this curve is convex to the axis $AB$, the quantity $r$ must be made negative. Hence $Ps = \frac{Ek^2}{r}$, or $rs = a^2$, which is the equation of the curve $amB$.

52. Therefore, since $\frac{1}{r} = \frac{s}{a^2}$, then $\int \frac{ds}{r} = \frac{s^2}{2a^2}$; or the amplitude of the arc $am$ will vary as the square of the arc itself. Hence the rectangular co-ordinates $x$ and $y$ for the curve $amB$ will be so defined that $x = \int ds \sin \frac{s^2}{2a^2}$, and $y = \int ds \cos \frac{s^2}{2a^2}$. In other words, in a circle whose radius is 1, the arc $\frac{s^2}{2a^2}$ will have to be cut off, the sine and cosine of which must be taken to determine the co-ordinates. Now from the fact that the radius of curvature constantly decreases the greater the arc $am = s$ is taken, it is manifest that the curve cannot become infinite, even if the arc $s$ be infinite. Therefore the curve will belong to the class of spirals, in such a way that after an infinite number of windings it will roll up at a certain definite point as a center, which point seems very difficult to find from this construction. Analysis therefore must be considered to gain no slight advantage if anyone should discover a method by the aid of which at least an approximate value can be assigned for the integrals $\int ds \sin \frac{s^2}{2a^2}$, and $\int ds \cos \frac{s^2}{2a^2}$, in the case where $s$ is taken as infinite. This seems to be a not unworthy problem upon which mathematicians may exercise their powers [21]).

53. Let $2a^2 = b^2$, and since

$$\sin \frac{s^2}{b^2} = \frac{s^2}{b^2} - \frac{s^6}{3!\,b^6} + \frac{s^{10}}{5!\,b^{10}} - \frac{s^{14}}{7!\,b^{14}} + \ldots,$$

$$\text{and } \cos \frac{s^2}{b^2} = 1 - \frac{s^4}{2!\,b^4} + \frac{s^8}{4!\,b^8} - \frac{s^{12}}{6!\,b^{12}} + \ldots,$$

the co-ordinates $x$ and $y$ of the curve sought can be conveniently expressed by infinite series; for

$$x = \frac{s^3}{3\,b^2} - \frac{s^7}{3!\,7\,b^6} + \frac{s^{11}}{5!\,11\,b^{10}} - \frac{s^{15}}{7!\,15\,b^{14}} + \ldots,$$

$$\text{and } y = s - \frac{s^5}{2!\,5\,b^4} + \frac{s^9}{4!\,9\,b^8} - \frac{s^{13}}{6!\,13\,b^{12}} + \ldots,$$

from which rapidly converging series, unless the arc $s$ be assumed to be very great, the approximate values of the co-ordinates $x$ and $y$ can be determined sufficiently closely. But what values $x$ and $y$ acquire if the arc $s$ be taken as infinitely great, can in no way be determined from these series.

54. Therefore, since putting $s = \infty$ makes a very great difficulty, aid can be brought to the inconvenience by the following method. Let $\frac{s^2}{b^2} = v$, then $s = b\sqrt{v}$, and $ds = \frac{b\,dv}{2\sqrt{v}}$ whence $x = \frac{b}{2}\int \frac{dv}{\sqrt{v}} \sin v$, and $y = \frac{b}{2}\int \frac{dv}{\sqrt{v}} \cos v$. And now I declare that the values for $x$ and $y$ when $s = \infty$ will be discovered by the following integral formulas:

$$x = \frac{b}{2} \int dv \left( \frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi + v}} + \frac{1}{\sqrt{2\pi + v}} - \frac{1}{\sqrt{3\pi + v}} + \ldots \right) \sin v,$$

$$\text{and } y = \frac{b}{2} \int dv \left( \frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi + v}} + \frac{1}{\sqrt{2\pi + v}} - \frac{1}{\sqrt{3\pi + v}} + \ldots \right) \cos v$$ [22])

if after integration $v$ be taken as equal to $\pi$, where $\pi$ denotes an angle equal to two right angles. In this way, therefore, the placing of $s = \infty$ is indeed avoided; but on the other hand, the infinite series

$$\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi + v}} + \frac{1}{\sqrt{2\pi + v}} - \frac{1}{\sqrt{3\pi + v}} + \ldots$$

is introduced into the calculation, and since the sum of this series is as yet unknown, the resolution of the knot is still subject to a great difficulty.

## The Curvature of an Elastic Ribbon at Individual Points Under the Action of Any Forces Whatsoever

55. It will be convenient also to study the curvature produced in an elastic ribbon by several forces, or indeed by an infinite number of forces, by the same method already given for studying the curvature of any elastic ribbon whatsoever if it be acted upon by a single force at a given point. But since it is not yet established just what expression in these cases is going to be either a maximum or a minimum, I shall use merely the direct method, in order that from the solution itself it may perchance be possible to discover that property which is either a maximum or a minimum. Therefore let the elastic ribbon that is straight in its natural state be brought into the position $AmM$ (Fig. 18), first by the finite forces $P$ and $Q$ acting in the directions $CE$ and $CF$ normal to each other, and then by the infinitely small forces applied to the single elements $m\mu$ of the ribbon, and acting in the directions $mp$ and $mq$ parallel to $CE$ and $CF$; all this being stipulated, the character of the curve produced in the ribbon $AmM$ is required.

56. Let the straight line $FCA$ produced be taken as an axis, and let $AC = c$, the abscissa $AP = x$, the ordinate $PM = y$, the arc of the curve $AM = s$, and the radius of curvature at $M$ be $R$; let the absolute constant of the elasticity of the ribbon be $Ek^2$; and the sum of the moments arising from all the acting moments with respect to the point $M$ must be equal to $\frac{Ek^2}{R}$.

Now in the first place, from the finite force $P$ acting in the direction $CE$ there arises the moment $P(c + x)$, acting in that direction in which the elastic forces are equilibrated. The moment $Qy$, arising from the other force $Q$, tends in the opposite direction, from which, due to the finite forces $P$ and $Q$ taken together, there arises the moment $P(c + x) - Qy$. Now let there be considered any intermediate element $m\mu$, and let its corresponding abscissa $Ap = \zeta$, and the ordinate $pm = \eta$; let the force acting upon the element $m\mu$ in the direction $mp$ be $dp$, and the force acting in the direction $mq$ be $dq$; then the moment of these forces about $M$ will be $(x - \zeta)\, dp - (y - \eta)\, dq$.

57. Therefore to find the sum of all the moments, the point $M$, and consequently $x$ and $y$, must be, for the time being, considered as constants, while only the co-ordinates $\zeta$ and $\eta$ with the forces $dp$ and $dq$ are regarded as variable. Therefore the sum of the moments arising from the forces acting upon the arc $Am$ will equal

$$xp - \int \zeta dp - yp + \int \eta dq,$$

where $p$ expresses the sum of all the forces acting upon the arc $AM$ applied in the direction parallel to $pm$, and $q$ expresses the sum of all the forces acting upon the arc $AM$ applied in the direction parallel to $Ap$. But $\int \zeta dp = \zeta p - \int p d\zeta$,
and $\int \eta dq = \eta q - \int q d\eta$,
whence the sum of the moments arising from the forces applied to the arc $AM$ will be $(x - \zeta)p + \int p d\zeta - (y - \eta)q - \int q d\eta$. Now let the point $m$ move to $M$; then $\zeta = x$, $\eta = y$, $d\zeta = dx$, and $d\eta = dy$; whence the sum of all the moments taken throughout the whole length of the arc $AM$ will equal $\int p dx - \int q dy$. Wherefore, for the curve sought, the following equation will be obtained:

$$\frac{Ek^2}{R} = P(c + x) - Qy + \int p dx - \int q dy.$$

Here $p$ expresses the sum of all the vertical forces, or those acting in the direction of the ordinates $MP$, and $q$ expresses the sum of all the horizontal forces, or those acting in the direction of $MQ$ parallel to $AP$, throughout the whole arc $AM$.

58. If the expressions $\int pdx$ and $\int qdy$ cannot be integrated, the equations found by differentiation will have to be freed from these integral expressions, whence the following equation will be had:

$$\frac{-Ek^2dR}{R^2} = Pdx - Qdy + pdx - qdy.$$

But if neither $p$ nor $q$ can be expressed in a finite number of terms, inasmuch as they already express the sums of an infinite number of forces infinitely small, then by a further differentiation, the finite values $p$ and $q$ will have to be eliminated, so that there remain only $dp$ and $dq$, with the differentials of the second order $d^2p$ and $d^2q$.

Now there will arise, after the first differentiation,

$$-Ek^2d\frac{dR}{R^2dx} = dp - (Q+q)\,d\frac{dy}{dx} - \frac{dy}{dx}dq.$$

Let $\frac{dy}{dx} = \omega$, and, when the equation has been differentiated again, we get

$$-Ek^2d\frac{d\frac{dR}{R^2dx}}{d\omega} = d\frac{dp}{d\omega} - 2dq - \omega\, d\frac{dq}{d\omega}.$$

This equation contains differentials of the fourth order.

59. In place of the vertical and horizontal forces $p$ and $q$, let two forces be applied to the ribbon—the one normal, $MN = dv$, and the other tangential, $MT = dt$ [FIG. 18]. Hence

$$dp = \frac{dx\,dv}{ds} + \frac{dy\,dt}{ds}, \text{ and } dq = \frac{dx\,dt}{ds} - \frac{dy\,dv}{ds},$$

and because $dy = \omega dx$, and $ds = dx\sqrt{1+\omega^2}$,

then $dp = \frac{dv}{\sqrt{1+\omega^2}} + \frac{\omega\,dt}{\sqrt{1+\omega^2}}$, and $dq = \frac{dt}{\sqrt{1+\omega^2}} - \frac{\omega\,dv}{\sqrt{1+\omega^2}}$.

When these values are substituted in the last equation of the preceding paragraph, the following equation will result:

$$-Ek^2\,d\frac{\frac{dR}{R^2\,dx}}{dv} = \frac{-dt}{\sqrt{1+\omega^2}} + \frac{2\,\omega\,dv}{\sqrt{1+\omega^2}} + \sqrt{1+\omega^2}\,d\frac{dv}{d\omega}.$$

This equation becomes integrable when multiplied by $\sqrt{1 + \omega^2}$. For the sake of brevity, let $z = \frac{dR}{R^2 dx}$;

then

$$A - t + \frac{dv\,(1 + \omega^2)}{d\omega} = -Ek^2 \left[\frac{dz\,\sqrt{1+\omega^2}}{d\omega} - \frac{\omega z}{\sqrt{1+\omega^2}} + \frac{1}{2R^2}\right].$$

$$= -Ek^2 \left[\frac{1+\omega^2}{d\omega}\, d\, \frac{dR}{R^2\, dx\,\sqrt{1+\omega^2}} + \frac{1}{2R^2}\right].$$

But since $R = -\frac{(1+\omega^2)^{\frac{3}{2}}}{d\omega}dx$, then $d\omega = \frac{-(1+\omega^2)^{\frac{3}{2}}}{R}dx$, and by substituting the value of $d\omega$ we shall have, because $dx\sqrt{1+\omega^2} = ds$,

$$A - t - \frac{Rdv}{ds} = -Ek^2\left[\frac{1}{2R^2} - \frac{R}{ds}\, d\, \frac{dR}{R^2\, ds}\right]$$

Therefore by transposing, the following equation will arise:

$$t + \frac{R\,dv}{ds} - A = Ek^2\left[\frac{1}{2R^2} - \frac{R}{ds}\, d\, \frac{dR}{R^2\, ds}\right].$$

60. Now in the first place, it is clear that if the elastic force $Ek^2$ should vanish, the ribbon would be transformed into a perfectly flexible filament; and hence all the curves which a perfectly flexible filament can form when acted upon by any forces whatsoever are included in these equations. Thus if a filament be merely drawn downwards by its own weight, then $q = o$, and $p$ will express the weight of the string $AM$, and therefore, by the first equation of section 58, $p\frac{dx}{dy} = Q =$ a constant, and $P = o$, which is the general equation for catenary curves of every kind. Now if a perfectly flexible filament be acted upon at various points by forces, the directions of which [FIG. 18] are normal to the curve itself, in such a way that, at the point $M$, the filament be drawn in the direction $MN$ by a force $dv$, then, because $t = o$, it follows that $\frac{Rdv}{ds} = A$, a constant.

This is the general property of trough-shaped curves, and of all curves in which acting forces of this kind appear.

## ON THE CURVATURE PRODUCED IN AN ELASTIC RIBBON BY ITS OWN WEIGHT

61. I return now to elastic ribbons about which there is offered the following investigation, which is especially worthy of note, viz., the kind of a figure an elastic ribbon takes on when curved by its own weight. Let *AmM* [FIG. 18] be the curve which is sought, and because only vertical forces due to gravity are acting upon it, $P = o$, $Q = o$, $q = o$, and $p$ will express the weight of the ribbon *AM*, wherefore, if $F$ be the weight of a ribbon of length $a$, because the ribbon is assumed to be uniform, $p = \frac{Fs}{a}$; whence the character of the curve will be expressed by the following equation [from sec. 58]:

$$\frac{-Ek^2 dR}{R^2} = \frac{Fs\,dx}{a}$$

Let the amplitude of the curve be $\int \frac{ds}{R} = u$; then $R = \frac{ds}{du}$, and $dx = ds \sin u$; whence, assuming the element $ds$ as a constant, the following equation will be found:

$$s\,ds \sin u - \frac{Eak^2}{F} \cdot \frac{d^2u}{ds} = 0$$

which, as far as appears at first glance, cannot be reduced further.

62. Now especially worthy of note is the curve which a fluid of considerable depth produces in an elastic ribbon (FIG. 19). Let *AMB* be the figure sought, and letting $AP = x$, $PM = y$, and $AM = s$, the element *Mm* will be drawn in the normal direction *MN* by a force proportional to $ds$; whence $dv = nds$, and $dt = o$. Hence the vertical force $dp = ndx$, and the horizontal force $dq = -ndy$; whence $p = nx$, and $q = -ny$; and therefore the equation [of section 57] becomes

$$\frac{Ek^2}{R} = P(c + x) - Qy + \tfrac{1}{2}nx^2 + \tfrac{1}{2}ny^2.$$

The co-ordinates $x$ and $y$ can be increased or diminished by constant quantities in such a way that the equation for the curve takes on the following form :

$$x^2 + y^2 = A + \frac{B}{R}.$$

Now if this equation be multiplied by $xdx + ydy$, and if we put $dy = \omega dx$, it becomes integrable, for

$$\int \frac{xdx + ydy}{R} = - \int \frac{x + y\omega}{(1 + \omega^2)^{\frac{3}{2}}} d\omega = \frac{y - \omega x}{\sqrt{1+\omega^2}} = \frac{ydx - xdy}{ds}.$$

Because of this, by changing the constant after integration, we shall have

$$(x^2 + y^2)^2 = A\,(x^2 + y^2) + B\,\frac{(ydx - xdy)}{ds} + C.$$

Let $\sqrt{x^2 + y^2} = z$, and $y = uz$; whence $x = z\sqrt{1 - u^2}$; therefore $ydx - xdy = -\frac{z^2du}{\sqrt{1 - u^2}}$, and $ds = \sqrt{dz^2 + \frac{z^2du^2}{1-u^2}}$.

Therefore by placing

$$\frac{du}{\sqrt{1 - u^2}} = dr, \text{ then } z^4 - Az^2 - C = -\frac{Bz^2dr}{\sqrt{dz^2 + z^2\,dr^2}},$$

and hence

$$dr = \frac{du}{\sqrt{1 - u^2}} = \frac{dz\,(z^4 - A\,z^2 - C)}{z\sqrt{B^2\,z^2 - (z^4 - A\,z^2 - C)^2}}.$$

Therefore this curve, if $A = o$ and $C = o$, will be algebraic, for we shall have the following equation

$$dr = \frac{du}{\sqrt{1 - u^2}} = \frac{z^2\,dz}{\sqrt{B^2 - z^6}} = \frac{3z^2\,dz}{3\sqrt{a^6 \quad z^6}},$$

which, being integrated, gives

$$\text{arc sin u} = \frac{1}{3}\text{ arc sin }\frac{z^3}{a^3},$$

$$\text{or } \frac{z^3}{a^3} = 3u - 4u^3 = \frac{3y}{z} - \frac{4y^3}{z^3},$$

whence $z^6 = 3a^3\,yz^2 - 4a^3\,y^3$.

Or, since $z^2 = x^2 + y^2$, $(x^2 + y^2)^3 = 3a^3x^2y - a^3y^3$, or $x^6 + 3x^4y^2 + 3x^2y^4 + y^6 = 3a^3x^2y - a^3y^3$.

## ON THE OSCILLATING MOTIONS OF ELASTIC RIBBONS

63. Now from all this the oscillating motion of elastic curves brought into motion in any manner whatsoever can be determined. The illustrious DANIEL BERNOULLI first began to investigate this assuredly most important topic, and some years ago sent me the problem of determining the oscillations of an elastic ribbon fastened at one end in a solid wall, the solution of which I have published in "Commentarii Petropolitani" Vol. VII, (1740). Since that time, not only has it been my good fortune to treat the problem in a more convenient fashion, but also, through consultation with the celebrated BERNOULLI, a number of other questions and considerations have been added, the elucidation of which, because of the relation of the subject matter, I shall here add. Now when the vibratory motion is sufficiently rapid, a musical tone is given by the vibrating ribbon, the pitch of which, and its relation to other tones, will be determined by these principles, with the aid of the theory of tones. And since the character of tones is very readily subject to experiment, by that fact the agreement of calculation with truth can be investigated, and the theory can be confirmed. In this fashion our knowledge of the nature of elastic bodies will be enlarged in no small measure.

64. Now it must first be noted that here our study is directed only to very small oscillations; and the interval through which the ribbon passes in oscillating is, as it were, infinitely small. But the utility and the application is not at all diminished by this limitation; for not only would oscillations be deprived of isochronism if they should take place through large spaces, but more than that, the formation of distinct tones, and that is what we are here primarily considering, requires very small oscillations. I therefore consider here, in the first place, a uniform elastic ribbon, naturally straight, one end of which is firmly fixed at $B$ (FIG. 20) in an immovable pavement, in such a way that the ribbon, when left to itself, has the upright position $BA$. Let the length of this ribbon be $AB = a$, and its absolute elasticity at each point be $Ek^2$; its true weight we either neglect, or else

we cause it to be fixed in such a fashion that its position cannot be disturbed by gravity.

## ON THE OSCILLATIONS OF AN ELASTIC RIBBON FIXED AT ONE END IN A WALL

65. Now this ribbon, acted on by any force whatsoever, performs very short vibrations passing through very small intervals *Aa* on either side of its natural position *BA*. Let *BMa* be any position whatsoever which the ribbon occupies while oscillating. Since this is only an infinitely short distance from its natural position *BPA*, the straight lines *MP* and *Aa* will at the same time represent the paths which the points *M* and *a* traverse, or rather these straight lines, when compared to the true paths, will differ from them by an infinitely small amount. Now to determine the oscillatory motion, it is absolutely necessary to know the character of the curve *BMa* which the ribbon takes on during oscillation. Therefore let $AP = x$, $PM = y$, the arc $aM = s$, the radius of curvature at $M$ be $R$, and the very small interval $Aa = b$; also, from the conditions mentioned, the arc $s$ will be approximately equal to the abscissa $x$, and accordingly $dx$ can be taken for $ds$; for in comparison with $dx$, $dy$ will be on the point of vanishing. And since, by assuming $dx$ as constant, the general expression for the radius of curvature $\frac{ds^3}{dxd^2y}$ is, in the present case $R = \frac{dx^2}{d^2y}$, for the curve *BMa* turns its convex side to the axis *BA*; and because the ribbon has been firmly fixed in a wall at *B*, the straight line *AM* will be tangent to the curve at *B*.

66. All this being stipulated, in order to determine not only the character of the curve *BMa*, but also its oscillatory motion, let $f$ be the length of a simple isochronous pendulum; for not only the nature of the case, but also the calculations to be instituted will show that the very small oscillations are isochronous. The acceleration by which the point $M$ of the ribbon is drawn toward $P$ will be $\frac{PM}{f} = \frac{y}{f}$. Wherefore, if the mass of the whole

ribbon be taken as $M$, which is expressed by the weight, the mass of the element $Mm = ds = dx$ is $\frac{Mdx}{a}$; whence the moving force drawing the element $Mm$ in the direction $MP$ is $\frac{Mydx}{af}$; and thus the forces by which the individual particles are actually brought into motion will be known, not only from the curve $BMa$, but also from the length $f$ of the simple isochronous pendulum. But, since the ribbon is, as a matter of fact, incited to motion by the elastic force, when this is known, and the nature of the curve is known, the length of the simple isochronous pendulum will also be determined by them.

67. Therefore, since the ribbon is moved exactly as if there had been applied to each element $Mm$ of it, in the direction $MP$, forces equaling $\frac{Mydx}{af}$, it follows that, if to the single elements $Mm$ of the ribbon, equal forces $\frac{Mydx}{af}$ should be applied in the opposite direction $M\pi$, the ribbon in the position $BMa$ would be in a state of equilibrium. Hence the ribbon while oscillating will undergo the same curvature which it would take on when at rest, if at the individual points $M$ it should be acted upon by the forces $\frac{Mydx}{af}$ in the direction $M\pi$.

Therefore by the rule discovered above in section 56 [and 57], let all these forces applied throughout the arc $aM$ be collected, and there will appear the sum $\frac{M}{af}\int ydx$, which must be substituted in the place of $p$. Wherefore, since the remaining forces $P$, $Q$, and $q$ which appeared there [sec. 56] are on the point of vanishing, the character of the curve will be expressed by the equation

$$\frac{Ek^2}{R} = \int pdx,$$

whence we shall secure $\frac{Ek^2}{R} = \frac{M}{af}\int dx \int ydx$.

But since $R = \frac{dx^2}{d^2y}$, then $\frac{Ek^2\, d^2y}{dx^2} = \frac{M}{af}\int dx \int ydx$.

Differentiating, $\frac{Ek^2\ d^3y}{dx^2} = \frac{M}{af} dx \int y dx,$

and by differentiating again, the following differential equation of the fourth order will appear: $Ek^2\ d^4y = \frac{My\ dx^4}{af}.$

68. By this equation, therefore, the character of the curves *BMa* is expressed, and from that, if it be adapted to the case presented, the length *f* will be determined. That being known, the oscillatory motion itself will become known. But first the equation must be integrated, and since it belongs to the class of differential equations of the higher orders, the general integration of which I have shown in Vol. VII of the "Miscell. Berol.," the following integral equation will be found by substituting, for the sake of brevity, $c^4$ for $\frac{Ek^2af}{M}$:

$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + \text{C}\sin\frac{x}{c} + \text{D}\cos\frac{x}{c}$, where *e* denotes a number the hyperbolic logarithm of which is 1, and $\sin\frac{x}{c}$, and $\cos\frac{x}{c}$ are assumed to denote the sine and cosine of the arc $\frac{x}{c}$ in a circle, the radius of which is 1. But then *A*, *B*, *C* and *D* are four arbitrary constants introduced by four integrations, which must be determined by adapting the calculus to the present case.

69. Now the determination of the constants will be insitituted in the following fashion. First, when $x = o$, $y = b$; hence the following equation will arise: $b = A + B + D$. Secondly, since $\frac{c^4d^2y}{dx^2} = \int dx \int y dx$, and $\int p dx = o$ for $x = o$, hence when $x = o$, $\frac{d^2y}{dx^2} = o$. But

$$\frac{d^2y}{dx^2} = \frac{A}{c^2}e^{\frac{x}{c}} + \frac{B}{c^2}e^{-\frac{x}{c}} - \frac{C}{c^2}\sin\frac{x}{c} - \frac{D}{c^2}\cos\frac{x}{c},$$

whence the second equation appears, namely $A + B - D = o$. Thirdly, since $\frac{c^4d^3y}{dx^3} = \int y dx$, then when $x = o$, $\frac{d^3y}{dx^3}$ disappears, and $\frac{c^3d^3y}{dx^3} = Ae^{\frac{x}{c}} - Be^{-\frac{x}{c}} - C\cos\frac{x}{c} + D\sin\frac{x}{c}$, whence the third equation $A - B - C = o$ appears. Fourthly, if $x = a$, $y = o$, and $Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + C\sin\frac{a}{c} + D\cos\frac{a}{c} = o$. Fifthly, because $AB$ is tangent to the curve at the point $B$, when $x = a$, $\frac{dy}{dx} = o$, whence the fifth equation $Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c} = o$ appears. From these five equations the four constants $A$, $B$, $C$, and $D$ will be determined; and that on which the hinge of the matter turns, the value of $c = \sqrt[4]{\frac{Ek^2af}{M}}$ will be found, from which the length of the simple isochronous pendulum will be secured, whereby the durations of the oscillations will become known.

70. From the second and third equation, the constants $C$ and $D$ will be expressed in terms of $A$ and $B$ thusly: $C = A - B$, and $D = A + B$. These values, substituted in the fourth and fifth equations, will give

$Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + (A - B)\sin\frac{a}{c} + (A + B)\cos\frac{a}{c} = o$, and

$Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + (A - B)\cos\frac{a}{c} - (A + B)\sin\frac{a}{c} = o$, from which we secure

$$\frac{A}{B} = \frac{-e^{-\frac{a}{c}} + \sin\frac{a}{c} - \cos\frac{a}{c}}{e^{\frac{a}{c}} + \sin\frac{a}{c} + \cos\frac{a}{c}} = \frac{e^{-\frac{a}{c}} + \cos\frac{a}{c} + \sin\frac{a}{c}}{e^{\frac{a}{c}} + \cos\frac{a}{c} - \sin\frac{a}{c}},$$

whence the following equation will be obtained:

$$2 + (e^{\frac{a}{c}} + e^{-\frac{a}{c}})\cos\frac{a}{c} = o, \text{ or } e^{\frac{2a}{c}}\cos\frac{a}{c} + 2e^{\frac{a}{c}} + \cos\frac{a}{c} = o.$$

This gives $e^{\frac{a}{c}} = \frac{-1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}$.

However, since $e^{\frac{a}{c}}$ is a positive quantity, then $\cos\frac{a}{c}$ will be negative, and the angle $\frac{a}{c}$ will be greater than a right angle.

71. From the last equation it is seen that an infinite number of angles $\frac{a}{c}$ will satisfy it, due to which angles an infinite number of diverse modes of oscillations of the same ribbon arise. For the curve can cut the axis *AB* at one or more points before it touches the axis at *B*; from this fact there is an infinite number of modes of oscillations equally possible. Since we are here primarily considering the case in which *B* is the first point when the ribbon crosses the axis *AB*, this case will be satisfied by a very small angle $\frac{a}{c}$ satisfying the discovered equation. Since this angle is greater than a right angle, let $\frac{a}{c} = \frac{\pi}{2} + \phi$, in which $\phi$ is less than a right angle. Hence, because $\sin\frac{a}{c} = \cos\phi$, and $\cos\frac{a}{c} = -\sin\phi$, we shall obtain the double equation $e^{\frac{a}{c}} = \frac{1 \pm \cos\phi}{\sin\phi}$,

which gives either $e^{\frac{a}{c}} = \tan\frac{1}{2}\phi$, or $e^{\frac{a}{c}} = \cot\frac{1}{2}\phi$.
The second equation will give the smaller value for the angle $\phi$, and will be adapted to the case proposed.

72. The following possible modes of oscillation will be found if the angle $\frac{a}{c}$ is greater than two right angles and less than three right angles. Let $\frac{a}{c} = \frac{3\pi}{2} - \phi$. Then $\sin\frac{a}{c} = -\cos\phi$, and $\cos\frac{a}{c} = -\sin\phi$, whence

$$e^{\frac{a}{c}} = \frac{1 \pm \cos\phi}{\sin\phi}, \text{ or } e^{\frac{a}{c}} = \tan\tfrac{1}{2}\phi, \text{ or } e^{\frac{a}{c}} = \cot\tfrac{1}{2}\phi.$$

In a similar way, other modes of oscillation will be found by letting $\frac{a}{c} = \frac{5\pi}{2} + \phi$; $\frac{a}{c} = \frac{7\pi}{2} - \phi$; etc.

From all these, if the hyperbolic logarithms be taken, there will arise the following equations:

I. $\frac{\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ II. $\frac{\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

III. $\frac{3\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$ IV. $\frac{3\pi}{2} - \phi = \log \tan \frac{1}{2}\phi$

V. $\frac{5\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ VI. $\frac{5\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

VII. $\frac{7\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$ VIII. $\frac{7\pi}{2} - \phi = \log \tan \frac{1}{2}\phi$

etc.

Now the third of these equations agrees with the second, for let $\frac{1}{2}\phi = \frac{\pi}{2} - \frac{1}{2}\theta$, then $\cot \frac{1}{2}\phi = \tan \frac{1}{2}\theta$, whence the third equation passes over into $\frac{\pi}{2} = \theta = \log \tan \frac{1}{2}\theta$, which is the second equation. In the same way the fourth equation agrees with the first; the fifth and the eighth agree; likewise the sixth agrees with the seventh, etc. Wherefore only the following different equations will appear:

I. $\frac{\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ II. $\frac{\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

III. $\frac{5\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ IV. $\frac{5\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

V. $\frac{9\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ VI. $\frac{9\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

etc.

73. Now the hyperbolic logarithm of any tangent or cotangent of an angle is found by taking the tabular [common] logarithm and subtracting ten from it [23]), and multiplying the remainder by 2.302585092994. In order to shorten the labor, it will be convenient to use logarithms again. Let $u$ be the hyperbolic logarithm of the tangent or cotangent of the angle $\frac{1}{2}\phi$ which is

sought. From the tables let the [common] logarithm of the same tangent or cotangent be taken, and let this value, diminished by ten [23]), be $v$. Therefore, since $u=2.302585092994\,v$, we shall get, by taking common logarithms, $\log u = \log v + 0.3622156886$. This logarithm being found, since $u = \frac{n\pi}{2} + \phi$, we have $\log u = \log (\frac{n\pi}{2} + \phi)$. To evaluate this, the angle $\phi$ must be expressed in radians, just as $\pi$ is expressed in the same fashion, where $\pi = 3.1415926535$, and hence $\frac{\pi}{2} = 1.57079632679$. Now the angle $\phi$ will be expressed in the same way if it be changed to seconds, and if from the logarithm of this number there be subtracted constantly 5.3144251332 [24]); for thus the logarithm of $\phi$ will appear, from which, by going back to numbers, the value of $\phi$ is secured. Now for every kind of oscillation, $\frac{a}{c}$ will constantly equal $u = \frac{n\pi}{2} + \phi$.

74. This advice having been given for instituting the calculations by approximations, the value of the angle $\phi$ will be secured without difficulty for any kind of oscillation. For by assigning any values you please to $\phi$ and determining by calculation $\frac{n\pi}{2} + \phi$, and $\log \tan \frac{1}{2}\phi$ or $\log \cot \frac{1}{2}\phi$, soon the approximate value of $\phi$ will be known.

If now the limits of the angle $\phi$ be removed as far as you please, then closer limits will be found immediately, and from these the true value of $\phi$. Thus for the first equation $\frac{a}{c} = \frac{\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$, I have secured the following limits for the angle $\phi$: $17^\circ\text{-}26'$ and $17^\circ\text{-}27'$. From these, by the following calculation, the true value of $\phi$ itself will be obtained.

| | | |
|---|---|---|
| $\phi$ | $= 17^\circ\text{-}26'\text{-}0''$ | $17^\circ\text{-}27'\text{-}0''$ |
| in sec. | $=$ 62760″ | 62820″ |
| log | $=$ 4.7976829349 | 4.7980979321 |
| subtract | $=$ 5.3144251332 | 5.3144251332 |

| | | |
|---|---|---|
| $\log \phi$ | $= 9.4832578017 - 10$ | $9.4836727989 - 10$ |
| $\phi$ | $= 0.3042690662$ | $0.3045599545$ |
| $\frac{1}{2}\pi$ | $= 1.5707963268$ | $1.5707963268$ |
| $\frac{1}{2}\pi + \phi = \frac{a}{c}$ | $= 1.8750653930$ | $1.8753562813$ |
| $\frac{1}{2}\phi$ | $= 8^\circ$ -43′ -0″ | $8^\circ$ -43′ -30″ |
| $v = \log \cot \frac{1}{2}\phi$ | $= 0.8144034109$ | $0.8139819342$ |
| $\log v$ | $= 9.9108395839 - 10$ | $9.9106147660 - 10$ |
| add | $= 0.3622156886$ | $0.3622156886$ |
| $\log u$ | $= 0.2730552725$ | $0.2728304546$ |
| $u = \frac{a}{c}$ | $= 1.8752331540$ | $1.8742626675$ |
| difference | $= + 1677610$ | $- 10936138$ |

From these errors of the two limits is is concluded that

$$\phi = 17^\circ \text{-}26' \text{-}7.98'', \text{ and } \tfrac{1}{2}\pi + \phi = \frac{a}{c} = 107^\circ \text{-}26' \text{-}7.98''.$$

But since

| | |
|---|---|
| $\phi$ | $= 62767.98''$ |
| log | $= 4.7977381525$ |
| subtract | $= 5.3144251332$ |
| $\log \phi$ | $= 9.4833130193 - 10$ |
| therefore $\phi$ | $= 0.3043077545$ |
| add $\frac{\pi}{2}$ | $= 1.5707963268$ |
| $\phi + \frac{\pi}{2} = \frac{a}{c}$ | $= 1.8751040813$ |

This being found, then $\frac{A}{B} = \tan \frac{1}{2}\phi = 0.1533390624$ [25]).

Therefore the ratio of the constants $A$ and $B$ is found. From which also the ratio of the remaining constants will be known in relation to them.

75. There still remains the first equation $b = A + B + D$. This equation, since $D = A + B$, becomes $b = 2A + 2B$, and therefore $A + B = \frac{1}{2} b$.

Since therefore $\frac{A}{B} = \tan \frac{1}{2}\phi$, $B\,(1 + \tan \frac{1}{2}\phi) = \frac{1}{2} b$, and

$$B = \frac{b}{2 + 2 \tan \frac{1}{2} \phi} = \frac{b}{2 (1 + \tan \frac{1}{2} \phi)}$$

Whence from $\tan \frac{1}{2} \phi = 0.1533390624$ the several constants of the equation will be determined in the following fashion:

$$\frac{A}{b} = \frac{\tan \frac{1}{2} \phi}{2 (1 + \tan \frac{1}{2} \phi)} = \frac{0.1533390624}{2.3066781248}$$

$$\frac{B}{b} = \frac{1}{2 (1 + \tan \frac{1}{2} \phi)} = \frac{1.0000000000}{2.3066781248}$$

$$\frac{C}{b} = \frac{-1 + \tan \frac{1}{2} \phi}{2 (1 + \tan \frac{1}{2} \phi)} = \frac{-0.8466609376}{2.3066781248}$$

$$\frac{D}{b} = \frac{1 + \tan \frac{1}{2} \phi}{2 (1 + \tan \frac{1}{2} \phi)} = \frac{1.1533390624}{2.3066781248}$$

These being found, the character of the curve *aMB* which the ribbon takes during oscillation will be expressed by the following equation:

$$\frac{y}{b} = \frac{A}{b} e^{\frac{x}{c}} + \frac{B}{b} e^{\frac{-x}{c}} + \frac{C}{b} \sin \frac{x}{c} + \frac{D}{b} \cos \frac{x}{c}.$$

76. As to the velocity of the oscillations, it will become known from the equation $\frac{a}{c} = 1.8751040813$. For the sake of brevity, put $n = 1.8751040813$, so that $a = nc$; and since $c^4 = \frac{Ek^2 af}{M}$ where $\frac{M}{a}$ expresses the specific gravity of the ribbon and $Ek^2$ the absolute elasticity, by the method which I have used hitherto, $a^4 = n^4 Ek^2 \frac{af}{M}$, and on that account $f = \frac{a^4}{n^4} \cdot \frac{1}{Ek^2} \cdot \frac{M}{a}$ from which the length of a simple isochronous pendulum will vary directly as the fourth power of the length of the ribbon, directly as the specific gravity, and inversely as the absolute elasticity. Let $g$ be the length of a simple pendulum oscillating in a single second in such a way that $g = 3.16625$ Rhenish feet. Since the durations of the oscillations of the pendulums are proportional to the square roots of the lengths of the pendulums, the time of one oscillation made by our elastic ribbon will be $\frac{\sqrt{f}}{\sqrt{g}}$ seconds $= \frac{a^2}{n^2} \sqrt{\frac{1}{g} \cdot \frac{1}{Ek^2} \cdot \frac{M}{a}}$. Whence the number of oscilla-

tions produced in one second will be

$$\frac{n^2}{a^2}\sqrt{g\,Ek^2\,\frac{a}{M}}$$

This number expresses the pitch of the tone which the ribbon produces. Therefore the sound produced by different elastic ribbons fastened at one end in a wall, will be proportional to the square root of the absolute elasticity, inversely proportional to the square root of the specific gravity, and inversely proportional to the square of the length. Wherefore if two elastic ribbons differ only in length, their tones are inversely proportional to the square of the lengths; in other words, a ribbon twice as long will give forth a tone two octaves lower. Now a tense chord twice as long gives forth a tone only one octave lower, if the tension remains the same. From this it is clear that the tones of elastic ribbons follow a very different ratio from that of the tones in tense chords [26]).

77. As to the character of the curve $aMB$ continued beyond the ends $a$ and $B$, it is clear, in the first place, that the curve beyond $a$ advances in such a way that it is continually diverging from the axis $BA$. For taking $x$ as a negative quantity,

$$y = Be^{\frac{x}{c}} + Ae^{-\frac{x}{c}} - \text{C}\sin\frac{x}{c} + \text{D}\cos\frac{x}{c}.$$

Now here, all the limits are positive, because only the coefficient $C$ previously had a negative value [sec. 75]; whence while $x$ increases, $y$ must also increase, because the number $B$ is greater than $A$, and so the term $Be^{\frac{x}{c}}$ prevails. Now as soon as $\frac{x}{c}$ has reached even a moderate value, then the term $Be^{\frac{x}{c}}$ increases in such a degree that the remaining terms, in comparison with it, disappear, as it were. For this reason, because the radius of curvature of the curve at $B$ does not equal infinity, for $\frac{E\,k^2}{R} = \frac{M}{af}\int dx \int y\,dx$ and hence the curve at $B$ will not have a point of contraflexure, and will advance further on the same side of the axis $AB$, and by increasing the abscissa $x$ beyond $BA = a$, the first term $Ae^{\frac{x}{c}}$ soon becomes so great that the remaining terms can be regarded as zero in comparison with it.

78. Therefore this is the first mode of oscillation among those innumerable ones to which the same ribbon may adapt itself. The second mode, represented in FIG. 21, whereby the ribbon fixed at $B$ crosses the axis $AB$ at one point $O$, will be deduced from the equation $\frac{a}{c} = \frac{1}{2}\pi + \phi = \log \tan \frac{1}{2}\phi$ or $\frac{3\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$. Here, by means of certain experiments, I have discovered that the angle is contained between the following limits: $1^\circ$ -2′ -40″ and $1^\circ$ -3′ -0″. From this as above the true value of $\phi$ itself will be secured.

| | | |
|---|---|---|
| $\phi$ | $= 1^\circ$-2′-40″ | $1^\circ$-3′-0″ |
| in sec. | = 3760″ | 3780″ |
| log | = 3.5751878450 | 3.5774917998 |
| subtract | = 5.3144251332 | 5.3144251332 |
| $\log \phi$ | = 8.2607627118 — 10 | 8.2630666666 — 10 |
| $\phi$ | = 0.0182289944 | 0.0183259571 |
| $\frac{3\pi}{2}$ | = 4.7123889804 | 4.7123889804 |
| $\frac{3\pi}{2} - \phi = \frac{a}{c}$ | = 4.6941599860 | 4.6940630233 |
| $\frac{1}{2}\phi$ | = 31′-20″ | 31′-30″ |
| $\log \cot \frac{1}{2}\phi$ | = 2.0402552577 | 2.0379511745 |
| $\log v$ | = 0.3096845055 | 0.3091937748 |
| add | = 0.3622156886 | 0.3622156886 |
| $\log u$ | = 0.6719001941 | 0.6714094634 |
| $u$ | = 4.6978613391 | 4.6925559924 |
| $\frac{a}{c}$ | = 4.6941599860 | 4.6940630233 |
| error | = + 37013531 | — 15070309 |

From these errors the true value of the angle $\phi$ is found to be $1^\circ$ -2′ -54.213″, and $\frac{a}{c} = 268^\circ$ -57′ -5.787″. Since therefore

| | |
|---|---|
| $\phi$ | = 3774.213″ |
| log | = 3.5768264061 |
| subtract | = 5.3144251332 |

$\log \phi \qquad = 8.2624012729 - 10$
$\phi \qquad = 0.0182979009$
$3\frac{\pi}{2} \qquad = 4.7123889804$

$\frac{3\pi}{2} - \phi = \frac{a}{c} \qquad = 4.6940910795$

Therefore the tone of a ribbon oscillating in the first case will be to the tone of the same ribbon vibrating in this case as the square of 1.8751040813 is to the square of 4.6940910795, or as 1 to 6.266891, or, in least integers, as 4 is to 25, or as 1 is to $6\frac{4}{15}$. Whence the latter tone will be just about 2 octaves plus a fifth plus a half tone higher than the former [27]).

79. For the following cases of oscillations of the same ribbon, in which the ribbon cuts the axis $AB$ at two or more points while oscillating, the angle $\phi$ becomes much smaller; thus for the third case the following equation is secured:

$$\frac{5\pi}{2} + \phi = \log \cot \tfrac{1}{2}\phi = \frac{a}{c}.$$

Therefore, since $e^{\frac{5\pi}{2}+\phi} = \cot \frac{1}{2}\phi$, because $\phi$ is an extermely small angle,

$$e^{\frac{5\pi}{2}+\phi} = e^{\frac{5\pi}{2}}\left(1 + \phi + \frac{\phi^2}{2} + \frac{\phi^3}{6} + \ldots\right),$$

$$\text{and } \cot \tfrac{1}{2}\phi = \frac{1 - \frac{1}{8}\phi^2}{\frac{1}{2}\phi - \frac{1}{48}\phi^3} = \frac{2}{\phi} - \frac{\phi}{6}.$$

Hence approximately,

$$e^{\frac{5\pi}{2}} = \frac{2}{\phi}, \text{ or } \phi = 2e^{\frac{-5\pi}{2}},$$

or more closely

$$\phi = \frac{1}{1 + \frac{1}{2}e^{\frac{5\pi}{2}}}\ {}^{28}),\ \text{whence } \frac{a}{c} = \frac{5}{2}\pi + \frac{2}{2 + e^{\frac{5\pi}{2}}},$$

The latter term is extremely small. In a similar manner for the fourth case of oscillations, approximately

$$\frac{a}{c} = \frac{7}{2}\pi = 2e^{\frac{-7\pi}{2}}$$

Since these second terms are on the point of vanishing, the values of $\frac{a}{c}$ will be $\frac{9\pi}{2}$, $\frac{11\pi}{2}$, etc., which will differ less from the true values the farther they proceed.

## Concerning the Oscillations of a Free Elastic Ribbon

80. Let us now consider an elastic ribbon fixed at no point, but free or lying upon an extremely smooth plane or, neglecting gravity, existing in a vacant space. Now it is readily apparent that a ribbon of this kind can receive an oscillatory motion, while the ribbon *acb* (Fig. 22), curving itself, passes alternately on one side and the other side of the position of rest *AB*. Therefore the oscillatory motion may be defined in the same way in which it was defined in the preceding case, provided only that the calculations be adapted to this case in the necessary manner. Therefore let *acb* be the curved shape of the ribbon which it assumes while oscillating, and *ACB* the shape of the same ribbon in the state of equilibrium through which it passes in each oscillation. As before, let the length of the ribbon $AB = a$, the absolute elasticity be $Ek^2$, and the weight or mass equal to $M$. Then let the abscissa $AP = x$, the ordinate $PM = y$, the arc $aM = s$, which will correspond with the abscissa $x$ in such a way that $ds = dx$; from this the radius of curvature at $M$ will be $R = \frac{dx^2}{d^2 y}$. Further, let the first ordinate $Aa = b$. All this being stipulated, by instituting the same process of reasoning as before [sec. 66 and 67], we shall arrive at the same equation

$$\frac{E\,k^2}{R} = \frac{M}{af}\int dx \int y\,dx = \frac{E\,k^2\,d^2 y}{dx^2}$$

81. Therefore, if we take $\frac{E\,k^2\,af}{M} = c^4$, where $f$, as before, expresses the length of a simple isochronous pendulum, we shall have, by integrating, the following equation for the curve:

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

This will be adapted to the present case as follows: First, when $x = o$, $y = b$, and hence $b = A + B + D$. Second.

since $\frac{c^4\, d^2 y}{dx^2} = \int dx \int y dx$, then $\frac{d^2 y}{dx^2} = o$ when $x = o$, whence arises $A + B - D = o$. Thirdly, since $\frac{c^4\, d^3 y}{dx^3} = \int y dx$, then $\frac{d^3 y}{dx^3} = o$ when $x = o$, whence $A - B - C = o$. Fourthly, if $x = a$, $\int y dx$, or $\frac{d^3 y}{dx^3}$ must vanish, because $\int y dx$ expresses the sum of all the forces drawing the ribbon in a direction normal to the axis $AB$, and if this sum were not equal to zero, the ribbon itself would undergo a local motion contrary to the conditions instituted; for this reason, therefore,

$$Ae^{\frac{a}{c}} - Be^{\frac{-a}{c}} - \text{C}\cos\frac{a}{c} + \text{D}\sin\frac{a}{c} = \text{o}.$$

Fifth, since the ribbon is free at the end $B$, it cannot have any curvature there, and therefore $\frac{d^2y}{dx^2} = o$ when $x = a$, whence

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - \text{C}\sin\frac{a}{c} - \text{D}\cos\frac{a}{c} = \text{o}.$$

By taking these five conditions into the computations, not only will the four constants $A$, $B$, $C$, and $D$ be determined, but also the value of the fraction $\frac{a}{c}$ will be found; from which the length of the simple isochronous pendulum $f$ will become known.

82. From the second and third of these equations, $D = A + B$, and $C = A - B$, and these values, substituted in the equations above, will give the following:

$$Ae^{\frac{a}{c}} - Be^{\frac{-a}{c}} - (\text{A}-\text{B})\cos\frac{a}{c} + (\text{A}+\text{B})\sin\frac{a}{c} = \text{o, and}$$

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - (\text{A}-\text{B})\sin\frac{a}{c} - (\text{A}+\text{B})\cos\frac{a}{c} = \text{o}.$$

From these it is found that

$$\frac{A}{B} = \frac{e^{\frac{-a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c}}{e^{\frac{a}{c}} - \cos\frac{a}{c} + \sin\frac{a}{c}} = \frac{-e^{\frac{-a}{c}} - \sin\frac{a}{c} + \cos\frac{a}{c},}{e^{\frac{a}{c}} - \sin\frac{a}{c} - \cos\frac{a}{c}.}$$

from which the following equation is secured:

$$2 - e^{\frac{a}{c}} \cos\frac{a}{c} - e^{\frac{-a}{c}} \cos\frac{a}{c} = 0, \text{ or } e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Whence the following equations will be found:

$$\text{I.} \qquad \frac{a}{c} = \frac{\pi}{2} - \phi = \log\tan\tfrac{1}{2}\phi$$

which gives $\frac{a}{c} = o$ for the natural position of the ribbon [29]),

$$\text{II.} \qquad \frac{a}{c} = \frac{\pi}{2} - \phi = \log\cot\tfrac{1}{2}\phi$$

$$\text{III.} \qquad \frac{a}{c} = \frac{3\pi}{2} + \phi = \log\cot\tfrac{1}{2}\phi$$

$$\text{IV.} \qquad \frac{a}{c} = \frac{5\pi}{2} - \phi = \log\cot\tfrac{1}{2}\phi$$

$$\text{V.} \qquad \frac{a}{c} = \frac{7\pi}{2} + \phi = \log\cot\tfrac{1}{2}\phi$$

$$\text{VI.} \qquad \frac{a}{c} = \frac{9\pi}{2} - \phi = \log\cot\tfrac{1}{2}\phi$$

etc.

83. These equations again indicate innumerable modes of oscillations. In the second of these the ribbon will cut the axis *AB* only once; in the third, twice; in the fourth, three times, in the fifth, four times, and so on. From this it is understood that the second, fourth, sixth, etc. modes are not adaptable to the present case. For since in these the number of intersections is uneven, the position of the ribbon, while oscillating in the second mode, would be such as FIG. 23 represents, in which mode, however small the sum of the forces acting throughout the whole ribbon tends to become, nevertheless the ribbon would acquire from them a rotary motion around the center point *C*, because the forces applied to each half *aC* and *bC* would combine to produce the same rotary motion in the ribbon. For this reason, since the rotary motion must be absolutely excluded, the shape of the ribbon which is taken on during oscillation ought to be of such a character that not only the sum of the acting forces

applied to the whole ribbon equals zero, but also that the sum of their moments tends to vanish; and this is obtained if the curve at the center point $c$ (FIG. 22) be given a diameter $cC$. This takes place if the curve cuts the axis $AB$ in two, in four, or in general, an even number of points; from which the 3rd, 5th, 7th, etc., equations only, will give us satisfactory solutions [36]).

84. This limitation will be found to be contained in the very statement of the problem if we admit only curves of the kind that have the straight line $Cc$ as a diameter, that is, in which the value of $y$ would be the same if $(a - x)$ should be written in place of $x$. Therefore let us substitue $(a - x)$ in place of $x$ in the general equation, whence

$$y = Ae^{\frac{a}{c}} e^{\frac{-x}{c}} + Be^{\frac{-a}{c}} e^{\frac{x}{c}} + C\sin\frac{a}{c}\cos\frac{x}{c} - C\cos\frac{a}{c}\sin\frac{x}{c}$$
$$+ D\cos\frac{a}{c}\cos\frac{x}{c} + D\sin\frac{a}{c}\sin\frac{x}{c}.$$

Since this equation must agree with the equation

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c},$$

therefore

$$Ae^{\frac{a}{c}} = B,\ C\left(1 + \cos\frac{a}{c}\right) = D\sin\frac{a}{c},\ \text{and } C\sin\frac{a}{c} = D\left(1 - \cos\frac{a}{c}\right).$$

The last two equations are identical. Since therefore $\frac{A}{B} = e^{\frac{-a}{c}}$ when this value is compared with the expression in section 82, there will appear

$$e^{\frac{-a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c} = 1 - e^{\frac{-a}{c}}\cos\frac{a}{c} + e^{\frac{-a}{c}}\sin\frac{a}{c},\ \text{or}$$

$$e^{\frac{-a}{c}} = \frac{1 + \cos\frac{a}{c} + \sin\frac{a}{c}}{1 + \cos\frac{a}{c} - \sin\frac{a}{c}} = \frac{1 + \sin\frac{a}{c}}{\cos\frac{a}{c}} = \frac{\cos\frac{a}{c}}{1 - \sin\frac{a}{c}}.$$

85. Therefore $e^{\frac{a}{c}} = \dfrac{1 - \sin\frac{a}{c}}{\cos\frac{a}{c}}$.

All this is contained in the equation previously found [sec. 82].

$$e^{\frac{a}{c}} = \frac{1 + \sin\frac{a}{c}}{\cos\frac{a}{c}}$$

Merely one half of the cases shown above [end of sec. 82], in other words those which have uneven numbers, will state the present problem. Wherefore, since the first equation comprises the natural state of the ribbon, all the modes of oscillations will be comprised in the following equations :

I. $\dfrac{a}{c} = \dfrac{3\pi}{2} + \phi = \log \cot \frac{1}{2} \phi$

II. $\dfrac{a}{c} = \dfrac{7\pi}{2} + \phi = \log \cot \frac{1}{2} \phi$

III. $\dfrac{a}{c} = \dfrac{11\pi}{2} + \phi = \log \cot \frac{1}{2} \phi$

etc.

Therefore the first of these equations will give us the first and principal mode of oscillation, for which the value of the angle will be found by approximation, in a way similar to that used before. Now the limits of the angle $\phi$ are soon found to be $1^\circ$-0′-40″ and $1^\circ$-1′-0″, from which, by the following calculation, the true value of $\phi$ is secured.

| | | |
|---|---|---|
| $\phi$ | = $1^\circ$-0′-40″ | $1^\circ$-1′ 0″ |
| in sec. | = 3640″ | 3660″ |
| log | = 3.5611013836 | 3.5634810854 |
| subtract | = 5.3144251332 | 5.3144251332 |
| $\log \phi$ | = 8.2466762504 — 10 | 8.2490559522 — 10 |
| $\phi$ | = 0.0176472180 | 0.0177441807 |
| $3\frac{\pi}{2}$ | = 4.7123889804 | 4.7123889804 |
| $3\frac{\pi}{2} + \phi = \frac{a}{c}$ | = 4.7300361984 | 4.7301331611 |
| $\frac{1}{2}\phi$ | = 30′-20″ | 30′-30″ |
| $v$ | = 2.0543424742 | 2.0519626482 |

| | | |
|---|---|---|
| $\log v$ | $= 0.3126728453$ | $0.3121694510$ |
| add | $= 0.3622156886$ | $0.3622156886$ |
| $\log u$ | $= 0.6748885339$ | $0.6743851396$ |
| $u = \frac{a}{c}$ | $= 4.7302983543$ | $4.7248186037$ |
| error | $= \quad + 636341$ | $+ 53145574$ |
| | | $636341$ |
| | Difference | $52509233$ |

From this it is seen that the true value of $\phi$ is not contained between those limits, but is somewhat less than $1^{\circ}$ -0′ -40″ [31]. None the less, however, it will be found from these errors. For let $\phi = 1^{\circ}$ -0′ -40″ $- n''$; then

$$20'' : 52509233 :: n'' : 636341, \text{ whence}$$

$$n = \frac{2423}{10,000}, \text{ and}$$

| | |
|---|---|
| $\phi$ | $= 1^{\circ}$ -0′ -39.7576″ |
| or | $= 3639.7576''$ |
| log | $= 3.5610724615$ |
| subtract | $= 5.3144251332$ |
| $\log \phi$ | $= 8.2466473283 - 10$ |
| $\phi$ | $= 0.0176460428$ |
| $3\frac{\pi}{2}$ | $= 4.7123889804$ |
| $3\frac{\pi}{2} + \phi = \frac{a}{c}$ | $= 4.7300350232$ [correct value 4.7300408] |

86. Let this number be equal to $m$, so that

$$c^4 = \frac{Ek^2af}{M}, \text{ and so that } a^4 = \frac{m^4Ek^2af}{M}, \text{ and } f = \frac{a^4}{m^4}\frac{1}{Ek^2}\cdot\frac{M}{a}.$$

Hence in the same way the number of oscillations produced by this ribbon in a single second will be

$$\frac{m^2}{a^2}\sqrt{g\,Ek^2\frac{a}{M}}$$

where $g = 3.16625$ Rhenish feet. Now if the same ribbon be made to produce a tone when it is either free, or has one end $B$ fixed in a wall, the tones will be in the ratio of $n^2 : m^2$, or as the square of the numbers 1.8751040813 and 4.7300350232 [correct value 4.7300408], or as 1 is to 6.363236. The ratio of these tones will be approximately 11 : 70. Therefore the interval between these tones will be two octaves plus a fifth plus a half tone. If the free ribbon be taken twice the length of the fixed ribbon, the interval between the tones will be about a minor sixth. $\left[\frac{8}{5} = \frac{72}{45} \text{ instead of } \frac{70}{44}\right]$

87. The value for the fraction $\frac{a}{c}$ being found, the equation for the curve which the ribbon forms during oscillations, hitherto indeterminate, may now be determined; for

$$e^{\frac{a}{c}} = \frac{1 - \sin\frac{a}{c}}{\cos\frac{a}{c}}, \text{ and } Ae^{\frac{a}{c}} = B; \text{ hence } B = \frac{1 - \sin\frac{a}{c}}{\cos\frac{a}{c}}\text{A},$$

$$C = A - B = \frac{A\left(\cos\frac{a}{c} + \sin\frac{a}{c} - 1\right)}{\cos\frac{a}{c}},$$

$$D = A + B = \frac{A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)}{\cos\frac{a}{c}},$$

Now

$$b = A + B + D = 2D = \frac{2A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)}{\cos\frac{a}{c}},$$

whence

$$A = \frac{b\cos\frac{a}{c}}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(+1 + \sin\frac{a}{c} - \cos\frac{a}{c}\right)}{4\sin\frac{a}{c}}$$

$$B = \frac{b\left(1 - \sin\frac{a}{c}\right)}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(-1 + \sin\frac{a}{c} + \cos\frac{a}{c}\right)}{4\sin\frac{a}{c}}$$

$$C = \frac{b\left(-1 + \sin\frac{a}{c} + \cos\frac{a}{c}\right)}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(1 - \cos\frac{a}{c}\right)}{2\sin\frac{a}{c}}$$

$$D = \frac{b}{2} = \frac{b\sin\frac{a}{c}}{2\sin\frac{a}{c}}.$$

These being substituted, the following equation will arise :

$$\frac{y}{b} = \frac{e^{\frac{x}{c}}\cos\frac{a}{c} + e^{-\frac{x}{c}}\left(1 - \sin\frac{a}{c}\right)}{2\left(1 - \sin\frac{a}{c} + \cos\frac{a}{c}\right)} +$$

$$\frac{(1 - \cos\frac{a}{c})\sin\frac{x}{c} + \sin\frac{a}{c}\cos\frac{x}{c}}{2\sin\frac{a}{c}}$$

88. Now because the straight line $Cc$ is the diameter of the curve, let there be taken, from the central point $C$, the abscissa $CP = z$, then $x = \frac{1}{2}a - z$, whence

$$e^{\frac{x}{c}} = e^{\frac{a}{2c}}\, e^{\frac{-z}{c}} \sqrt{\frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}}, \text{ and } e^{\frac{-x}{c}} = e^{\frac{z}{c}} \sqrt{\frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}},$$

whence

$$\frac{Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}}}{b} = \frac{(e^{\frac{z}{c}} + e^{\frac{-z}{c}})\sqrt{\cos\frac{a}{c}\,(1 - \sin\frac{a}{c})}}{2\,(1 - \sin\frac{a}{c} + \cos\frac{a}{c})} = \frac{e^{\frac{z}{c}} + e^{\frac{-z}{c}}}{2\,(e^{\frac{a}{2c}} + e^{\frac{-a}{2c}})}$$

$$\textit{Further}\ (1 - \cos\frac{a}{c})\sin\frac{x}{c} + \sin\frac{a}{c}\cos\frac{x}{c} = \sin\frac{x}{c} + \sin\frac{a-x}{c}$$

$$= \sin(\frac{a}{2c} - \frac{z}{c}) + \sin(\frac{a}{2c} + \frac{z}{c}).$$

$$= 2\sin\frac{a}{2c}\cos\frac{z}{c}.$$

These values being substituted, the following equation will arise :

$$\frac{2y}{b} = \frac{e^{\frac{z}{c}} + e^{\frac{-z}{c}}}{e^{\frac{a}{2c}} + e^{\frac{-a}{2c}}} + \frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}$$

which is the most simple form whereby the character of the curve $aMcb$ can be expressed. Now it is manifest that whether $z$ be taken as positive or negative, the same value for $y$ will appear.

It is also true that $e^{\frac{a}{2c}} + e^{\frac{-a}{2c}} = -\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}$ [32])

We have found that the angle $\frac{a}{c} = 271^{\circ}$ -0′ -39 3/4″ [correct value is $271^{\circ}$ -0′ -40.94″].

89. Setting $z = o$, $y$ will express the value of the ordinate $Cc$. This gives

$$\frac{2Cc}{b} = -\frac{2\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}} + \frac{1}{\cos\frac{a}{2c}} \text{ or } \frac{Cc}{Aa} = \frac{1-\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}$$

$$= \frac{1}{2}\sec\frac{a}{2c} - \frac{1}{2}\sec\frac{a}{2c}\sqrt{\cos\frac{a}{c}}.$$

But $\cos\frac{a}{c} = \sin 1^\circ\text{-}0'\text{-}39\ 3/4''$ [correct value $1^\circ\text{-}0'\text{-}40.94''$]

and $\cos\frac{a}{2c} = -\sin 45^\circ\text{-}30'\text{-}19\ 7/8''$ [correct value $45^\circ\text{-}30'\text{-}20.47''$].

Hence it is found that $\frac{Cc}{Aa} = -0.607815$ [correct value $-0.607841$].

Then if $y = 0$, the points $E$ and $F$ at which the curve intersects the axis will be found; therefore

$$e^{\frac{z}{c}} + e^{\frac{-z}{c}} = -\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}\left(e^{\frac{a}{2c}} + e^{-\frac{a}{2c}}\right) = \frac{2\cos\frac{z}{c}}{\sqrt{\cos\frac{a}{c}}},$$

from which, by approximation

$\frac{CE}{CA} = 0.551685$, and $\frac{AE}{AC} = 0.448315$.

Therefore, while the ribbon is performing these oscillations, these points $E$ and $F$ will remain motionless. Therefore the oscillatory motion of this kind, which otherwise, it would seem, could scarcely be produced in reality, can be easily produced. For if the ribbon should be fixed at the points $E$ and $F$ defined in this fashion, it would oscillate exactly as if it were free.

90. If the second of the equations found above, viz. $\frac{a}{c} = \frac{7\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$ be treated in this same fashion, in which case $\phi$ will be approximately zero, then the second mode by which a free ribbon can perform vibrations will appear, that is, by cutting the axis $AB$ at four points. Then the ribbon will oscillate precisely as if it had been fixed at these four points. Conversely, therefore, if the ribbon be fixed at these four points, or merely at any two of them, it will oscillate just as if it were free, and it will produce a much higher tone, inasmuch as it will be in about the same ratio to the preceding tone as $7^2 : 3^2$; that is, the interval will be of two octaves plus a fourth plus

the half of a semitone. The third mode of oscillation, in which $\frac{a}{c} = \frac{11\pi}{2} + \phi = \log \cot \frac{1}{2} \phi$, will have six intersections of the curve *acb* with the axis *AB*. The tone produced will be higher by one octave plus a minor third, $[\frac{121}{49} = 2\frac{2}{5}$, approximately] and the ribbon will produce this tone if it be fixed at two of the six points. Hence it is clear how different tones can be produced by the same ribbon, according to the different ways in which it is fixed at two points; and if the two points at which it is fixed coincide with its intersections in the first, second, or third, etc. modes, the oscillations adapt themselves to some one of the following modes down to an infinite value. In the latter case the tone will be so high that it cannot be heard at all, or what amounts to the same thing, the ribbon will be absolutely unable to take on an oscillatory motion; or at all events, as in the case of a vibrating chord under which a bridge is so placed that its parts have no rational ratio to one another, an indistinct tone will be produced.

## On Oscillations of an Elastic Ribbon Fixed at Both Ends

91. Now let the elastic ribbon be fixed at both ends *A* and *B* (Fig. 24), but in such a way that the tangents of the curve at these points are not fixed. To produce this case in experiment, let extremely sharp points $A\alpha$, and $B\beta$ be fixed to the extremities of the ribbon; these sharp points, when fastened to a wall, will render the extremities *A* and *B* of the ribbon immovable. In order to investigate the oscillatory motion of this elastic ribbon, let us take, as above, the absolute elasticity of the ribbon to be equal to $Ek^2$, its length $AB = a$, its weight equal to *M*, and the length of the simple isochronous pendulum equal to *f*. Let *AMB* be a curvilinear figure which the ribbon takes on while performing oscillations, and let the abscissa $AP = AM = x$, the ordinate $PM = y$, and the radius of curvature at *M* be equal to *R*. Furthermore, let *P* be the force which the sharp point $A\alpha$ supports in the direction $A\alpha$. Because the force by which the element *Mm* must be acted upon in the direction $M\mu$ in order

that the ribbon be kept in this position, is equal to $\frac{Mydx}{af}$, the following equation, by the rules described above [sec. 57, 66, 67], will result : $\frac{Ek^2}{R} = Px - \frac{M}{af}\int dx \int ydx.$

But $R = \frac{-dx^2}{d^2y}$, because the curve is concave to the axis; hence

$$\frac{Ek^2d^2y}{dx^2} = \frac{M}{af}\int dx \int ydx - Px.$$

Therefore, when $x = o$, the radius of curvature $R$ at $A$ will be infinite, that is, $d^2y = o$.

92. If this equation be differentiated twice, the same equation which we have found in the preceding case will appear, namely

$$Ek^2d^4y = \frac{M}{af}\,ydx^4.$$

But if $\frac{Ek^2af}{M}$ be put equal to $c^4$, the integral equation will be

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + \text{Csin}\,\frac{x}{c} + \text{Dcos}\,\frac{x}{c}.$$

To determine this, let $x = o$, and since $y = o$ at the same time, then $A + B + D = o$. Second, let $x = a$, and since $y$ again must be zero,

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} + \text{Csin}\,\frac{a}{c} + \text{Dcos}\,\frac{a}{c} = 0.$$

Third, since $\frac{d^2y}{dx^2}$ must vanish when $x = o$, and when $x = a$, we have $A + B - D = o$, and

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - \text{Csin}\,\frac{a}{c} - \text{Dcos}\,\frac{a}{c} = 0.$$

Now the equations $A + B - D = o$, and $A + B + D = o$, give $D = o$, and $B = -A$, which values, when substituted in the remaining two equations, give

$$A\,(e^{\frac{a}{c}} - e^{\frac{-a}{c}}) + \text{Csin}\,\frac{a}{c} = 0, \text{ and } A\,(e^{\frac{a}{c}} - e^{\frac{-a}{c}}) - \text{Csin}\,\frac{a}{c} = 0.$$

These equations cannot be satisfied unless $A = o$, since it is not possible for $e^{\frac{a}{c}} = e^{\frac{-a}{c}}$, except in the case when $\frac{a}{c} = o$. Then indeed $C\sin\frac{a}{c} = o$, and here, since $C$ cannot be put equal to zero, because there would be no oscillatory motion, $\sin\frac{a}{c}$ will be equal to zero. Therefore $\frac{a}{c} = \pi$, or $\frac{a}{c} = 2\pi$, etc., whence again there arises an infinite number of modes of oscillation, according as the curve *AMB* cuts the axis either nowhere except at the end points *A* and *B*, or at one point, or at two points, or at several points. This is deduced from the equation $y = C\sin\frac{x}{c}$; and however many points of intersection there are, they will be at equal intervals from one another.

93. Since, therefore, for the first and principal mode of oscillation $\frac{a}{c} = \pi$, $a^4 = c^4\pi^4 = \pi^4 Ek^2 \frac{af}{M}$, whence $f = \frac{a^4}{\pi^4} \cdot \frac{1}{Ek^2} \cdot \frac{M}{a}$. Wherefore as far the length of the ribbon is concerned, the tones again will be inversely proportional to the square of the length [sec. 76]. Now the tones of this ribbon, produced in this fashion, will be to the tones of the same ribbon, if it be fastened at one end *B* in a wall, as $\pi^2$ is to the square of the number 1.8751040813, that is, as 2.807041 is to 1, or, in least integers, as 160 is to 57, an interval which is about one octave plus the third half tone. If the oscillations are related according to the second mode, in which $\frac{a}{c} = 2\pi$, the tone will be higher by 2 octaves, but if $\frac{a}{c} = 3\pi$, the tones will be higher by 3 octaves and a whole tone than in the case where $\frac{a}{c} = \pi$, and so on [33]). In order to adapt this more readily to experiment, it must be noted that here extremely small oscillations must be taken, so that there is no essential elongation of the ribbon. Wherefore, since the tenacity of the ribbon, by which it resists even a slight extension, without which oscillations of this kind cannot be

produced, introduces an alteration here, those points ought to be fixed in such a way that such a minute extension is not impeded. This results if they rest on a perfectly smooth plane. Thus the elastic ribbon *AB*, equipped at *A* and *B* with the cusps $A\alpha$ and $B\beta$, if these cusps be placed upon a mirror, will give a sound which conforms to the calculations.

## On Oscillations of an Elastic Ribbon Fastened at Both Ends in a Wall

94. The preceding case having been cleared up, let the discussion of elastic ribbons come to a close with the oscillatory motion of an elastic ribbon fastened in a wall at both ends *A* and *B* (Fig. 25), in such a way that during oscillations, not only do the points *A* and *B* remain motionless, but also the straight line *AB* is constantly tangent to the curve *AMB* at the points *A* and *B*.

Here we must again be careful that the bolts fastening *A* and *B* are not absolutely firm, but allow as much extension as is required for curvature. Whatsoever be the forces requisite to hold the band fixed at the points *A* and *B*, therefore, we shall arrive at the following differential equation of the 4th order :

$$Ek^2d^4y = \frac{M}{af}\, ydx^4.$$

the integral of which is, as above, letting $\frac{Ek^2af}{M} = c^4$,

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + \text{C}\sin\frac{x}{c} + \text{D}\cos\frac{x}{c}.$$

95. The constants *A*, *B*, *C*, and *D* must be so defined that, taking $x = o$, not only $y$ disappears, but also $dy$ becomes zero, because at *A* the curve is tangent to the curve *AB*. Now the same thing must also take place if $x = a$, whence the following four equations will arise :

I. $A + B + D = o.$

II. $A - B + C = o.$

III. $Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} + \text{C}\sin\frac{a}{c} + \text{D}\cos\frac{a}{c} = 0$

$$\text{IV. } Ae^{\frac{a}{c}} - Be^{\frac{-a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c} = 0.$$

From the first and second of these equations, it follows that $C = -A + B$, and $D = -A - B$, which values, substituted in the other two equations, will give

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - (A - B)\sin\frac{a}{c} - (A + B)\cos\frac{a}{c} = 0, \text{ and}$$

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - (A - B)\cos\frac{a}{c} + (A + B)\sin\frac{a}{c} = 0,$$

The sum and difference of these give, respectively,

$$Ae^{\frac{a}{c}} + B\sin\frac{a}{c} - A\cos\frac{a}{c} = 0, \text{ or } \frac{A}{B} = \frac{\sin\frac{a}{c}}{\cos\frac{a}{c} - e^{\frac{a}{c}}},$$

$$\text{and } Be^{\frac{a}{c}} - A\sin\frac{a}{c} - B\cos\frac{a}{c} = 0, \text{ or } \frac{A}{B} = \frac{e^{-\frac{a}{c}} - \cos\frac{a}{c}}{\sin\frac{a}{c}},$$

whence

$$2 = \left(e^{\frac{a}{c}} + e^{\frac{-a}{c}}\right)\cos\frac{a}{c}, \text{ or } e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

This equation, since it agrees with the one found in section 82, will be satisfied by the following solutions, infinite in number:

$$\text{I. } \frac{a}{c} = \frac{\pi}{2} - \phi = \log\cot\tfrac{1}{2}\phi \quad \text{II. } \frac{a}{c} = \frac{3\pi}{2} + \phi = \log\cot\tfrac{1}{2}\phi$$

$$\text{III. } \frac{a}{c} = \frac{5\pi}{2} - \phi = \log\cot\tfrac{1}{2}\phi \quad \text{IV. } \frac{a}{c} = \frac{7\pi}{2} + \phi = \log\cot\tfrac{1}{2}\phi$$

etc.

96. It is impossible to satisfy the first of these equations unless $\frac{a}{c} = 90^{\circ}$, and therefore $\frac{a}{c} = 0$ [29]); whence the first mode of oscillation arises from the equation $\frac{a}{c} = \frac{3\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$, and since this has been treated above [sec. 85], $\frac{a}{c} = 4.7300350232$ [correct value 4.7300408]. Wherefore the elastic ribbon, both ends of which are held fast in a wall, will make its vibrations exactly as if it were absolutely free. Now this agreement concerns only the first mode of oscillation [34]); for the second mode of

oscillation, in which $\frac{a}{c} = \frac{5\pi}{2} - \phi = \log \cot \frac{1}{2} \phi$, and the ribbon cuts the axis $AB$ at one point during the oscillation, does not have its equivalent in a free ribbon. The third mode of a ribbon fastened at both ends will agree with the second mode of a free ribbon, and so on.

97. The latter two kinds of oscillations [sec. 91 and 94] cannot, for the reason given, be investigated in any suitable fashion by experiment. The first kind, however, [sec. 65] is not only extremely well suited for experiment, but also it can be applied to the study of the absolute elasticity, which we have called $Ek^2$, of any proposed ribbon. If the tone which a ribbon of this kind produces when fastened at one end in a wall be noted, and a similar tone be produced at the same time in a chord, the number of oscillations produced in a second will become known. If this number be put equal to the expression $\frac{n^2}{a^2} \sqrt{g \frac{Ek^2 a}{M}}$, since $n$ is known, and the quantities $g$, $a$, and $M$ have been found by measurements, then the value of the expression $Ek^2$ will become known, and so also the absolute elasticity. This latter value can be compared with that absolute elasticity which we have already shown how to find from the curvature. [35]) [sec. 38]

NOTES ON THE MONOGRAPH OF

## LEONHARD EULER

# Concerning Elastic Curves

1744

**Additamentum I to the ad Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes.**

(Original notes by H. Linsenbarth in " Ostwald's Klassiker der exakten Wissenschaften " number 175. Translation and changes from the original German by Donald M. Brown.)

For a pertinent understanding of the older, fundamental works on elastics, it is necessary to know the connections of the statements contained in them with the methods of the Mechanics of Solids and the Mechanics of Continua.

The following introductory remarks should make possible, even to those readers who have not thoroughly studied general mechanics, a critical judgment of the most important of the original works on this subject.

*a*) Suppose the originally straight elastic wire (lamina) to be replaced by a chain of infinitely small, stiff body elements, which are connected with one another by spigot-joints, all of whose axes are and remain perpendicular to a fixed plane. At the joints $C$ and $C'$ of any element $K$, the forces $-\bar{r}$ and $\bar{r}'$ are transmitted from the preceding and following elements respectively. In addition to these single forces, the moments $-\bar{R}$ and $\bar{R}'$ will be transmitted at $C$ and $C'$, if the joints offer resistance to rotation of the elements about the spigot-joint axes. Let the resultant of the external forces acting on $K$ be $\overline{dk}$, and its moment with respect to the point of rotation $C$ of the element $K$ be $\overline{dM}$. The moments of reaction $\bar{R}$ and $\bar{R}'$ can be referred to a suitable pole $O$ in the plane.

Putting $\overline{OC} = \bar{c}$, $\overline{OC'} = \bar{c}'$, $\bar{c}' - \bar{c} = \overline{dc}$, $\bar{r}' - \bar{r} = \overline{dr}$, and $\bar{R}' - \bar{R} = \overline{dR}$, the principles of elementary statics give, for the equilibrium of the forces on the body element $K$, the conditions :

$$\overline{dr} + \overline{dk} = o, \text{ and } \overline{dR} + \overline{dc}.\bar{r} + \overline{dM} = o \text{ (Pole at } C).$$

Introducing the element of arc $ds$ of the equilibrium curve (axis of the curved inextensible wire), which connects the joint-point $C$, the specific quantities $\bar{\kappa} = \frac{\overline{dk}}{ds}$ and $\bar{m} = \frac{\overline{dM}}{ds}$, which are related to the constant length of the axis, may be used instead of the absolute quantities $\overline{dk}$ and $\overline{dM}$. In addition, taking $\frac{d\bar{c}}{ds} = \bar{\sigma}$, then the static relation takes the form 1) $\frac{\overline{dr}}{ds} + \bar{\kappa} = o$, and 2) $\frac{d\bar{R}}{ds} + \overline{\sigma r} + \bar{m} = o$.

These equations appear frequently in modern literature, since they are not present in this explicit form in the works of the older writers (Jacob Bernoulli, Euler). Euler knew these equations in their corresponding form for chains whose links are of finite dimensions (compare the statements by Routh in his "Dynamik," German edition, vol. 2, page 71.) He merely failed to take the transitional step. Compare also Euler's statement in section 57 (page 116).

Equations 1) and 2) are advanced by Clebsch, "Elastizität fester Körper," Leipzig 1862, pp. 204-222, and were used to establish Kirchhoff's theory of wires. They are also found in Thomson and Tait "Natural Philosophy" Part 2, 1st ed., Oxford, 1867; 2d ed., Cambridge, 1895, pp. 152-155; in Love "Theory of Elasticity," 2d ed. Cambridge, 1906, pp. 370-372; and, in their direct relation to the theory of body chains with finite links, in the "Zeitschrift für Math. und Phys." vol. 56, 1908, pp. 68 ff. by K. Heun. From a general point of view, they have been treated in great detail by E. and F. Cosserat in "Théorie des corps déformables" Paris 1909, pp. 6-65.

*b*) Let the departure of the axis element $\overline{dc}$ in the direction of the $x$-axis of a fixed set of axes $Oxy$ be defined by the angle $\theta$, and let the contingent angle of the elastic curve be denoted in magnitude and direction by $d\theta$. By this stipulation, $\frac{d\bar{\theta}}{ds}$ is the specific rotation of the axis element $\overline{dc}$. In addition to the direction of the tangent $(\bar{\sigma})$, let the direction of the normal to the curve, $\bar{\nu}$, be introduced. Then $\bar{\eta} = \bar{\sigma\nu}$ is the binormal (of unit length) on the plane curve, so that $\frac{d\theta}{ds} = \bar{\omega} = \frac{1}{a_1}\bar{\eta}$, where $a_1$, is the radius of curvature of the elastic curve.

According to the hypothesis of DANIEL BERNOULLI, $\bar{R} = P\bar{\omega}$, where $P$ is a constant depending upon the dimensions of the cross section and the coefficient of elasticity.

Further, let $$\kappa_x = \frac{\partial u}{\partial c_x},\ \kappa_y = \frac{\partial u}{\partial c_y},\ m = \frac{\partial u}{\partial \sigma}.$$

The function $u$ can be represented as the potential of the external force. From the static fundamental equations 1) and 2), it now follows that

$$\frac{d\bar{r}}{ds}\frac{d\bar{c}}{ds} + \frac{d\bar{R}}{ds}\frac{d\bar{\theta}}{ds} + \bar{\sigma r}\frac{d\bar{\theta}}{ds} + \frac{du}{ds} = o$$

or $$\frac{d\bar{r}}{ds}\bar{\sigma} + \bar{\sigma r}\,\bar{\omega} + P\bar{\omega}\frac{d\omega}{ds} + \frac{du}{ds} = o.$$

Now putting $$\bar{r} = r_\sigma\bar{\sigma} + r_\nu\bar{\nu},$$

then $$\frac{d\bar{r}}{ds} = \frac{dr_\sigma}{ds}\bar{\sigma} + \frac{dr_\nu}{ds}\bar{\nu} + \omega r_\sigma\bar{\nu} + r_\nu\frac{d\bar{\nu}}{ds}$$

and $$\bar{\sigma}\frac{d\bar{r}}{ds} = \frac{dr_\sigma}{ds} + r_\nu\bar{\sigma}\frac{d\bar{\nu}}{ds}$$

However, $$\frac{d\bar{\nu}}{ds} = -\omega\bar{\sigma},$$

and it follows that $$\bar{\sigma}\frac{d\bar{r}}{ds} = -\omega r_\nu + \frac{dr_\sigma}{ds},\text{ and } \sigma r = r_\nu\bar{\eta}.$$

From this it is seen that the equation $\frac{dr_\sigma}{ds} + P\bar{\omega}\frac{d\bar{\omega}}{ds} + \frac{du}{ds} = o.$

is integrable, so that

3) $$\tfrac{1}{2}P\omega^2 + r_\sigma + u = h^o.$$

This equation shows a certain analogy with the principle of the living force in kinetics.

*c*) For elastic bodies with no external forces acting, $u$ can be taken as zero. In this case, equation 3) takes the simplified form

3') $$\tfrac{1}{2}P\omega^2 + r_\sigma = h^o$$

The virtual work of bending is $R\delta\theta$. From this the equation

$\frac{d}{ds}(R\delta\theta) = \frac{dR}{ds}\delta\bar{\theta} + R\frac{d\delta\theta}{ds}$ is formed, or with the use of equation 2),

$$\frac{d}{ds}(R\delta\theta) = -r_\nu\delta\theta + R\delta\omega$$

Denoting the end points of the elastic curve by $A$ and $B$, the integration along the axis of the wire gives

$$\Big[R\delta\theta\Big]_A^B = \int_A^B (R\delta\omega - r_\nu\delta\theta)\,ds.$$

Setting the virtual distortion equal to zero at the boundaries $A$ and $B$ gives

$$\int_A^B (R\delta\omega - r_\nu\delta\theta)\, ds = o, \text{ or, since } r_\nu\delta\theta = dr_\sigma,$$

$$\delta\int_A^B (\tfrac{1}{2}P\omega^2 - r_\sigma)\, ds = o.$$

By equation 3′), however, $-r_\sigma = \frac{1}{2}P\omega^2 - h^o$. Hence it follows that

$$4) \qquad \delta\left[ \int_A^B P\omega^2\, ds - h^o \int_A^B ds = o. \right.$$

According to EULER's conception, the integral $\int_A^B P\omega^2\, ds$ is therefore a maximum-minimum with the isoperimetric condition $\int_A^B ds = l$ (constant).

From equation 1) follows $\overline{r} = \overline{r^o}$. Therefore $r_\sigma = r_x^o \cos\theta + r_y^o \sin\theta$, and $r_\nu = -r_x^o \sin\theta + r_y^o \cos\theta$.

Usually the axes $Ox$, $Oy$ are so selected that $r_y^o = o$.

*d*) In equation 3) the quantity $\frac{1}{2}P\omega^2 = e$, can be considered as an energy. The sum $u + r_\sigma = u'$ can be considered as a modified potential energy. Setting $e - u' = f$, and defining $f$ as the static LAGRANGIAN Function (in analogy to the kinetics of solid bodies), the static analogue to the LAGRANGIAN kinetic equations has the form $\frac{d}{d\tau}\frac{df}{d\omega} - \frac{df}{d\theta} = o$, and in the present case is identical with the equation $\frac{dR}{ds} + r_\nu + m = o$.

Thereby the analogy of KIRCHHOFF is presented. Further discussions on this analogy are found in LOVE " Elasticity " 2d ed., p. 382, and W. HESS, " Math. Ann." Vol. 25, 1885.

*e*) EULER gives the method of treating the isoperimetric problem in chapter 5 of " Methodus inveniendi," the German treatment of which, by P. STÄCKEL, is contained in Number 46 of " Ostwald's Klassiker der exakten Wissenschaften."

(K. HEUN).

*Note* 1, *page* 78.

DANIEL BERNOULLI pointed out the potential force to EULER in a letter dated October 20, 1742 (letter 26 in Vol. 2 of FUSS " Correspondance mathémat. et physique," Petersburg, 1843). He says at the close of this letter : " Since no one is so completely the master of the isoperimetric method (i.e., the calculus of variations, which EULER founded as a special branch of analysis) as you are, you will very easily solve the following problem in which it is required that $\int \frac{ds}{R^2}$ shall be a minimum." DANIEL BERNOULLI knew of EULER's " Method of finding Curves," together with the supplement on elastic curves, before its appearance; for he spoke about these with great interest in his letters to EULER in 1743. See section 63 and note 30.

*Note* 2, *page* 78.

For the special formulas employed here, compare Chapters II. and V. of EULER's "Methodus inveniendi lineas curvas etc." (Vol. 46 of "Ostwald's Klassiker der exakten Wissenschaften," edited by P. STÄCKEL.)

*Note* 3, *page* 81.

For this transformation of the co-ordinates, let the following facts be noticed: The new axes are again at right angles; the new $x$-axis forms an angle $\phi$ with the old $x$-axis defined by $\tan\phi = \frac{\gamma}{\beta}$. Putting $P = \frac{dY}{dX}$, this gives $p = \frac{\beta P - \gamma}{\beta + \gamma P}$, and therefore $1 + p^2 = \frac{(\beta^2 + \gamma^2)(1 + P^2)}{(\beta + \gamma P)^2}$

Substituting this value in EULER's last equation, namely

$$\frac{2\sqrt{a\sqrt{1 + p^2} + \beta p + \gamma}}{(1 + p^2)^{\frac{1}{4}}} = \beta x - \gamma y + \delta, \text{ gives the result}$$

$$\frac{2\sqrt{a\sqrt{1 + P^2} + P\sqrt{\beta^2 + \gamma^2}}}{(1 + P^2)^{\frac{1}{4}}} = X\sqrt{\beta^2 + \gamma^2}$$

Let $\beta_1 = \sqrt{\beta^2 + \gamma^2}$

Introducing lower case letters instead of capital letters gives

$$2\sqrt{a\sqrt{1 + p^2} + \beta_1 p} = \beta_1 x\,(1 + p^2)^{\frac{1}{4}}$$

Again, writing $\beta$ for $\beta_1$ this last equation reduces to the equation given in the text.

*Note* 4, *page* 82.

Put $n = \gamma$, $x = x_1 + \frac{\beta}{2\gamma}$, $m = \frac{1}{a^2}\left(\frac{\beta^2}{4} - a\gamma\right)$.

The quantities $a$, $\beta$, $\gamma$, used here are, of course, different from those given at the beginning of this section. This gives

$n^2x^2 - ma^2 = \gamma\,(a + \beta x_1 + \gamma x_1^2)$, and therefore

$$dy = \frac{dx_1\,(a + \beta x_1 + \gamma x_1^2)}{\sqrt{a^4 - (a + \beta x_1 + \gamma x_1^2)^2}}$$

Omitting the subscript on $x$ gives the next to the last equation in section 4 of the text.

*Note* 5, *page* 89.

Here EULER is thinking of NEWTON's famous enumeration of curves of the third order. In the following discussion, $AP$ (Fig. 6) is always the direction of the positive $x$-axis, and $AB$ the direction of the positive $y$-axis. $AB$ is also the direction of the external force. In section 5 the direction of the external force is parallel to the *negative* $y$-axis.

*Note* 6, *page* 91.

The shape of the curve in the neighborhood of $C$ can be also derived as $u = a\sqrt{\frac{t}{c}}$, so that $u^2 = a^2\frac{t}{c}$ represents a parabola. The curve, near $C$, is approximately a parabola. If $x$ is made very small in the original equation, then $dy = \frac{(a^2 - c^2)\,dx}{c\sqrt{2a^2 - c^2}}$, and therefore $y = \frac{a^2 - c^2}{c\sqrt{2a^2 - c^2}}\,x$, that is, the curve has the form of a straight line in the neighborhood of $A$. This also follows from the fact that $A$ is a point of flexion of the curve; for we have

$$\frac{d^2y}{dx^2} = 2x \frac{a^4}{\sqrt{(c^2 - x^2)^3\,(2a^2 - c^2 + x^2)^3}},$$ which vanishes for $x=0$.

The curve has flexion points for no other values of $x$.

*Note* 7, *page* 92.

The elastic curve has been treated in a few places, although not in detail, in "Methodus inveniendi lineas curvas" by EULER. (Vol. 46 of "Ostwald's Klassiker der exakten Wissenschaften," pp. 110, 111, 127, 131.) In chapter 5, par. 46, EULER demonstrates the important property, that of all the curves of the same length which all pass through the same two points, the elastic curve is that one which, when rotated about an axis, generates the solid of greatest volume. He also mentions there the relation $R = \frac{a^2}{2x}$: the radius of curvature is inversely proportional to the abscissa.

*Note* 8, *page* 92.

If $\frac{x}{c} = u$, and $\frac{c^2}{c^2 - 2a^2} = k^2$, then $s = \frac{a^2}{\sqrt{2a^2 - c^2}} \int_0 \frac{du}{\sqrt{(1-u^2)\,(1-k^2u^2)}}$ that is, according to the definition of LEGENDRE, $s$ is an elliptic function of the first order. By the same substitution, $y$ is transformed into

$$y = \sqrt{2a^2 - c^2} \int_0 \frac{\sqrt{1-k^2u^2}\,du}{\sqrt{1 - u^2}} - \frac{a^2}{\sqrt{2a^2 - c^2}} \int_0 \frac{du}{\sqrt{(1 - u^2)\,(1 - k^2u^2)}}$$

The first member is an elliptic integral of the second order, and the second member is another such integral of the first order. Therefore the integration for $s$ and $y$ cannot be put into a condensed form.

*Note* 9, *page* 93.

The notation of the limits of the definite integral has been added for brevity, although they are not found in EULER's work, the notation being first adopted by FOURIER in 1822, in the "Traité analytique de la chaleur."

*Note* 10, *page* 94.

$y = \frac{a}{\sqrt{2}} \text{arc sin} \frac{x}{c}$ gives $x = c \sin \frac{y\sqrt{2}}{a}$. In modern terminology, the curve represented by this equation is called a sinusoidal curve. By the term trochoid is now meant a curtate or prolate cycloid; the sine curve can be considered as a special case of a curtate cycloid.

*Note* 11, *page* 95.

With the aid of the Legendrian relation $KE' + K'E - KK' = \frac{1}{2}\pi$, the EULER relation $4bf = \pi a^2$ is easily derived. (For the necessary formulas on the elliptical integrals for this, compare for example, E. PASCAL "Repertorium der höheren Mathematik," German ed. by A. SCHEPP, p. 156.) We have

$$f = \int_0^a \frac{a^2 dx}{\sqrt{(a^2 - x^2)(a^2 + x^2)}}.$$

Putting $x = a \cos\phi$ gives

$$f = \frac{a}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}} K. \text{ Further,}$$

$$b = \int_0^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(a^2 + x^2)}} = \int_0^a \frac{dx \sqrt{a^2 + x^2}}{\sqrt{a^2 - x^2}} - \int_0^a \frac{a^2 dx}{\sqrt{(a^2 - x^2)(a^2 + x^2)}}.$$

Again, putting $x = a \cos\phi$, then

$$b = a\sqrt{2} \int_0^{\frac{\pi}{2}} d\varphi \sqrt{1 - \tfrac{1}{2}\sin^2\varphi} - \frac{a}{\sqrt{2}} K = \frac{a}{\sqrt{2}} (2E - K).$$

As the formulas show, the complete integrals $K$ and $E$ belong, in this case, to the modulus $k^2 = \frac{1}{2}$, and therefore $1 - k^2 = \frac{1}{2}$ also. Putting $1 - k^2$ for $k^2$ in $E$ and $K$ transforms them into $E'$ and $K'$. Here, therefore, $K = K'$, and $E = E'$. The above LEGENDRIAN relation gives as a result the equation: $KE' + K'E - KK' = 2KE - K^2 = K(2E - K) = \frac{1}{2}\pi$. But $bf = \frac{1}{2}a^2K(2E - K)$, and therefore $4bf = \pi a^2$. Another proof is found in TODHUNTER, "A History of the Theory of Elasticity," Cambridge, 1886, Vol. 1, p. 36.

*Note* 12, *page* 96.

In this calculation, the author has made an error. EULER puts $b = 1.1803206 \frac{a}{\sqrt{2}}$, instead of $b = \frac{a}{1.1803206 \sqrt{2}}$ (this has been corrected in the text).

It then follows that $\frac{b}{a} = 0.59896$, that is, approximately 0.6. From this follows the formula $f = \frac{5a}{6} . \frac{\pi}{2}$. In EULER's text is given the incorrect value $\frac{b}{a} = 0.834612$.

*Note* 13, *page* 97.

W. HESS treats the problem of elastic curves as an analogue to the oscillation of a pendulum, and gives a series of figures on the possible forms. (Mathem. Annalen 25, 1885).

The direction of the force in EULER's curves of the fifth class, that is, the perpendicular from $A$ on $AP$, (Fig. 8), forms, in EULER's work, the angle $90° + 40° 41' = 130° 41'$ with the curve. Hess gives 129.3°. He took this angle from LEGENDRE's Table of Elliptic Integrals; it consists of finding the value of the modulus $k^2$ for which $2E - K$ vanishes. H. LINSENBARTH, editor of Volume 175 of "Ostwald's Klassiker der exakten Wissenschaften," says, in Note 25, p. 117 of that work, "As the editor has convinced himself, this gives also in this manner the value 130° 41', and in addition the verification of the calculation from EULER's equation for $v$ shows that EULER calculated correctly, since otherwise the value 0.8261 would obtain for $v$, instead of EULER's value 0.8259."

*Note* 14, *page* 99.

Here the well known relation $\log_e n = \log_{10} n . \log_e 10$ has been employed. $\log_e 10 = 2.302585$. At the end of the section, $\angle QOM$ is set equal to $2\theta - 90^\circ$. To derive this, let $\angle QOM = \varphi$. Therefore $\tan\varphi =$

$$\frac{dy}{dx} = \frac{\frac{1}{2}c^2 - x^2}{x\sqrt{c^2 - x^2}}.$$ Setting $x = c\cos\theta$ gives $$\tan\varphi = \frac{1 - 2\cos^2\theta}{2\sin\theta\cos\theta} = -\cot 2\theta.$$

Therefore $\tan\varphi = \cot(180^\circ - 2\theta) = \tan(2\theta - 90^\circ)$. Hence $\varphi = 2\theta - 90^\circ$.

*Note* 15, *page* 100.

If $c$ is interchanged with $g$, the curve remains unchanged. Therefore the shape of the curve near $G$ must be similar to the shape near $C$, except that the curvature at $C$ is greater than at $G$. The perpendicular from $G$ to $Dd$ is therefore also a diameter of the curve, as is the perpendicular from $C$ to $Dd$.

*Note* 16, *page* 102.

Equation I is the equation at the end of section 5, if $x$ is replaced by $t$, $h$ by $c$, and if $mEk^2 = -Pf$. This relation is derived at the end of section 10. If the origin is translated from $A$ to $C$, then I goes into II. The normal equation (III) of the elastic curve does not change, since in it only $dy$, and not $y$ is moved when the x-axis is displaced parallel to itself. The origin will then be a suitable point of the straight line $AB$ (Fig. 6). In this figure, the external force acts in the direction $AB$, as is the case in II at the point $C$. The change of sign before the $dx$ in II is accounted for by the fact that in figure 3, section 5, the force acts in the direction of the negative y-axis, while in figure 6 it acts in the direction of the positive y-axis. (See note 5.) Therefore, since II and III are referred to the same co-ordinate axes, these equations can be brought into agreement.

*Note* 17, *page* 104.

In this calculation EULER has made an error (corrected in the text). He used only the first four terms of the series expansion given in the text, and obtained the following value for $f$:

$f = g - \frac{37g^5}{30c^4}$, which is less than $g$. From a glance at the figure, it is evident that $f$ is greater than $g$. EULER seems to have overlooked this fact. The value for $h$ is the same in both cases. The detailed calculation given in the text is not given in the original.

*Note* 18, *page* 106.

Here EULER refers to the formulas given in his " Methodus inveniendi lineas curvas," Chapter IV, sec. 7, II, p. 132.

*Note* 19, *page* 108.

This relationship can be proved in detail as follows : Let any straight line $CP$, (Fig. 3) whose equation is $Ax + By + C_1 = o$, be the axis. Then the moment of the force $P$ at the point $M$ (corresponing to the developments in sec. 5) equals

*P.CP*. Let the point *M* have, for the time being, the co-ordinates $\xi$ and $\eta$, giving the equation of *MP* as $A(y-\eta) - B(x-\xi) = o$. If the point *C* be given the co-ordinates $x = k$, $y = l$, then $Ak + Bl + C_1 = o$, and the length of the perpendicular from *C* to *MP*, i.e., *CP*, is given by $\frac{A(l-\eta) - B(k-\xi)}{\sqrt{A^2+B^2}}$. The moment *P.CP* is, if $\xi$ and $\eta$ be again replaced by $x$ and $y$, $\frac{P(Al - Bk)}{\sqrt{A^2+B^2}} + \frac{BP}{\sqrt{A^2+B^2}}x - \frac{AP}{\sqrt{A^2+B^2}}y$, corresponding to $\alpha + \beta x - \gamma y$ in the text.

*Note* 20, *page* 112.

The monograph of JOHN BERNOULLI "De Motu Reptorio" is found in the *Acta Erudit.*, Aug. 1705 (Works I, p. 408.)

*Note* 21, *page* 113.

EULER himself further exercised his powers on both of these definite integrals, for he says in Vol. 4 of the "Institutiones Calculi Integralis" (Petersburg, 1794) on p. 339 : "I recently found by a happy chance, with the aid of a quite unique method, that (see also sec. 54) $\int_o^\infty \frac{d\theta \cos\theta}{\sqrt{\theta}} = \frac{\pi}{2}$, and also $\int_o^\infty \frac{d\theta \sin\theta}{\sqrt{\theta}} = \frac{\pi}{2}$." The curve analyzed here is the one whose natural equation is $rs = a^2$.

This has been called the Klothide by CESARO. Particulars on this, and figures, which appear also in the above mentioned work by EULER, are found in LORIA "Spezielle algebraische und transzendente Kurven der Ebene," German ed. by F. SCHÜTTE, Leipzig, 1902, p. 458.

*Note* 22, *page* 114.

Namely, consider the interval from zero to infinity to be divided up into the following parts : 1. from 0 to $\pi$, 2. from $\pi$ to $2\pi$, 3. from $2\pi$ to $3\pi$, etc. Then

$$x = \frac{b}{2}\left[\int_o^\pi \frac{dv \sin v}{\sqrt{v}} + \int_\pi^{2\pi} \frac{dv \sin v}{\sqrt{v}} + \int_{2\pi}^{3\pi} \frac{dv \sin v}{\sqrt{v}} + \ldots\ldots\right].$$

Putting $v = v_1 + \pi$ in the second integral, $v = v_2 + 2\pi$ in the third integral, etc., gives

$$x = \frac{b}{2}\left[\int_o^\pi \frac{dv \sin v}{\sqrt{v}} - \int_o^\pi \frac{dv_1 \sin v_1}{\sqrt{v_1+\pi}} + \int_o^\pi \frac{dv_2 \sin v_2}{\sqrt{v_2+2\pi}} - \ldots\ldots\right].$$

Since the relation between the variables is immaterial in the definite integral, $v$ can be put equal to $v = v_1 = v_2 = v_3 = \ldots$ Similarly the value for $y$ may be obtained.

*Note* 23, *page* 127.

The Latin text reads "...; indeque auferendo logarithmum sinus totius,...," or "and subtracting from it the logarithm of the entire sine," instead of "and subtracting ten from it" as given in the text. The original is apparently a misprint of some nature. The reading in the text conforms to the method of procedure as given in sec. 31.

*Note* 24, *page* 128.

Let $\varphi$ transformed into seconds give the number $\beta$. Then, in radians, $\phi=\beta \frac{\pi}{180.60^2}$. Therefore $\log \phi = \log \beta - \log \frac{180.60^2}{\pi}$. This last logarithm is 5.3144... This explains the appearance of this number in sections 73, 74, and 85. In order to get $\log_e \cot\frac{1}{2}\phi$ EULER used the known formula $\log_e \cot\frac{1}{2}\phi = \frac{1}{M}$ $\log_{10} \cot\frac{1}{2}\phi$. To carry out the multiplication on the right side, he again used common logarithms. $\text{Log}_{10} \frac{1}{M} = 0.362215...$ From this the appearance of this number in the second part of the tabular calculations is understood. At the end of the calculations, EULER used the reguli falsi, as also in sec. 85.

*Note* 25, *page* 129.

The formula $\frac{A}{B} = \tan\frac{1}{2}\varphi$, which has not as yet been derived, is easily found in the following manner : Adding the first two formulas of sec. 70, which contain only $A$ and $B$, gives

$2Ae^{\frac{a}{c}} - 2B\sin\frac{a}{c} + 2A\cos\frac{a}{c} = o$, or, since $\sin\frac{a}{c} = \cos\varphi$, and

$\cos\frac{a}{c} = -\sin\varphi$, and $e^{\frac{a}{c}} = \cot\frac{1}{2}\varphi$,

$$\cos\varphi = \frac{A}{B}(\cot\tfrac{1}{2}\varphi - \sin\varphi) = \frac{A}{B}\cot\tfrac{1}{2}\varphi\,(1 - 2\sin^2\tfrac{1}{2}\varphi)$$

or $\cos\varphi = \frac{A}{B}\cot\frac{1}{2}\varphi\ \cos\varphi$. Therefore $\frac{A}{B} = \tan\frac{1}{2}\varphi$.

*Note* 26, *page* 131.

Here EULER refers to the difference between the oscillations of bodies which are elastic due to stretching—a taut chord, corda elastica, and those which are elastic due to stiffness—an elastic ribbon, lamina elastica.

*Note* 27, *page* 133.

If the lower tone is $C$, then the higher tone is slightly lower than $G$ sharp. If $C$ has the frequency $N$, then $G$ sharp has the frequency $\frac{25}{4} N$ (that is, $6\frac{4}{16}$ instead of $6\frac{4}{15}$ as according to EULER's calculations). See the note on section 79.

*Note* 28, *page* 133.

Approximately $e^{\frac{5\pi}{2}}(1+\varphi) = \frac{2}{\varphi}$. Therefore $e^{\frac{5\pi}{2}} + \phi\, e^{\frac{5\pi}{2}} = \frac{2}{\phi}$.

By the first approximation in the text, however, $\phi\, e^{\frac{5\pi}{2}} = 2$.

Therefore $\frac{2}{\phi} = e^{\frac{5\pi}{2}} + 2$, i. e., $\phi = \frac{1}{1 + \frac{1}{2}e^{\frac{5\pi}{2}}}$

Letting $v = \frac{1}{a^2}\sqrt{gEk^2 \frac{a}{M}}$ , then to the various modes of vibration correspond the tones having the frequencies

$1.815^2\ \nu,\ 4.69^2\ \nu,\ ,\ \left(\frac{5\pi}{2}\right)^2 \nu,\ \left(\frac{7\pi}{2}\right)^2 \nu, \ldots$ . All these tones have been sought out experimentally by CHLADNI, ("Akustik," Leipzig 1802, pp. 94-103). They are, as he also found for the following cases, in the best agreement with EULER's results.(See note 30.)

*Note* 29, *page* 136.

The case $\frac{a}{c} = o$, which appears frequently in the following sections, is derived as follows: Since $a$ is not zero, then $c = \infty$. That is, since $c^4 = \frac{Ek^2\ af,}{M}$, $f$ must equal $\infty$. The corresponding isochronous pendulum is infinitely long, and the time for one oscillation is infinite. An infinitely long time is needed to produce an oscillation in the ribbon, i.e., it remains at rest.

*Note* 30, *page* 137.

Only the case of the oscillation with one knot point, represented in Fig. 23, is excluded in free elastic ribbons, not the others. DANIEL BERNOULLI expressed his astonishment at EULER's error in a letter of Sept. 4, 1743. (Letter 30 in the " Correspondance math. et physique," by FUSS). " These oscillations arise freely, and I have determined various conditions, and have performed a great many beautiful experiments on the position of the knot points and the pitch of the tone, which agree beautifully with the theory. I hesitated whether I should not strike out from the supplement the few words which you say about the matter." In the " Acta Acad. Petrop. " 1779, Part 1, p. 103, EULER again takes up the investigation of oscillating ribbons from other points of view, and admits of oscillations with an uneven number of knot points without more ado. There he also treats six classes of oscillations, with consideration of the ends of the ribbon —whether free, fixed on supports, or set in a wall—while here he treats only four classes. LORD RAYLEIGH, in Chapter VIII of the " Theorie des Schalles " (German ed. by Fr. NEESEN, Braunschweig 1879), gives a comprehensive presentation of the transversal oscillations of elastic ribbons. In that very place, for instance, he considers the free oscillations with three knot points. See also STREHLKE, " Poggendorf's Annalen " vol. 27, and A. SEEBECK, " Abhandl. d. Kgl. Sächs. Gesellschaft der Wissensch." 1852.

*Note* 31, *page* 139.

The deviation in the left table is incorrect. It should read — 2621559, namely 4.73003... — 4.73029... The angle $\underline{\phi}$, then, does lie between $1^\circ$-0′-40″ and $1^\circ$-1′-0″. Letting $\phi = 1^\circ\text{-}0'\text{-}40'' + n$, the rule of false values gives $\frac{n''}{20''} = \frac{2621559}{53145574 + 2621559}$. This gives $n = 0.9402$, and hence $\phi = 1^\circ\text{-}0'\text{-}40.9402''$. This value appears, for example, in RAYLEIGH's " Theorie des Schalles," vol. 1, p. 298 ff. Then $\frac{a}{c} = 4.7300408$ for the correct value. Since the mistake has no influence until the fifth decimal place, the further numerical values are correct to that place. In this and the following section, the correct values are inserted.

*Note* 32, *page* 141.

EULER had, incorrectly, $e^{\frac{a}{2c}} + e^{-\frac{a}{2c}} = + \frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}$. Since

$\frac{a}{c} = 270^\circ\text{-}0'\text{-}40.94''$, then $\cos\frac{a}{2c} = \cos 135^\circ\text{-}0'\text{-}20.47''$ is necessarily negative. Therefore the right side of the formula has a positive value, as must be true since $e^{\frac{a}{2c}} + e^{-\frac{a}{2c}}$ is positive. The formula can be derived as follows:

$$e^{\frac{a}{2c}} + e^{-\frac{a}{2c}} = \sqrt{e^{\frac{a}{c}}} + \sqrt{e^{-\frac{a}{c}}} = \frac{(1 - \sin\frac{a}{c}) + \cos\frac{a}{c}}{\sqrt{\cos\frac{a}{c}\left(1 - \sin\frac{a}{c}\right)}}.$$

This value has already been used to calculate $\frac{1}{b}(A\, e^{\frac{x}{c}} + Be^{-\frac{x}{c}})$. Introducing half angles,

$$e^{\frac{a}{2c}} + e^{-\frac{a}{2c}} = \frac{2\cos^2\left(\frac{a}{2c}\right) - 2\cos\frac{a}{2c}\sin\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}\sqrt{\sin^2\frac{a}{2c} + \cos^2\frac{a}{2c} - 2\sin\frac{a}{2c}\cos\frac{a}{2c}}}$$

$$= \frac{2\cos\frac{a}{2c}\left(\cos\frac{a}{2c} - \sin\frac{a}{2c}\right)}{\sqrt{\cos\frac{a}{c}\left(\sin\frac{a}{2c} - \cos\frac{a}{2c}\right)}} = - \frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}.$$ For the second root in the denominator, $\sin\frac{a}{2c} - \cos\frac{a}{2c}$ is used, since this value is positive.

The formulas of sec. 89 are changed from EULER's text to the correct values derived here. The last formula is correct in the text, and hence the numerical values are correct. $\frac{Cc}{Aa}$ is negative, since in Fig. 22 these ordinates have opposite directions.

*Note* 33, *page* 145.

It should be noticed about the tone intervals under consideration that: 1) the first interval, $\frac{160}{57}$ is, in the C-major scale, the interval from the base note *C* to *F* sharp of the next higher octave (2.78 instead of 2.81); 2) the second interval, 4:1, is that of the base tone *C* to the *C* two octaves higher; 3) the third interval, 9:1, reaches from *C* as the base note to the note *D* which lies three octaves higher. In connection with the practical working out of these oscillations, which EULER considered difficult, see CHLADNI, "Akustik," p. 99.

*Note* 34, *page* 147.

This agreement takes place for all modes of vibration. EULER's differing statement comes from the fact that he rejects, for free ribbons, the oscillations for an uneven number of knots. See note 30.

*Note* 35, *page* 148.

These experiments, which are very important for technique, have been performed in a most fruitful fashion. However, the formulas given here do not lead to useful results, since they do not take into consideration the cross section of the elastic ribbon. See, for example, KUPFFER, "Recherches expérimentales sur l'élasticité des métaux." St. Pétersbourg, 1860.