INTRODUCTION

- <u>What is active matter?</u>

  collection of interacting active particles, each ~~generating~~ self-driven and capable of converting stored energy in motion/forces, and collectively generating coordinated motion

- <u>How does it differ from other noneq. system?</u>

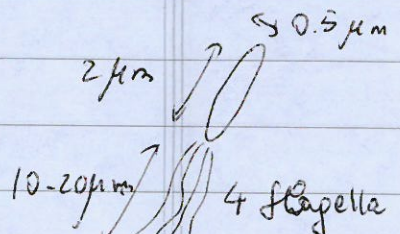
  drive is local on each unit, not global (field) or at boundary → reverse energy cascade

- SP motion is "force free" (≠ sedimentation under gravity)

⟹ emergent behavior
(not mechanisms of motility)

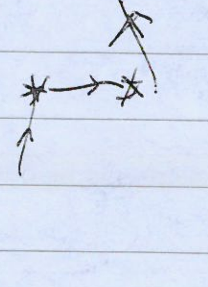
EXAMPLE OF ACTIVE PARTICLE

E. coli rod-shaped
1% gut flora



converts chemical energy into motion via an internal <u>cyclic</u> transformation

run-and-tumble $v_0 \sim 10-40\ \mu m/s$
not a conventional RW $\alpha \sim 1\ s^{-1}$



<u>MANY E. coli</u> : swarming, turbulent flow, pattern formation, biofilms

- not motility mechanisms, but collective behavior
- time >> cycle (but some recent work on activity and synchronization)

EXAMPLES ON MANY SCALES → {movies}

- inside a cell : cytoskeleton → cell motility, division, mechanics (Joanny)
- many cells → tissues : mechanics, collective migration, wound healing (Manning)
- fish, birds, people (Toner)
- synthetic ~~[illegible]~~ microswimmer

What do they have in common?

- drive on each unit / symmetry broken locally, breaks TRS
- emergent behavior, order/disorder transition
- [liquid xstalline order] → living LC

{GOALS} Use methods from noneq. stat mech + Soft CM

- Which new states of active matter are possible?
- Can we classify behaviors and identify generic properties?
- What do we tune to change from one state to another?

MCM et al, RMP 85, 1143 (2013)

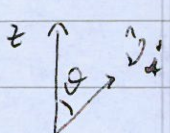
CLASSIFICATION

- Types of orientational order

POLAR : bacteria, fish, …

ferromagnetic order $\langle \vec{v} \rangle \neq 0$
moving state

O.P. vector
velocity / polarization

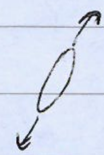


$$\vec{P} = \sum_i \hat{\nu}_i \delta(\vec{r} - \vec{r}_i)$$

$P_z \sim \langle \cos\vartheta_i \rangle$

→ TONER

APOLAR : melanocytes, rods, …



O.P. tensor

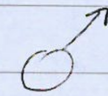
$$Q_{\alpha\beta} = \left\langle \sum_i \left(\hat{\nu}_{\alpha i} \hat{\nu}_{\beta i} - \frac{1}{2}\delta_{\alpha\beta}\right) \times \delta(r - r_i) \right\rangle$$

$Q_{zz} \sim \langle \cos^2\vartheta_i \rangle$

no state with mean motion

→ Dogic

SPHERICAL : active colloids



No orientational order, but surprising collective behavior

→ MCM

- Role of medium : "dry" vs "wet"

- Forces on environment : contractile vs extensile

contractile	extensile
puller	pusher
→ ←	← →

VICSEK MODEL OF FLOCKING 1995 Craig Reynolds 1987

inspired by analogy with ferromagnetism

→ flying XY spins

- N point particles
- fixed speed v_0
- align w/neighbors with noisy rules
- overdamped dynamics

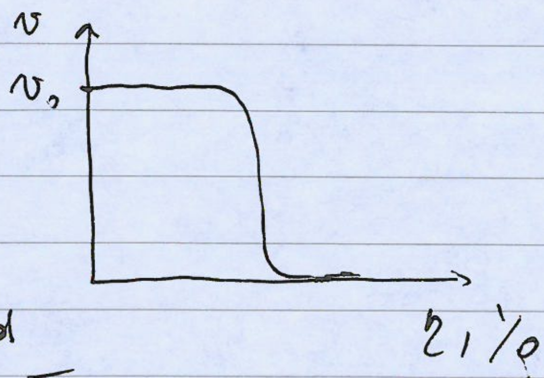
$v_0 \hat{e}_i$, θ_i

$$\hat{e}_i = (\cos\theta_i, \sin\theta_i)$$

$$\begin{cases} \vec{r}_i(t+\Delta t) = \vec{r}_i(t) + v_0 \hat{e}_i \Delta t \\ \theta_i(t+\Delta t) = \langle \theta_i(t) \rangle_R + \eta_i(t) \end{cases}$$

η_i random # uniform in $[-\eta/2, \eta/2]$

OP $\quad v = \left| \frac{1}{N} \sum_i \vec{v}_i(t) \right|$



- first order
- spontaneous breakup of continuous symmetry in 2d ($\neq$ Mermin-Wagner) ⟶ Toner

Agent or rule-based model ⟶ demo

→ continuum : Toner-Tu model

separation of time scales :

- most fluctuations decay on microscopic time scales
- some are 'slow' : $\underbrace{\omega(k)}_{\text{decay rate}} \to 0 \quad k \to \infty$

TUTORIAL : from Langevin dynamics to Smoluchowski

Brownian particle

$$m \frac{d\vec{v}}{dt} = \underbrace{- \zeta \vec{v}}_{\text{mean drag}} + \underline{\vec{\eta}(t)}$$

random component of effect of collisions with fluid atoms

$\zeta = 6\pi\eta a$ (3d)

$\langle \vec{\eta}(t) \rangle = 0$

time scale $\quad \tau = m/\zeta$

$\langle \eta_\alpha(t)\eta_\beta(t') \rangle = 2\Delta\,\delta_{\alpha\beta}\,\delta(t-t')$

Gaussian, white

$$\langle |\vec{v}(t)|^2 \rangle \xrightarrow[t \gg m/\zeta]{} \frac{\Delta}{\zeta m} d = \langle v^2 \rangle_{th} = \frac{d K_B T}{m}$$

equilibrium

$\boxed{\Delta = K_B T \zeta}$ FD theorem

$$\langle [\Delta \vec{r}(t)]^2 \rangle = 2d\frac{K_B T}{\zeta}\left\{ t - \tau\left(1 - e^{-t/\tau}\right)\right\}$$

balance of dissipation and noise

$t \ll \tau \qquad \langle [\Delta \vec{r}(t)]^2 \rangle = \frac{d K_B T}{m} t^2 \qquad$ ballistic

$t \gg \tau \qquad \langle [\Delta \vec{r}(t)]^2 \rangle = 2Ddt \qquad$ diffusive

$$D = \frac{K_B T}{\zeta} \qquad \text{Einstein}$$

Many particle, interactions : Langevin dynamics hard, not well suited to analytics

$\rightarrow$ eq. for probability distribution

Overdamped dynamics $\quad t \gg m/\zeta$

$\zeta \vec{v} = \vec{\eta}(t)$

$\langle [\Delta \vec{r}(t)]^2 \rangle = 2Ddt \qquad$ all times

Derive Eq. for noise-averaged probability distribution

- overdamped dynamics
- 1 d

$$v = \frac{dx}{dt} = -\frac{1}{\zeta} U'(x) + \eta(t)$$

$\langle \eta(t) \rangle = 0$

$\langle \eta(t) \eta(t') \rangle = 2\Delta \delta(t-t')$

Gaussian, white but no FD

$\hat{\varphi}(x,t)$ probability density

all t $\int_V dx\, \hat{\varphi}(x,t) = 1$ conservation law $\Rightarrow$ $\partial_t \hat{\varphi}$ is the divergence of a flux $J = \hat{\varphi} v$ (cf fluid dynamics)

$\downarrow$

$$\partial_t \hat{\varphi} = -\partial_x v \hat{\varphi}$$

$$\partial_t \hat{\varphi} = \overbrace{-\partial_x \left[-\frac{1}{\zeta} U' \hat{\varphi} \right]}^{L\hat{\varphi}} - \partial_x (\eta \hat{\varphi})$$

We want an eq. for $\varphi = \langle \hat{\varphi} \rangle$

$$\hat{\varphi}(t) = e^{-\hat{L}t} \hat{\varphi}(0) - \int_0^t ds\, e^{-\hat{L}(t-s)} \partial_x (\eta(s) \hat{\varphi}(s))$$

$\hat{\varphi}(t)$ only depends on noise at time $s < t$

$$\partial_t \hat{\varphi} = -L\hat{\varphi} - \partial_x \eta(t) e^{-Lt} \hat{\varphi}(0)$$

$$+ \partial_x \int_0^t ds\, \eta(t) e^{-L(t-s)} \partial_x (\eta(s) \hat{\varphi}(s))$$

noise average

- assume $\hat{\varphi}(0)$ does not depend on noise $\langle \eta(t) \hat{\varphi}(0) \rangle = 0$
- $\langle \eta(t) \eta(s) \hat{\varphi}(s) \rangle$ Gaussian : only terms even in $\eta \neq 0$ Wick's theorem

$$\langle \eta(t)\,\eta(s)\,\psi(s)\rangle \;\begin{cases} \langle \eta(t)\eta(s)\rangle\langle\psi(s)\rangle \sim \delta(t-s) \\ \langle \eta(t)\eta(s')\rangle \sim \delta(t-s') \end{cases}$$

↑ from $\psi(s) \Rightarrow s' < s$

but $t > s > s'$ $\quad t=s'$ vanishes

Smoluchowski equation

$$\partial_t \psi = -\partial_x \Big[\underbrace{-\frac{1}{\zeta} U'}_{\text{force}} \psi - \Delta \partial_x \psi \Big]$$

when FD holds $\Delta = D$

$\boxed{\text{Note: noise was \underline{not} assumed to be small}}$

<u>Many interacting particles</u>: hierarchy of Smoluchowski eqns. for $\psi_s(\vec r_1, \dots, \vec r_s, t)$

$\mu = 1/\zeta$

$$\partial_t \psi_1 = \vec\nabla_1 \cdot \Delta \vec\nabla_1 \psi_1 - \vec\nabla_1 \cdot \Big[\underbrace{\mu \int d\vec r_2 \left(-\vec\nabla_1 U(\vec r_{12})\right) \psi_2(\vec r_1, \vec r_2, t)}_{\text{interaction force density}} \Big]$$

ψ_2 couples to ψ_3

etc.

molecular chaos $\qquad \psi_2 = \psi_1 \psi_1$

$$\boxed{\partial_t \psi_1 = \Delta \nabla_1^2 \psi_1 - \mu \vec\nabla_1 \cdot \int d\vec r_2 \left(-\vec\nabla_1 U(\vec r_{12})\right) \psi_1(r_2)\, \psi_1(r_1)}$$

Non-Gaussian noise $\Rightarrow$ higher order gradient terms

MICROSCOPIC → HYDRODYNAMICS

Describe large scale dynamics in terms of a small number of continuum fields that are 'slow' → field theory

- conserved densities $\omega(k) \to 0 \quad \lambda \to \infty$
- broken symmetry fields

3 ways of constructing dynamical eqs:

- phenomenological (symmetry) → TONER
- entropy production (near eq.) → JOANNY
- derive by coarse graining microscopic dynamics

EXAMPLE : Vicsek model (continuous time)*
→ Toner-Tu eqs.

$$\frac{d\vec{r}_i}{dt} = v_0 \hat{e}_i$$

neglect translational thermal noise

$$\frac{d\theta_i}{dt} = \gamma \sum_j \underbrace{F(\theta_j - \theta_i, \vec{r}_j - \vec{r}_i)}_{} + \sqrt{2D_r}\, \eta_i(t)$$

$\eta_i \in [-1/2, 1/2]$

$= \sin(\theta_j - \theta_i)/\pi R^2 \qquad r_{ij} < R$

$= 0 \qquad$ otherwise

Now $\psi = \psi(\vec{r}, \theta, t)$

$$\partial_t \psi = -\vec{\nabla}\cdot(v_0 \hat{e}\psi) - D_r \partial_\theta^2 \psi$$

$$-\gamma \partial_\theta \int d\theta' \int d\vec{r}\,' \underbrace{F(\theta' - \theta, \vec{r}\,' - \vec{r})}_{\sin(\theta' - \theta)\,\delta(\vec{r} - \vec{r}\,')} \psi(\vec{r}\,', \theta')\psi(\vec{r}, \theta)$$

* E. Bertin, M. Droz, G. Gregoire, J Phys ~~Math~~ A: Math Th. 42, 445001 (2009)
S. Mishra, A. Baskaran, MCM, PRE 81, 061916 (2010) (banding)

Continuum fields such as density etc. as moments of ψ

2d.

$$\rho(\vec{r},t) = \int \frac{d\hat{e}}{2\pi}\,\psi \qquad\qquad \psi_k(\vec{r},t) = \int \frac{d\theta}{2\pi}\, e^{ik\theta}\,\psi$$

$$\rho\,\vec{P} = \int \frac{d\hat{e}}{2\pi}\,\hat{e}\,\psi \qquad\qquad \psi_0 = \rho$$

$$\psi_1 = \rho(P_x + iP_y)$$

$$\rho\, Q_{\alpha\beta} = \int \frac{d\hat{e}}{2\pi}\left(\hat{e}_\alpha \hat{e}_\beta - \frac{1}{2}\delta_{\alpha\beta}\right)\psi \qquad\qquad \psi_2 = \rho(Q_{xx} + iQ_{xy})$$

$\dots \qquad\qquad \dots$

Eqn for $\psi(\vec{r},t) \longrightarrow$ infinite set of Eqs. $\{\psi_k\}$

$$\partial_t \psi_k + \frac{v_0}{2}\partial_x(\psi_{k+1} - \psi_{k-1}) + \frac{v_0}{2i}\partial_y(\psi_{k+1} - \psi_{k-1})$$

$$= \underbrace{-k^2 D_r \psi_k}_{\text{decay}} + \frac{i\gamma k}{2\pi}\sum_s \psi_s F_{-s}\psi_{k-s}$$

$$-\frac{i}{2}\left(\psi_1^* \psi_{k+1} - \psi_1 \psi_{k-1}\right)$$

need closure! $\qquad F_1 = -i/2 \quad F_{-1} = F_1^* \quad$ others $\emptyset$

$$\partial_t \psi_0 + \frac{v_0}{2}(\partial_x - i\partial_y)\psi_1 = 0 \longrightarrow \partial_t \rho = -\vec{\nabla}\cdot(v_0 \rho \vec{P})$$

$\psi_0 \sim \rho$ conserved field ✓

$\psi_1 \sim \vec{P}$ order parameter ✓

Assume $\psi_k = 0 \quad k \geq 3$

$\dot{\psi}_2 = 0$

Can be made systematic as an expansion in a small parameter ε near MF transition where

$\rho - \rho_0 \sim \varepsilon \qquad \psi_1 \sim P \sim \varepsilon \qquad \partial_t \sim \nabla \sim \varepsilon \qquad f_2 \sim \varepsilon^2$

[see Peshkov et al., EPJ Special Topics 223 1315 (2014)]

Toner-Tu Eqs of flocking:

$\boxed{\vec{\nabla}(\text{pressure})}$

$$\partial_t \rho = -\vec{\nabla}\cdot(v_0 \rho \vec{p})$$

$$\partial_t \vec{p} + \lambda_1 (\vec{p}\cdot\vec{\nabla})\vec{p} = -\left[\alpha(\rho) + \beta p^2\right]\vec{p} - \frac{v_0}{2}\vec{\nabla}\rho$$

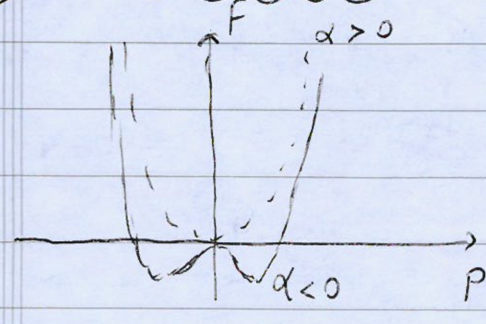
$$+ \frac{\lambda_3}{2}\vec{\nabla}p^2 + \lambda_2 \vec{p}(\vec{\nabla}\cdot\vec{p}) + K_3 \nabla^2 \vec{p} + (K_1 - K_3)\vec{\nabla}(\vec{\nabla}\cdot\vec{p})$$

all parameters given in terms of D_r, γ, v_0, ρ_0

$$\alpha(\rho) = D_r - \frac{1}{2\pi}\gamma\rho$$

$$\beta = \gamma^2 \rho^2 / 32 D_r \pi^2$$

Dual role of $\vec{p}$



nonlinear friction

$$-\frac{1}{\gamma}\frac{\delta F}{\delta \vec{p}}$$

$$F = \frac{1}{\gamma}\int_{\vec{r}} \left\{ \frac{\alpha}{2} p^2 + \beta/_4\, p^4 \right\}$$

$\lambda_1 = v_0$ if galileian invariance

substrate $\rightarrow$ non universal paramete

Disordered $|\vec{p}| = 0$ $\longrightarrow$ ordered $p_0^2 = -\alpha/\beta$

$\uparrow$

$$\boxed{\begin{array}{c}\alpha = 0 \\ \rho_c = 2D_r\pi/\gamma\end{array}}$$

Banding instability

ordered phase linearly unstable for $\alpha \to 0^-$ $\rho_0 \to 0^+$

$|\delta \vec{p}|$ decay at rate $|\alpha_0| \to 0$

$\delta\rho$, $|\delta\vec{p}|$ fluctuations unstable along $\vec{p}$

at a wavelength $\lambda_c \sim (\rho_c - \rho_0)^{-3/2}$ (see Refs. on p. 5)

$\Rightarrow$ bands, observed ubiquitously

Fluctuations about ordered state

$\rho = \rho_0 + \delta\rho$

$\vec{p} = p_0 \hat{x} + \delta\vec{p}$ $\qquad\qquad \vec{p} = p\,\hat{p}$

$\qquad\qquad\qquad\qquad \delta\vec{p} = \delta p\,\hat{x} + p_0\,\delta\hat{p}_y$

spatial variations along x

$\delta\rho$, δp decouple $\sim e^{iqx}$

$$\partial_t \delta\rho = -iq v_0 \rho_0 \delta p - iq v_0 p_0 \delta\rho$$

$$\partial_t \delta p = a\,\delta\rho - 2|\alpha_0|\,\delta p - \frac{v_0}{2} iq\,\delta\rho$$

generic instability

propagating wave $\qquad \delta\rho, \delta p \sim e^{st}$

$$\mathrm{Re}\, s = -s_2 q^2 - s_4 q^4 \qquad a \sim -\frac{\partial\alpha}{\partial\rho}$$

$$s_2 = \frac{v_0^2}{2|\alpha_0|}\left[1 - \frac{v_0 a^2 \rho_0^2}{4 v_0 |\alpha_0| \beta}\right]$$

← unstable even for small v_0 near MFT

no noise : propagating solitary waves as bands

→ seen in simulation and experiments.

Two comments:

1) Method can be used to derive eqs for model with other symmetry, and to include flow

- SP rods w/ steric repulsion Baskaran + MCM PRL 2008
- dry "Vicsek nematic" Bertin et al, NJP 2013
- polar and nematic w/ flow, Liverpool + MCM 2008 in "Cell motility", P. Lenz ed.
- active polymers: Ahmadi, MCM, Liverpool, PRE 2006

3) Eq. for one-particle distribution with ~~no~~ noise:
David Dean, J. Phys. A: Math Gen 29, L613 (1996)
white, Gaussian noise in microscopic dynamics
→ multiplicative noise in coarse-grained dynamics.