Lecture 3

- Review yesterday
- Wrinkles in 2D geometries
- Crumples – quick overview
- Large deformation - wrapping

Last time recap

1D folds - progress / puzzles

2D wrinkles

Setup for 2D axisymmetric problems

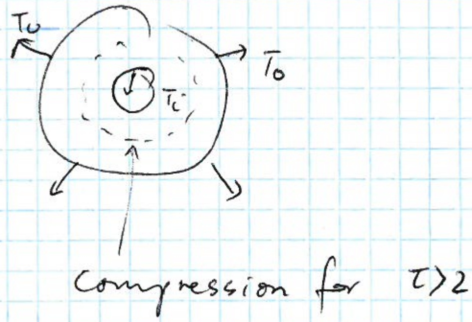
Two in-plane FvK equations

radial σ_{rr} $\partial_i \sigma_{ij} = 0$
azimuthal $\sigma_{\theta\theta}$

Normal FvK

$$B \Delta^2 \zeta + \sigma_{ij} \varkappa_{ij} = f_{normal}$$

Solved a planar problem



Found solutions that have compressive hoop stress ($\sigma_{\theta\theta} < 0$) for some

(1) values of confinement, $\tau > 2$

(2) A second parameter prescribes the buckling threshold

Bendability, $\boxed{\epsilon^{-1} = \frac{R_i^2 T_o}{B}}$

Some comments on bendability:

$$\epsilon = \frac{B \, T}{W^2}$$

(0) Unlike vK number, bendability includes mechanical informat

(1) $$\epsilon^{-1} = \frac{W^2 T}{B} \to \frac{T}{Y} \left(\frac{W}{t}\right)^2$$

$$= \text{Characteristic strain} \times \cancel{F}_v K$$

vK for thin sheets $\gg 1$

$\frac{T}{Y}$ for linear elasticity typically small.

Thus ϵ^{-1} for a plate can be either large or small.

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(2) Small ϵ^{-1} and large ϵ^{-1} represent two limits of FvK equations.

$$B \nabla^4 \zeta - \sigma_{ij} \partial_i \partial_j \zeta = f_{normal}.$$

Bendability $\epsilon^{-1} \sim \frac{T \zeta / W^2}{B \zeta / W^4} \sim \frac{T W^2}{B}$.

For large ϵ, bending resists forces

small ϵ, in-plane force resist normal forces

NT - Near threshold for annulus problem

Get L_{NT} $(\epsilon = 0)$ by solving the planar Lamé problem (only uses in-plane FvK eqns).

Take the normal FvK equation

① plug in $\sigma_{\theta\theta}$ and σ_{rr} from Lamé

② assume wrinkles are $\zeta(r,\theta) = f(r)\cos(m\theta)$

where f is small

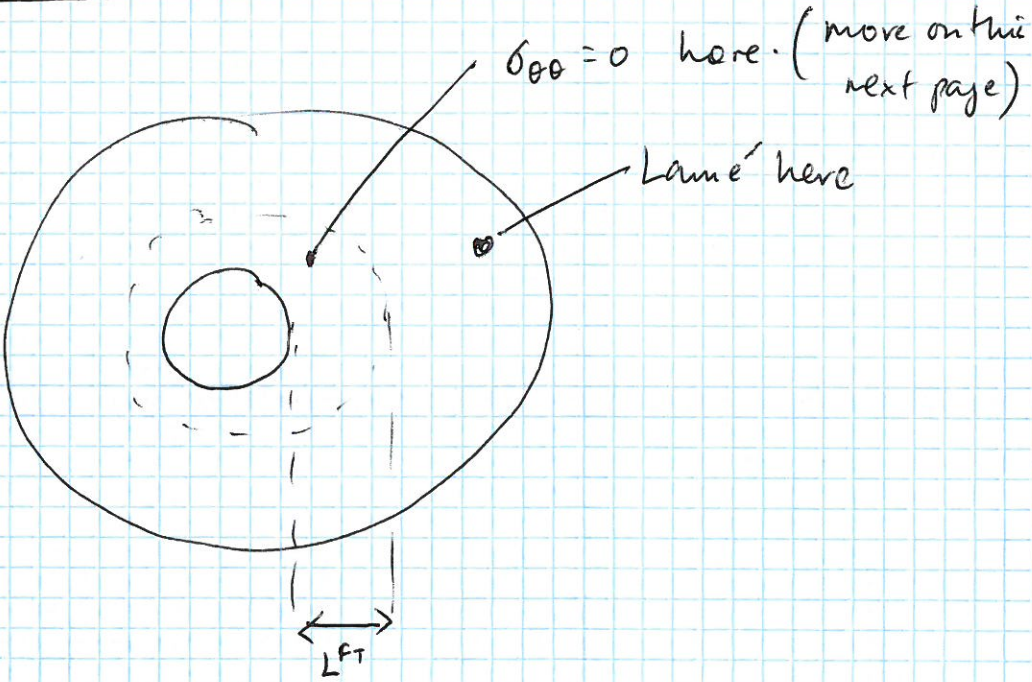
③ Solve, or extract scaling by balancing terms of bending ($\cancel{\Delta}^2 \zeta$) and stretching ($\sigma \nabla^2 \zeta$)

Get $\tau_c(\epsilon) \cancel{=} 2 \sim \epsilon^{1/4}$

$L_{NT}/R_i \cancel{=} 1 \sim \epsilon^{1/4}$

$m(\epsilon) \sim \epsilon^{-3/8}$

FT - far from threshold



$\boxed{\text{Far from threshold}}$

For $r > L^{FT}$, Lamé form of stress

For $r < L^{FT}$

Use $\sigma_{\theta\theta} = 0$

$$\sigma_{rr} = \frac{T_i R_i}{r}$$ - from force balance, FvK radial.

Why $\sigma_{\theta\theta} = 0$, rather than inextensibility?

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{Y} \qquad \varepsilon_{\theta\theta} = -\Lambda \varepsilon_{rr} = -\frac{\Lambda \sigma_{rr}}{Y} \neq 0.$$

(Thus mf is finite)

<u>Length</u> Get L_{FT} by minimizing

E = Radial stretching inside + Radial and azimuthal stretch outside.

[does not depend on f]

$$\frac{L_{FT}}{R_i} = \tau/2 \quad \leftarrow \text{Completely different from NT}$$

<u>Number of wrinkles</u>: Now assume $\zeta(r,\theta) = f(r)\cos m\theta$

[mf does not satisfy inextensibility as $\varepsilon_{\theta\theta} \neq 0$]

Now minimize smaller energy terms inside the wrinkled region

azimuthal bending + azimuthal compression inside + radial stretch inside.

Forces $\rightarrow$ $Bm^4 f/L^4$ $\qquad \sigma_{\theta\theta}\frac{m^2}{L^2} f$ $\qquad \frac{\sigma_{rr} f}{L^2}$

Analogy from hydrodynamics

Fluids (Navier-Stokes equations)

Stokes

linear (viscous) theory

(expansion in *Re*)

Laminar flow

Euler

non-linear (inertial) theory

inertial flow

Re

Elastic sheets (FvK equations)

Near threshold

compression

(amplitude expansion)

low bendability

Far from threshold

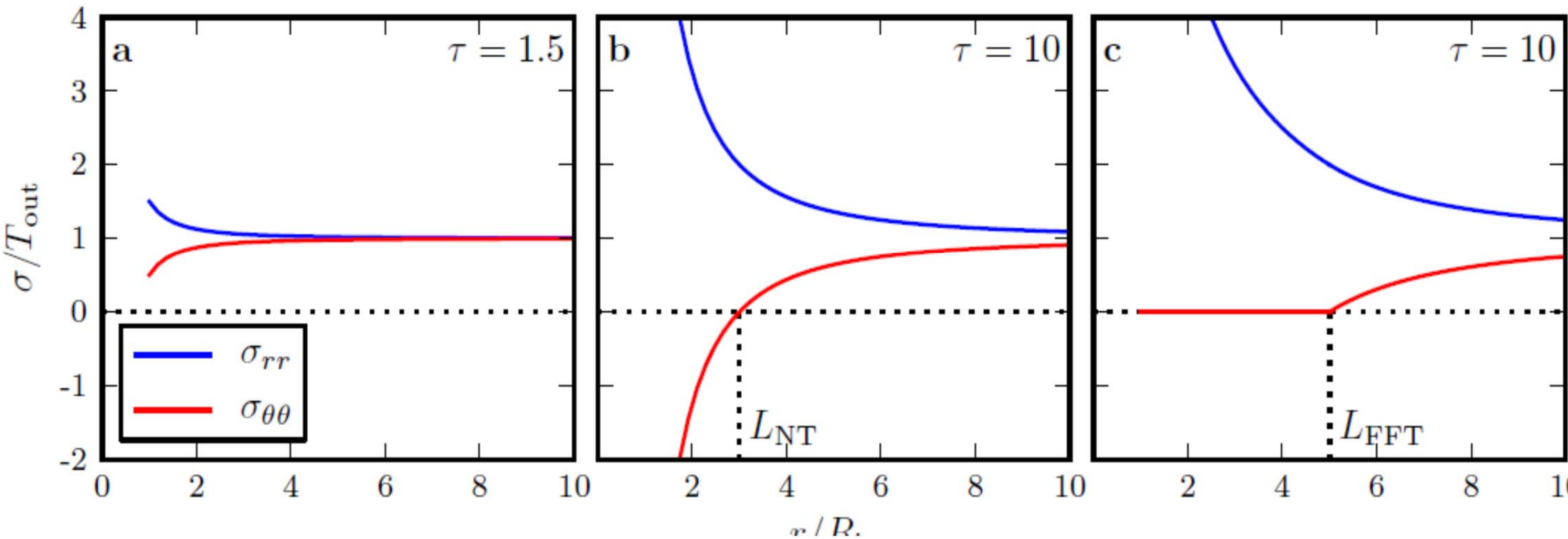
compression-free, tension-field theory

(bendability expansion)

ε^{-1}

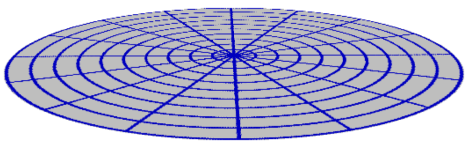
high bendability

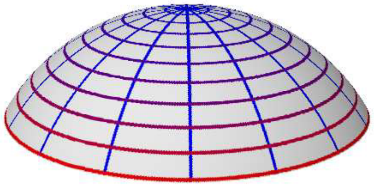
Slide from Benny Davidovitch

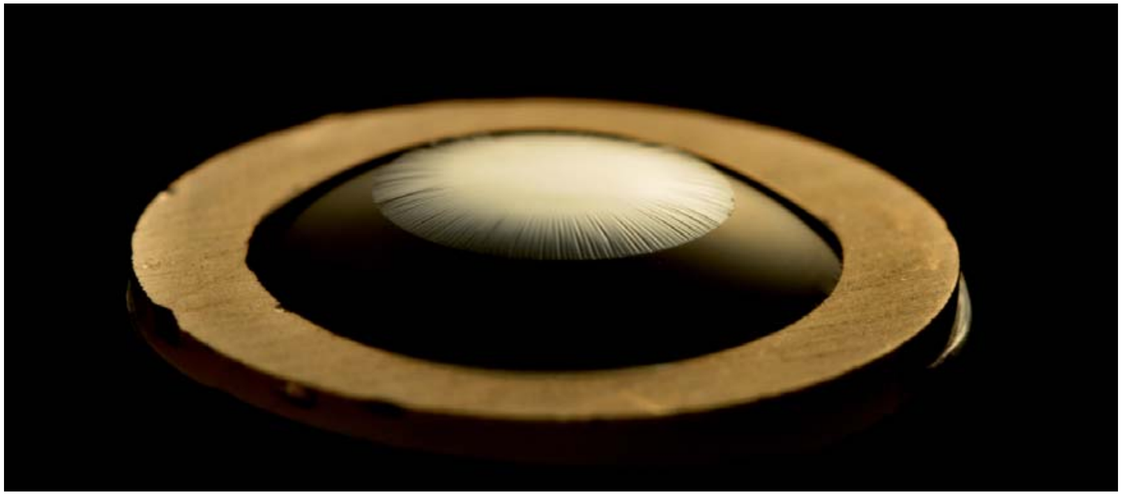


a
$\tau = 1.5$
b
$\tau = 10$
c
$\tau = 10$
$\sigma / T_{\mathrm{out}}$
σ_{rr}
$\sigma_{\theta\theta}$
L_{NT}
L_{FFT}

Flat sheet on curved surface







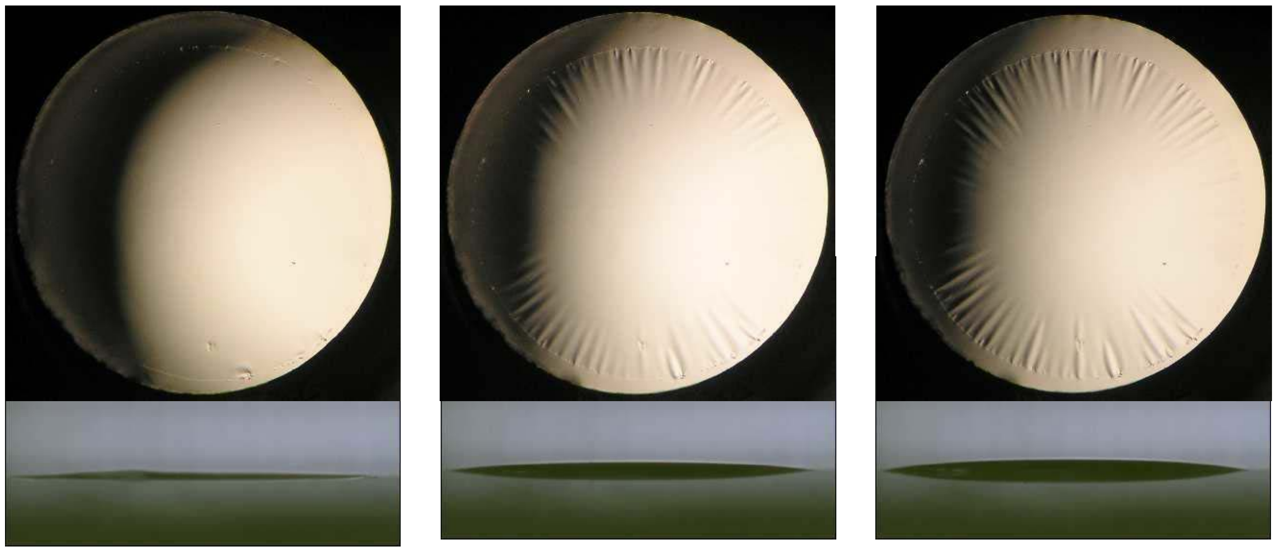
Confinement into smaller perimeters governed by

$\alpha = Y/\gamma\ (W/R)^2$ *Y: stretching modulus*

Wrinkles grow inward from edge

Top view

Side view

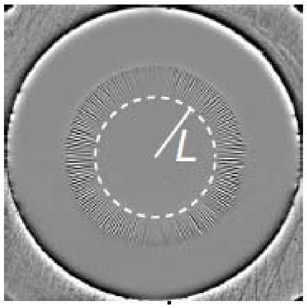


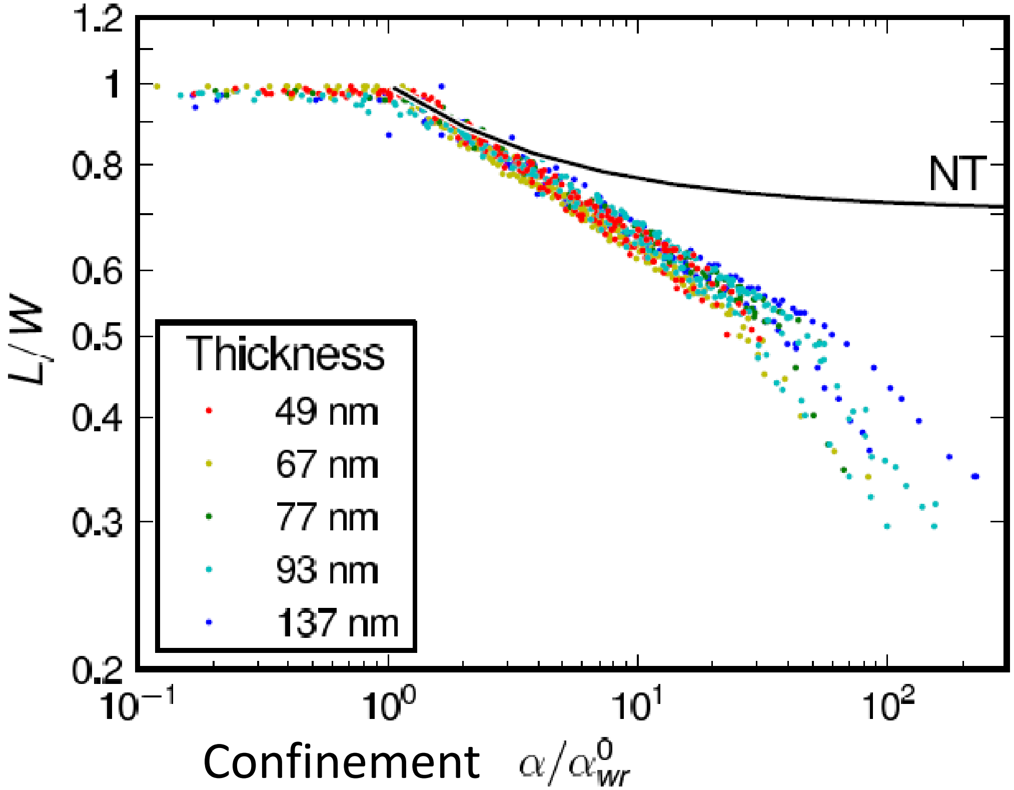
increase drop curvature
(increase confinement α)

How long are the wrinkles? How many are there?

Wrinkle length

Normalized size of unwrinkled zone

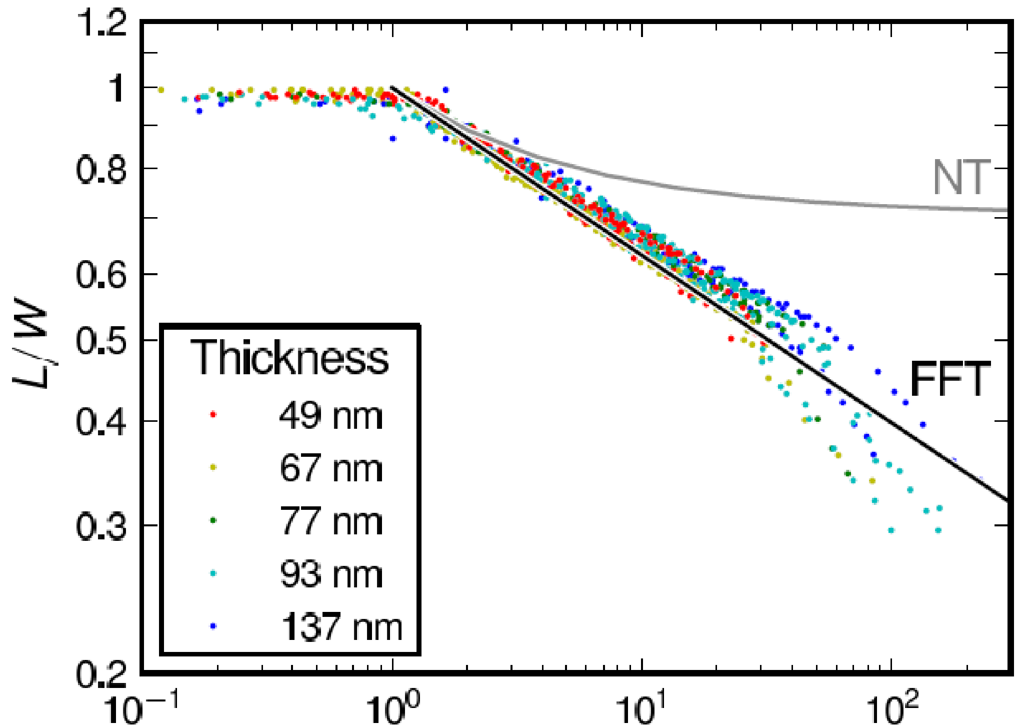




Post-buckling (NT) calculation gets wrinkle length entirely wrong

Wrinkle length

Normalized size of unwrinkled zone



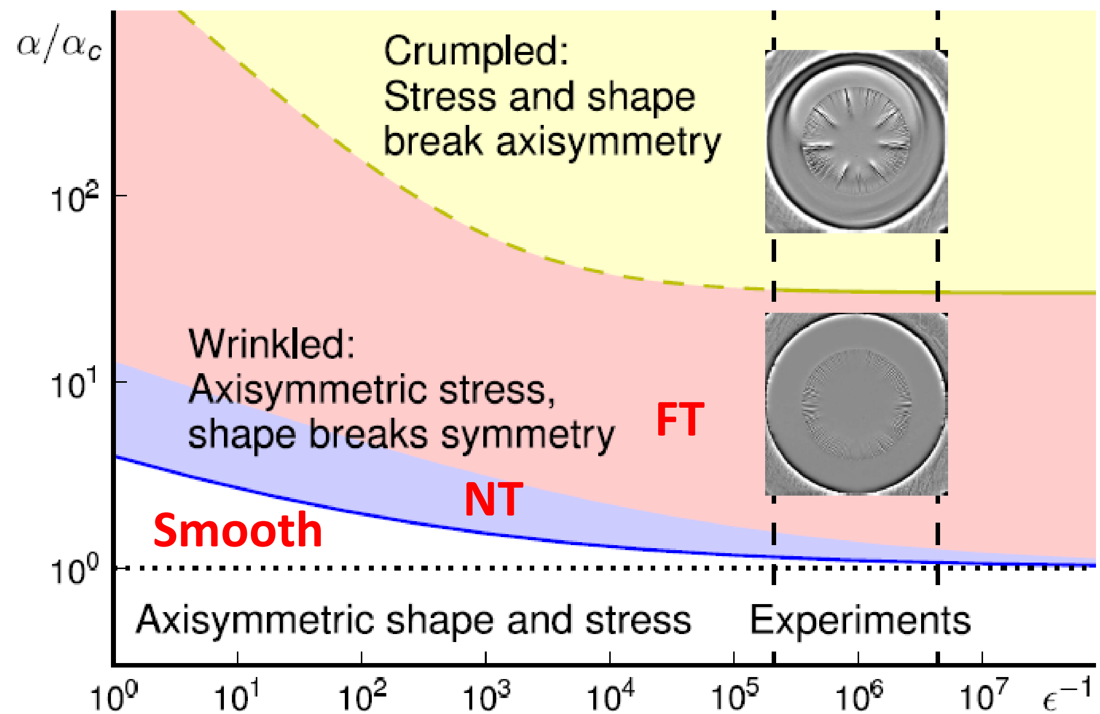
Confinement α/α_{wr}^{0}

FT calculation predicts wrinkle length successfully

Wrinkling for sheet on drop – “phases”

$\gamma/Y\ (W/R)^2$

“Confinement”



“Bendability” $\gamma W^2/B$

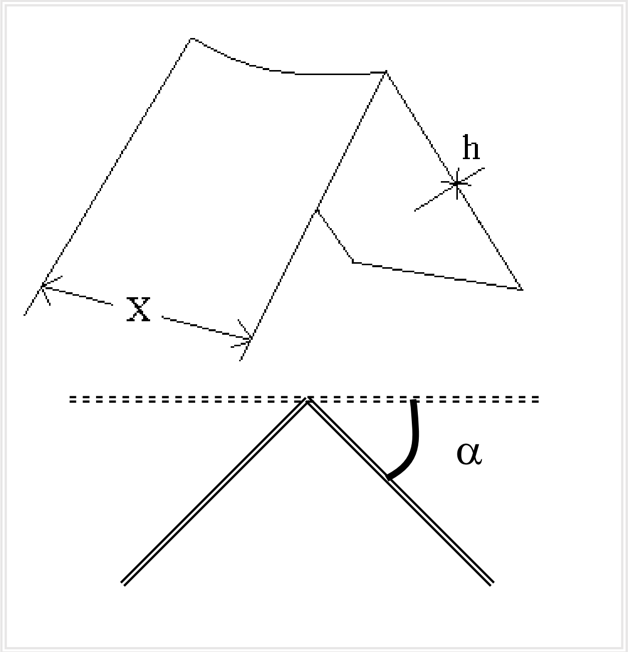
Almost no regime
of near threshold

King et al. PNAS 2012

A little about crumples - ridges

Localizes strain and bending

- Stored energy E ~ $\alpha^{7/3}\, k(X/t)^{1/3}$
- Mid-ridge radius ~ $\alpha^{-4/3}\, X^{2/3} t^{1/3}$
- Bending energy ~ Stretching energy

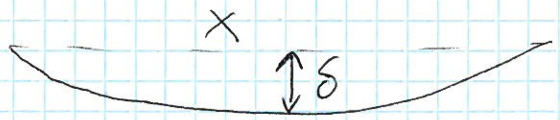


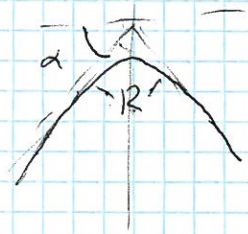
Lobkovsky et al. 1997

Ben-Amar, Pomeau

Witten RMP 2009

Energy of a ridge (scaling analysis of Witten, Lobkovsky etc)





Strain $\sim \dfrac{\sqrt{X^2+\delta^2} - X}{X} \sim \dfrac{1}{2}\dfrac{\delta^2}{X^2}$

$\dfrac{R+\delta}{R} = \sin\alpha$

$\Rightarrow \dfrac{\delta}{R} \sim (\sin\alpha - 1) = f(\alpha$

$E_{stretch} \sim Y \dfrac{\delta^4}{X^4} \cdot (RX) \sim \dfrac{YR^5}{X^3}$

$E_{bend} \sim B \dfrac{1}{R^2} \cdot XR \sim \dfrac{BX}{R}$

Minimize to get answers

Strain is localized in a region

$$\underline{\frac{RX}{X^2} \sim \frac{X^{2/3}\, t^{1/3}\, X}{X^2}} \sim \left(\frac{X}{t}\right)^{-1/3}$$

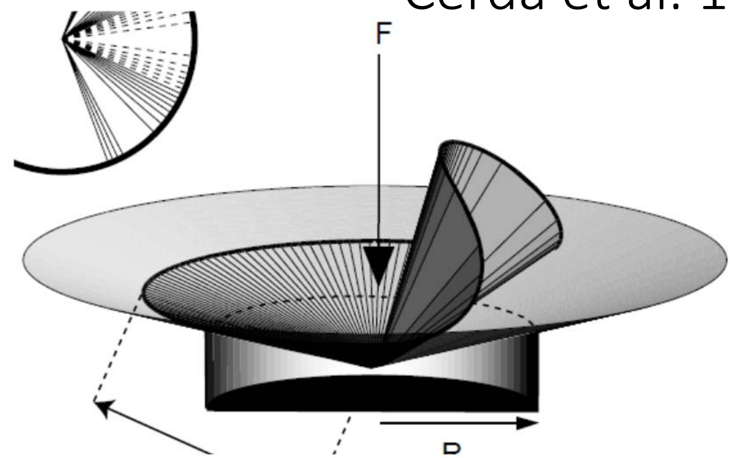
as the sheet becomes thinner this is a vanishingly small number

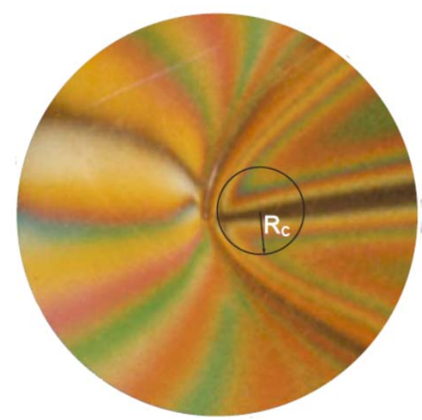
A little about crumples : d-cones

Only bending outside core

Core has comparable stretching and bending

Cerda et al. 1999





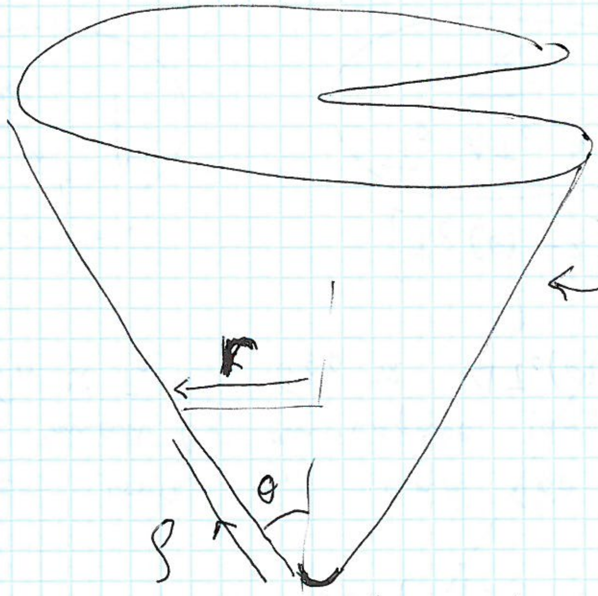
NMenonBoulder2015

Figure 2 Geometry of a real conical dislocation. Cross-polarizers are used to view the reflected light from a painted sheet deformed into a conical dislocation. Isochromatic lines

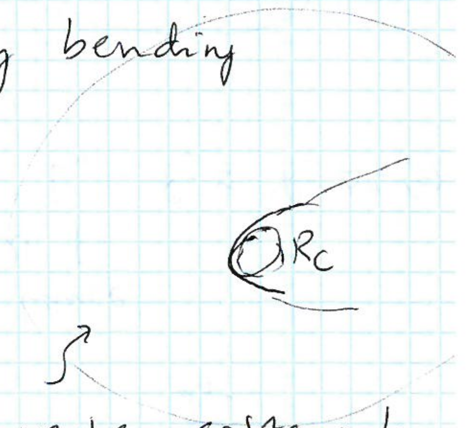
Energetics of a d-cone

"developable"

following Cerda et al. (Nature 1999)



only bending



CONE

sharp cone vertex softens by stretching.

$r = \rho \sin\theta$

$$E_{\text{bending}} = \frac{1}{2}\int_{R_c}^{R} B\,\frac{1}{r^2}\,\rho\, d\rho\, 2\pi = \frac{1}{2}\int_{R_c} B\,\frac{\rho}{\rho^2}\,\frac{2\pi}{\sin^2\theta}\,d\rho$$

(away from vertex)

$$\simeq \frac{1}{2}\cdot\frac{2\pi}{\sin^2\theta}\,B\,\ln\left(\frac{R}{R_c}\right)$$

logarithmic in size
and dependent on R_c, the radius at the cone tip

CORE OF CONE VERTEX

R_c found to depend on system size! $R_c \sim R^{2/3}$
(not yet carefully checked in expts, I think)

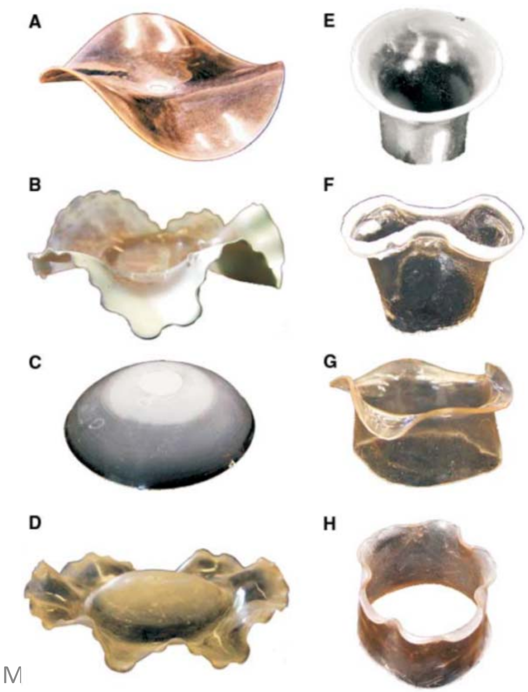
Finding energy by making bending & stretch comparable.

A little about crumples : e-cones

- An example with too much material

- Emerges naturally in growth problems

Muller et al 2008



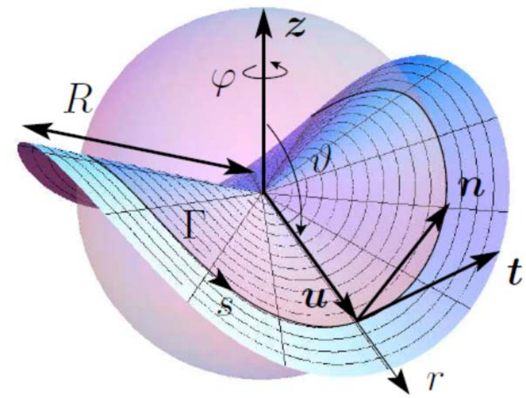


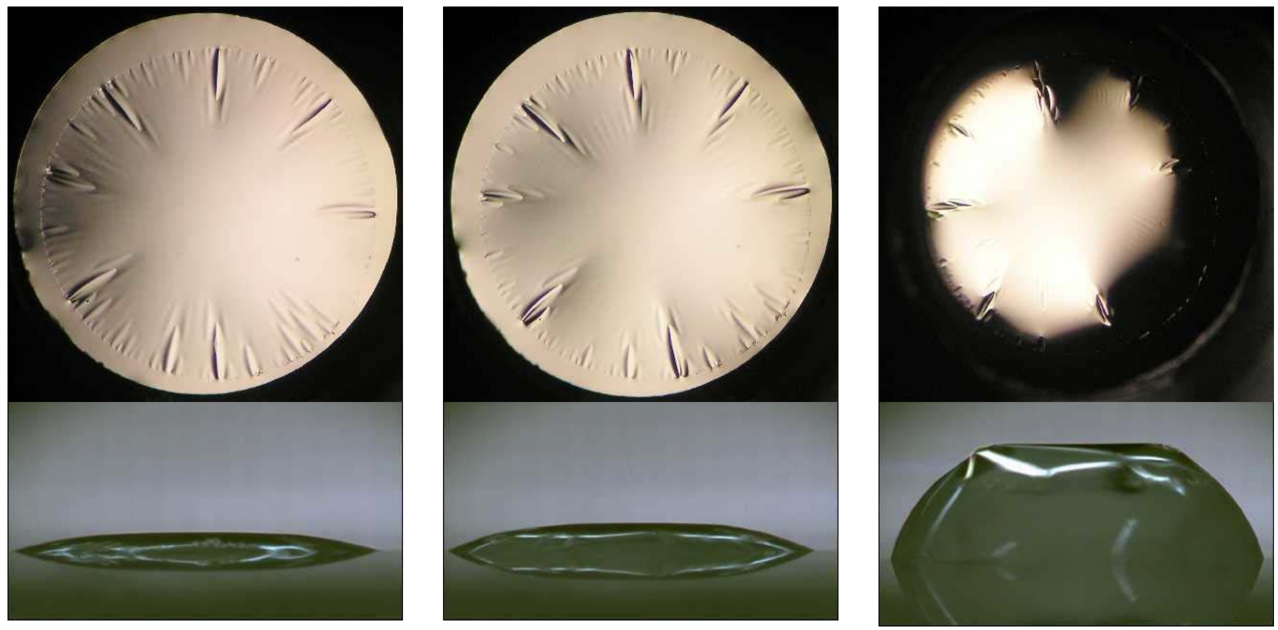
FIG. 1: Geometry of the e-cone with $\varphi_e = \frac{2\pi}{9}$.

- Klein, Efrati, Sharon 2007

Delocalized modes get localized

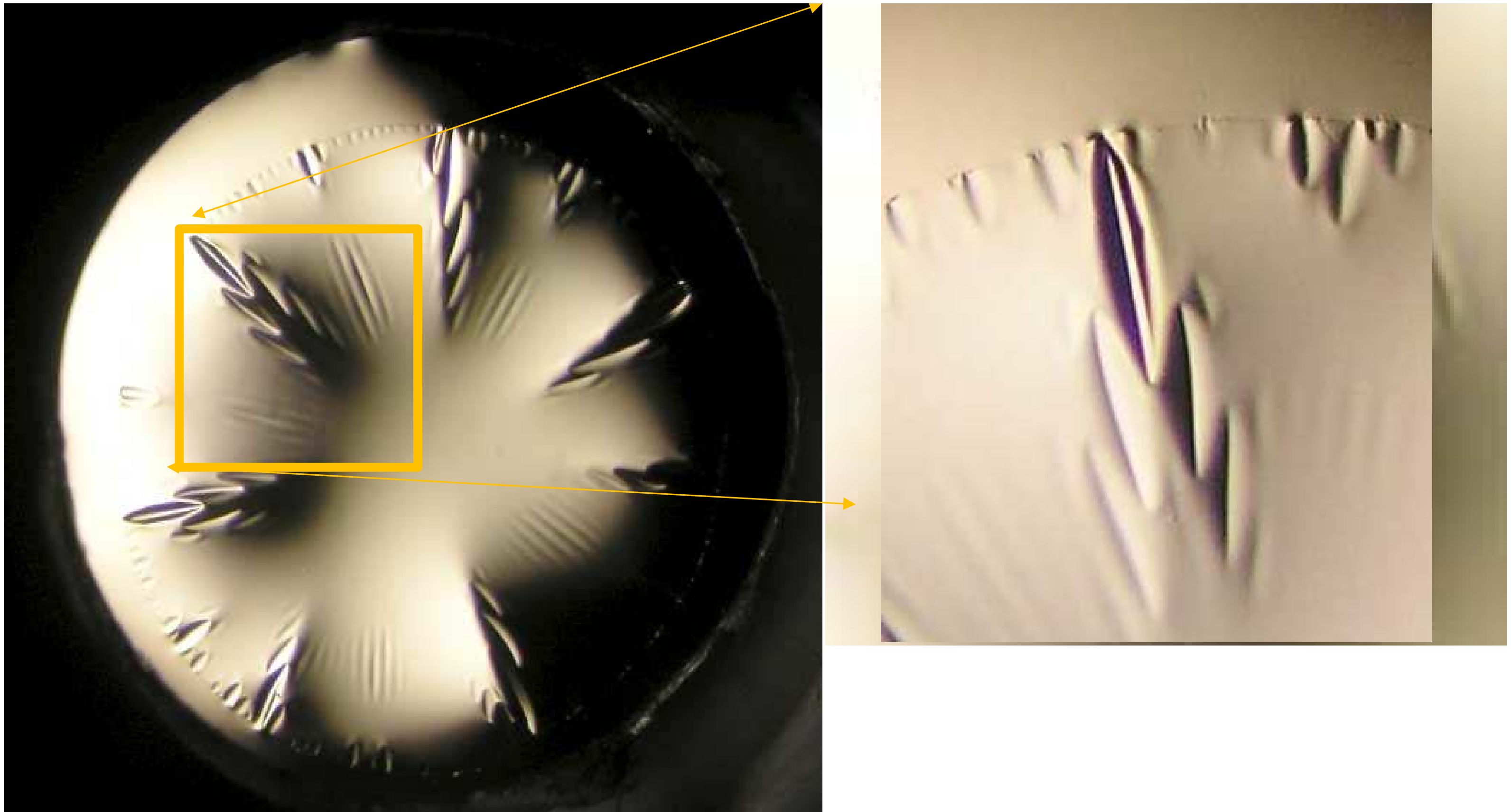
Top view

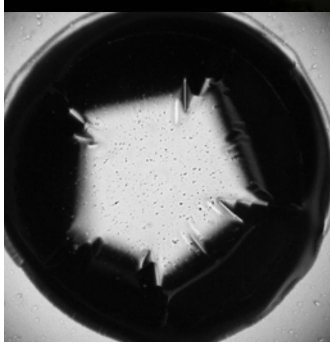
Side view



increase drop curvature →

A few wrinkles grow, and sharpen into “crumples”
The others recede





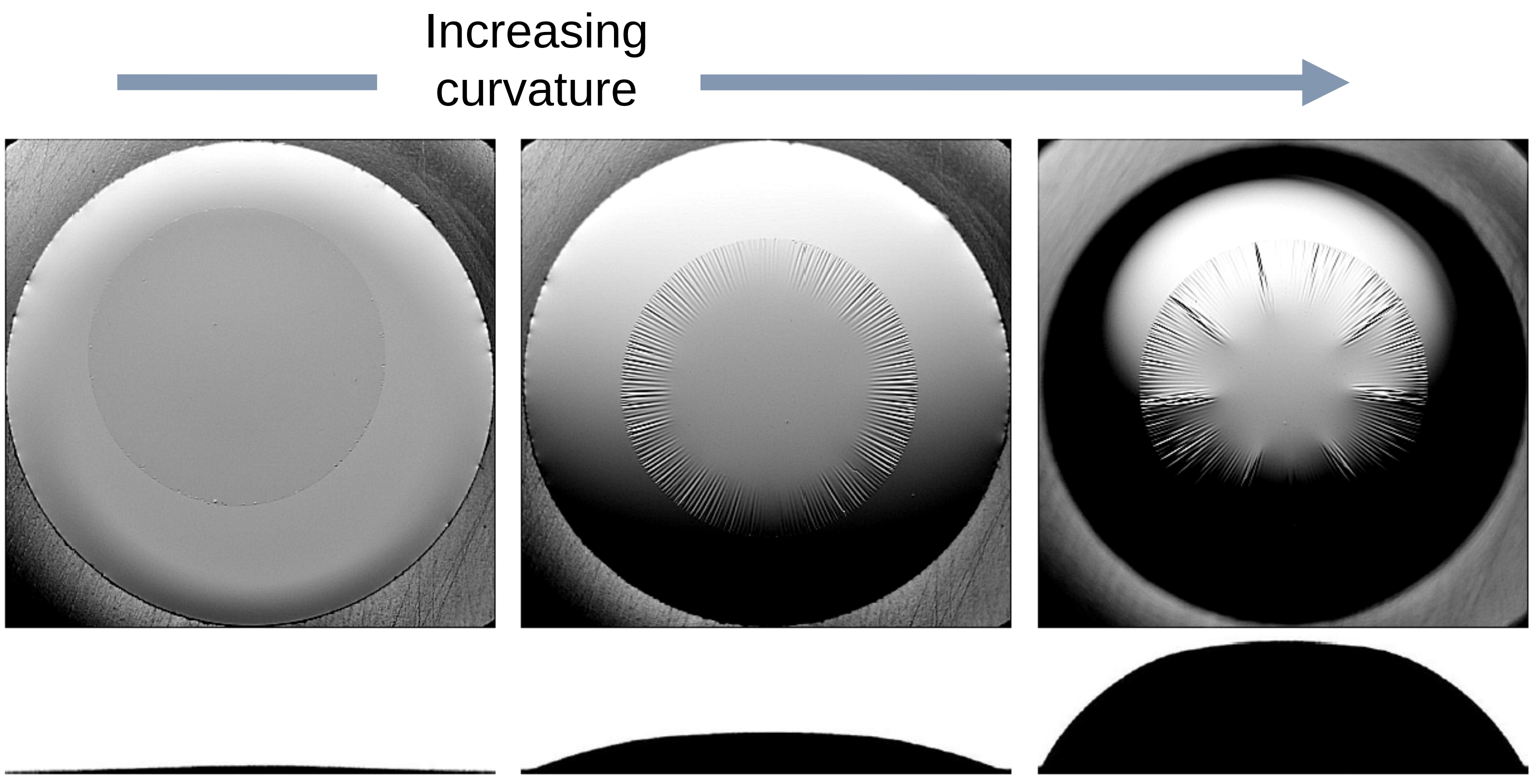
A different way of hiding material

Analogous to scars/disclinations

but fold, not cut, material

emerges from continuum elasticity, no discrete charge

Continuous, reversible, wrinkle-to-crumple transition

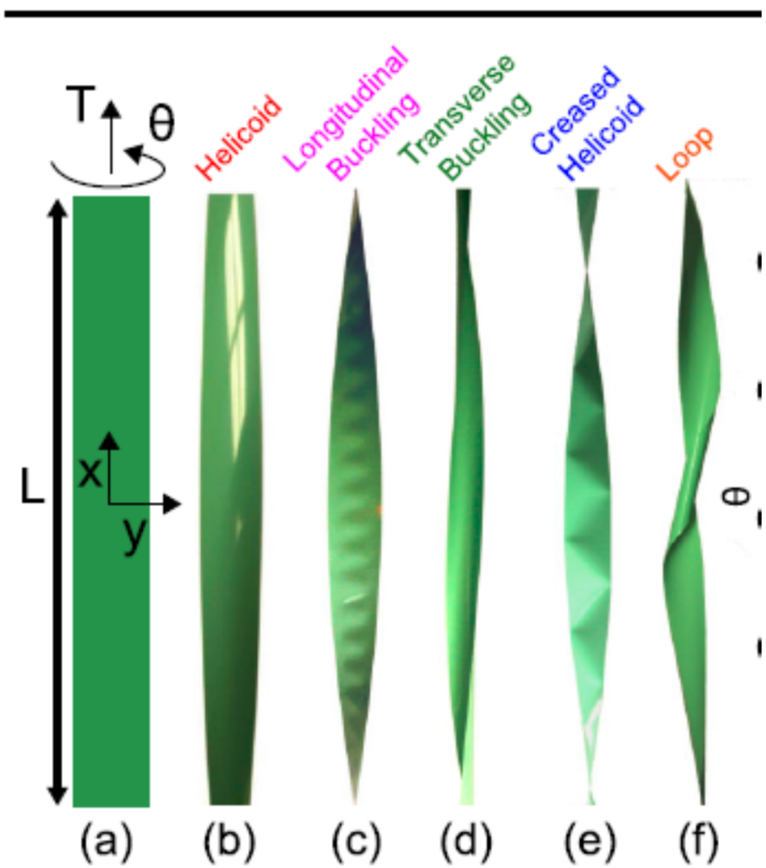


Axisymmetric shape,
Stretched 'cap',
Axisymmetric stress

Broken symmetry shape,
Smooth wrinkles,
Axisymmetric stress

Further symmetry breaking,
Localized features

Another wrinkle-to-crumple transition



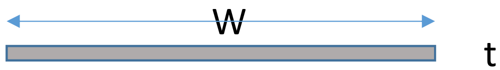
Chopin Kudrolli 2013

Large deformation

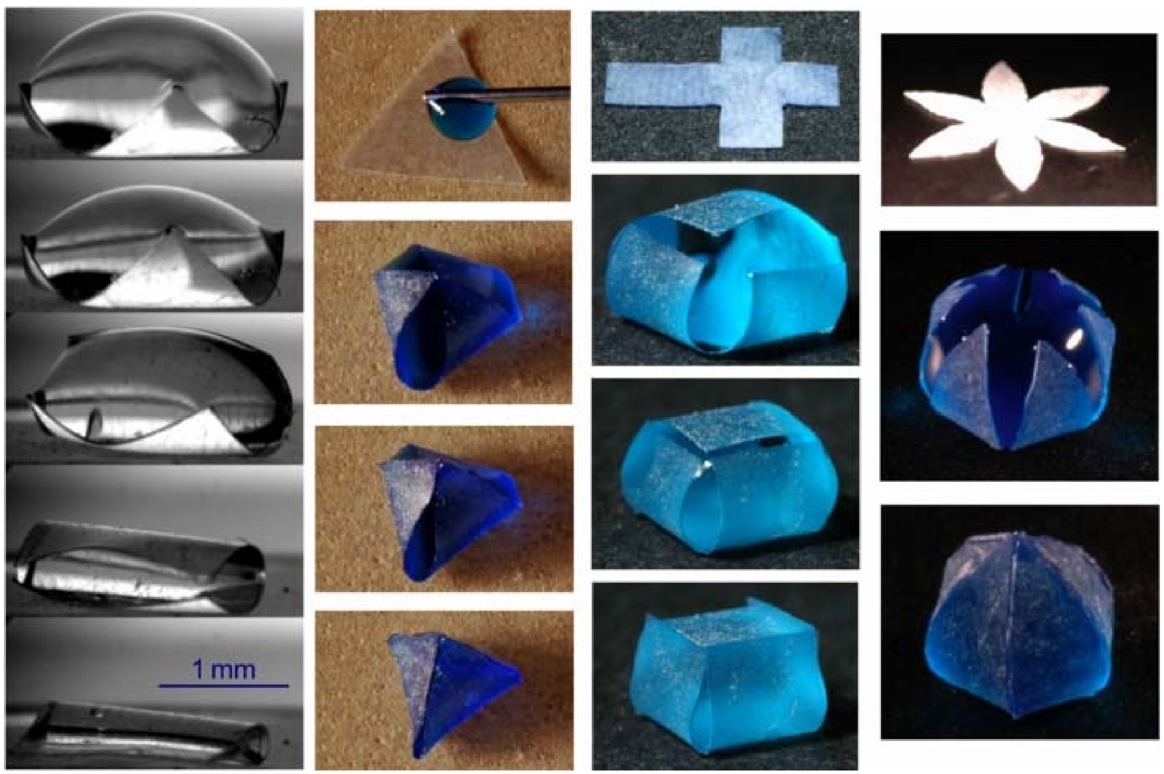
What can wrinkles, folds and crumples do for you?

When they are cheap (large bendability), they can achieve amazing shapes

Wrapping a drop



“Capillary origami” Py, Reverdy, Baroud, Roman, Bico 2006 *t=50μm; W ~ few mm, PDMS*



Bending balances torques created by capillary forces

Shapes with flaps cut to allow pure bending

Wrapping with thin sheets

J. Paulsen, V Démery

Polystyrene sheet (t=79nm) wrapping a fluid drop

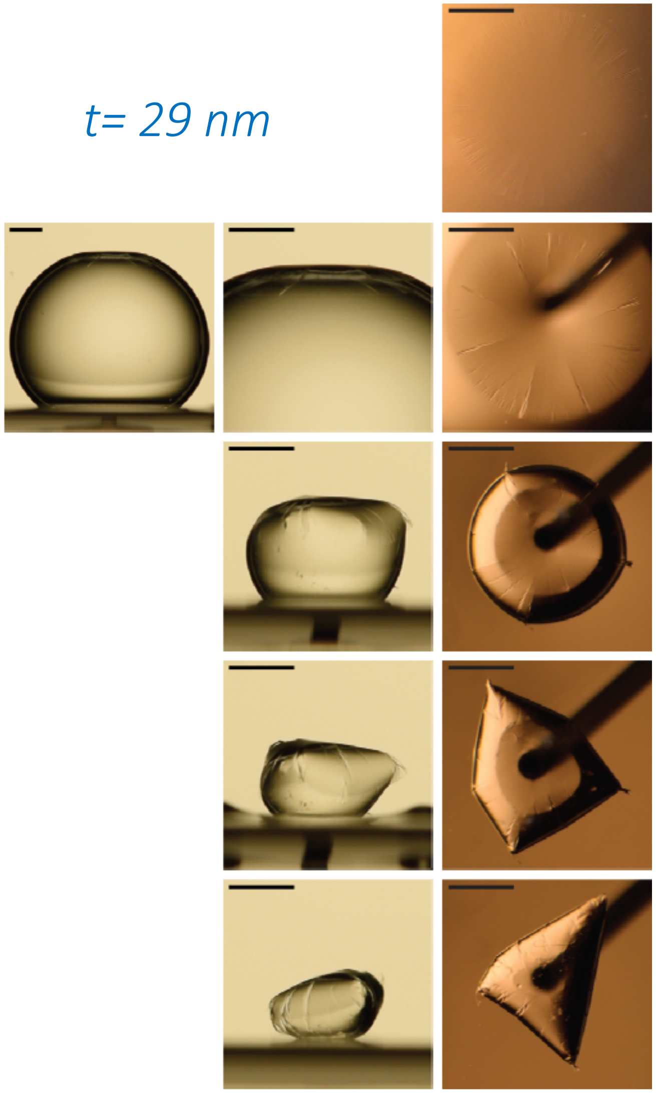
- Watch our movie for APS-DFD 2104 at

http://tinyurl.com/dfd-wrap

Side view

Top view

t= 29 nm

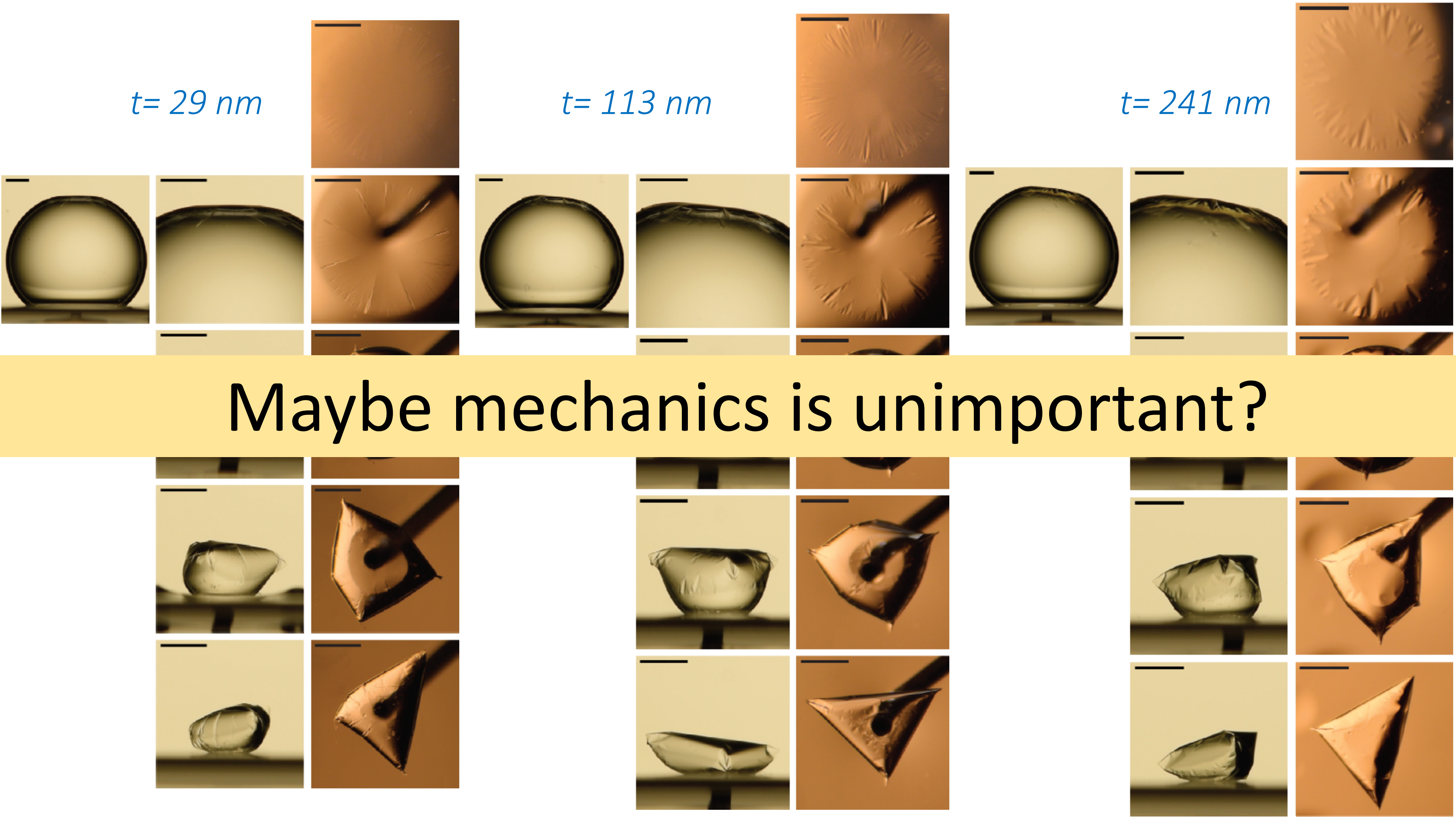


Axially symmetric   wrinkles, crumples

Polygonal shapes   folds, crumples

How to understand this sequence of shapes?

Wrinkles,folds, crumples, all interacting on a curved surface



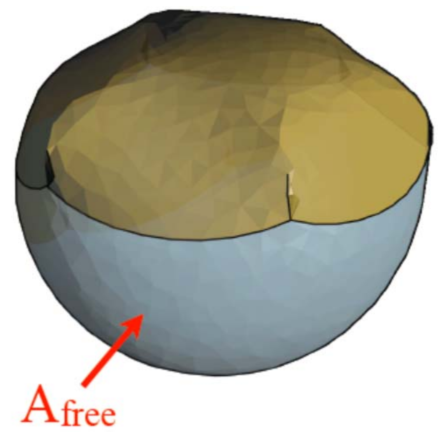
Maybe mechanics is unimportant?

Wrapping with thin sheets

Describe all shapes with a simple equation:

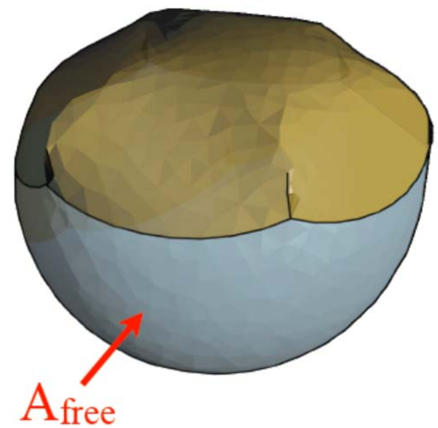
Energy, $U = \gamma A_{free}$

Constraint: free ‘compression’, but no stretching



Pure geometry, no material parameters!

Wrapping with thin sheets



Describe all shapes with a simple equation:

Energy, $U = \gamma\, A_{free}$

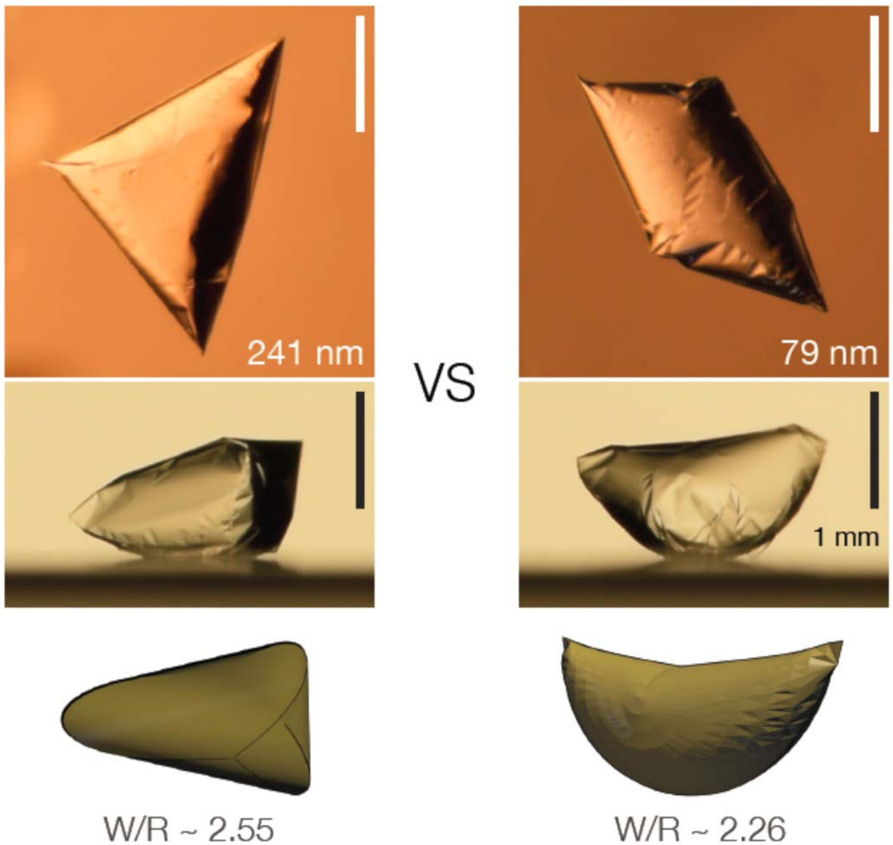
Constraint: free ‘compression’, but no stretching

Works when energy scales are separated (the first inequality ins high bendability):

bending << surface << stretching

Et^3/W^2 γ **Et**

Predicts non-axisymmetric shapes



Samosa less efficient than empañada

Implications

- A ‘thin’ sheet spontaneously achieves the highest wrapping efficiency

  No need for careful design

- Doesn’t rely on material parameters (in this regime)

- Shows possibilities of near-isometric deformation if you have high enough bendability (see also Vella 2015)

<u>2D wrinkles (again)-</u>

Huang, J., Juszkiewicz, M., De Jeu, W. H., Cerda, E., Emrick, T., Menon, N., & Russell, T. P. (2007). Capillary wrinkling of floating thin polymer films. *Science*, *317*(5838), 650-653.

King, H., Schroll, R. D., Davidovitch, B., & Menon, N. (2012). Elastic sheet on a liquid drop reveals wrinkling and crumpling as distinct symmetry-breaking instabilities. *Proceedings of the National Academy of Sciences*, *109*(25), 9716-9720. <u>The SI is useful.</u>

Davidovitch, B., Schroll, R. D., Vella, D., Adda-Bedia, M., & Cerda, E. A. (2011). Prototypical model for tensional wrinkling in thin sheets. *Proceedings of the National Academy of Sciences*, *108*(45), 18227-18232.

<u>Crumples-</u>

Witten, T. A. (2007). Stress focusing in elastic sheets. *Reviews of Modern Physics*, *79*(2), 643.

Lobkovsky, A., Gentges, S., Li, H., Morse, D., & Witten, T. A. (1995). Scaling properties of stretching ridges in a crumpled elastic sheet. *Science*

Cerda, E., Chaieb, S., Melo, F., & Mahadevan, L. (1999). Conical dislocations in crumpling. *Nature*, *401*(6748), 46-49.

<u>Wrapping etc</u>

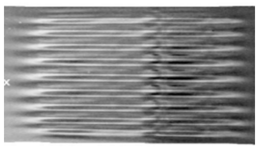
Py, C., Reverdy, P., Doppler, L., Bico, J., Roman, B., & Baroud, C. N. (2007). Capillary origami: spontaneous wrapping of a droplet with an elastic sheet. *Physical Review Letters*, *98*(15), 156103.

Vella, D., Huang, J., Menon, N., Russell, T. P., & Davidovitch, B. (2015). Indentation of ultrathin elastic films and the emergence of asymptotic isometry. *Physical review letters*, *114*(1), 014301.

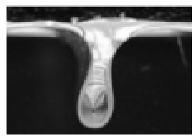
JD Paulsen, V. Demery et al. (2015) Optimal wrapping of liquids with ultrathin sheets .. Ask me if you want this paper. In referee process.

Overall goals of our discussion

Wrinkles

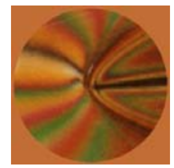


Folds

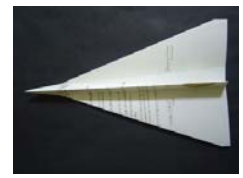


Ridges

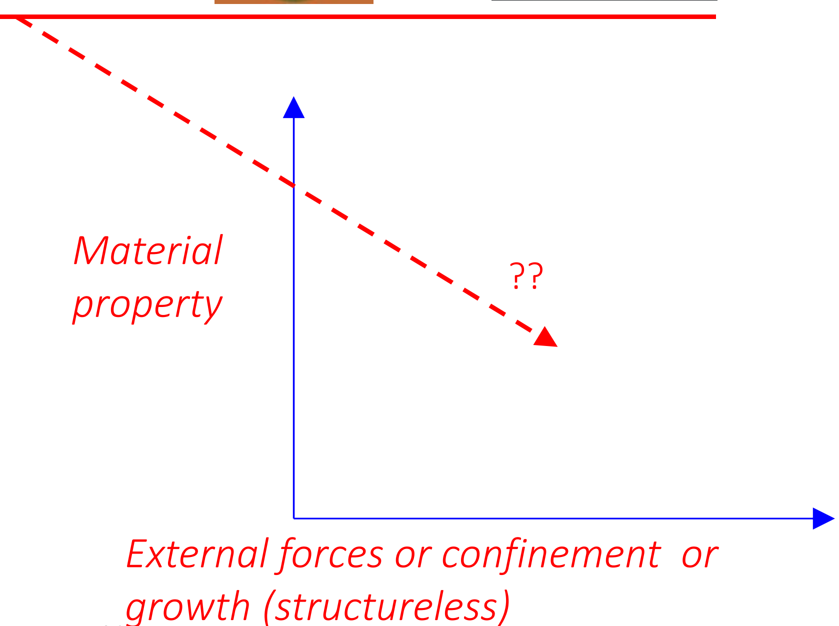
d-cones



Creases (plasticity)



- These structures are generated by elastic instabilities
- What are the energetics and stability of these constructs?
- Where do all these structures belong?
- How to specify these axes?



Thanks

- Audience

- School organizers

  Especially Leo R, Xiao, Han

- Collaborators

  Expts – J. Huang, H. King, KB Toga, JD Paulsen, Tom Russell

  Theory – B. Davidovitch, R Schroll, V. Demery

  E. Cerda, D. Vella