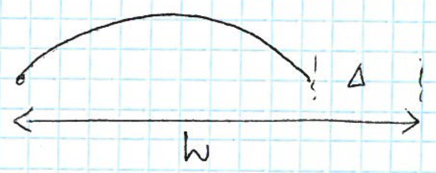
# Lecture 2

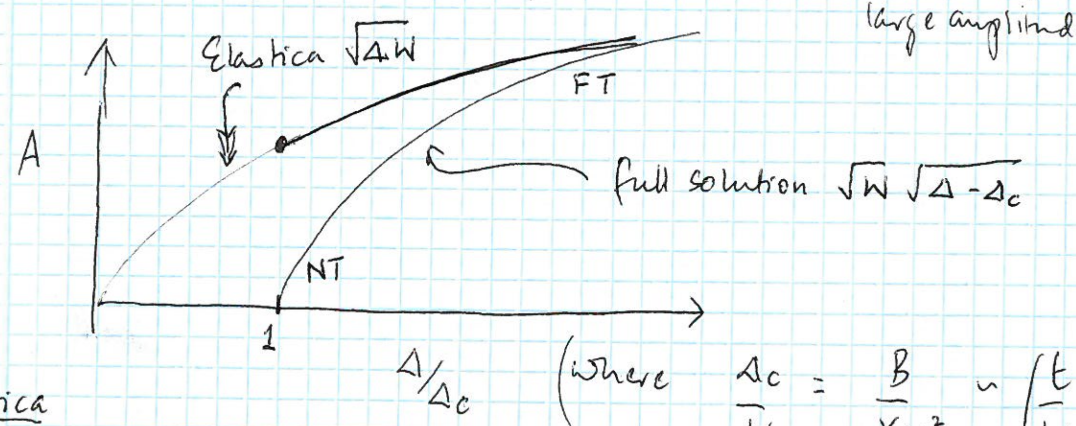
- Review yesterday
- Folds – bending localization in 1D
- Wrinkles in 2D geometries

# Last time recap ①

Euler buckling



Consider small slopes $t \ll \Delta \ll W$
large amplitude.



$\left(\text{where } \frac{\Delta_c}{W} = \frac{B}{YW^2} \sim \left(\frac{t}{W}\right)^2 = vK^{-1}\right)$

Elastica

$e =$ Bending energy $+ \sigma \cdot$ Inextensibility constraint

Full solution

$e =$ bending energy $+$ stretching energy $\rightarrow$ $\partial_y \sigma_{yy} = 0$ (in-plane)

$B\xi'''' - \sigma_{yy}\xi'' = 0$ (normal)

(We only did first term in series solution)

NT : Near threshold → can solve in powers of $\left(\frac{\Delta}{\Delta_c} - 1\right)$ or A

(Post-buckling / Landau / amplitude expansion)

FT : Far from threshold → can perturb about elastica in powers of $(\Delta/\Delta_c)^{-1}$

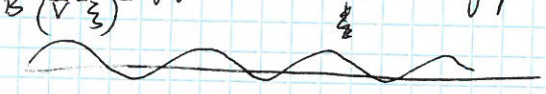
Only parameter here is the confinement parameter : $\Delta/\Delta_c$

Last time recap - ②

Wrinkling (a' la Cerda Mahadevan 2003)

$e$ = bending energy + substrate energy + $\sigma$ Inextensibility constraint

$\frac{1}{2}B(\nabla^2\xi)^2$



$\xi = A \sin qx$

$\downarrow$

We worked out case of fluid substrate where $e_{sub} = \frac{1}{2}K\xi^2$ and $K = \rho g$

$qA = 2\sqrt{\Delta/W}$

leads to $q \sim \left(K/B\right)^{1/4}$

---

For tensional case see C-M 2003 solution

$$e_{sub} = \frac{1}{2} T (\partial_x \xi)^2 \rightarrow q \sim \left(K/B\right)^{1/4} \text{ with } K = \frac{T}{L^2}$$

---

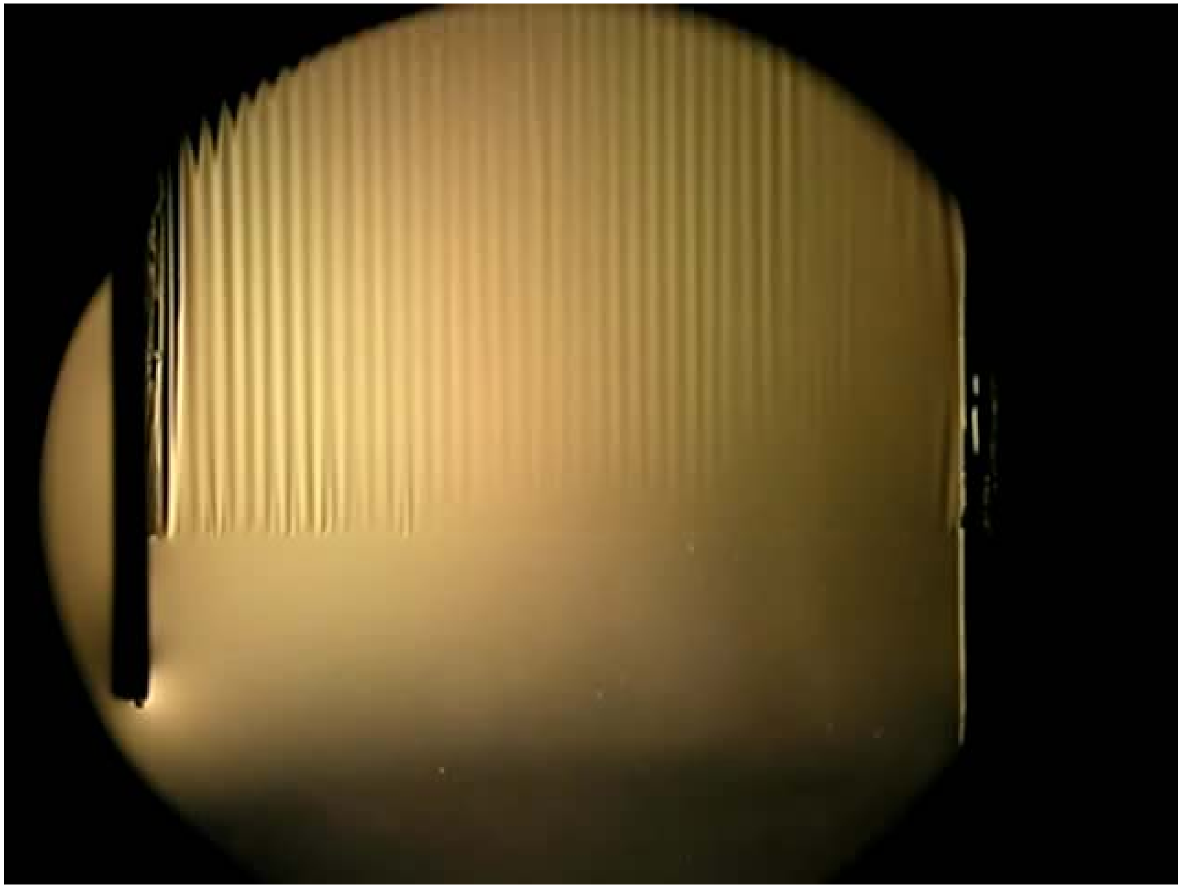
Soft solid substrate

nasty problem but inspired handwaving in C-M 2003

leads to $\boxed{\text{deep substrate}}$ $K = E_s/\lambda$ ($E_s$: Youngs modulus of substrate.)

$\boxed{\text{shallow substrate}}$ and $K = \frac{E_s \lambda^2}{H^3}$

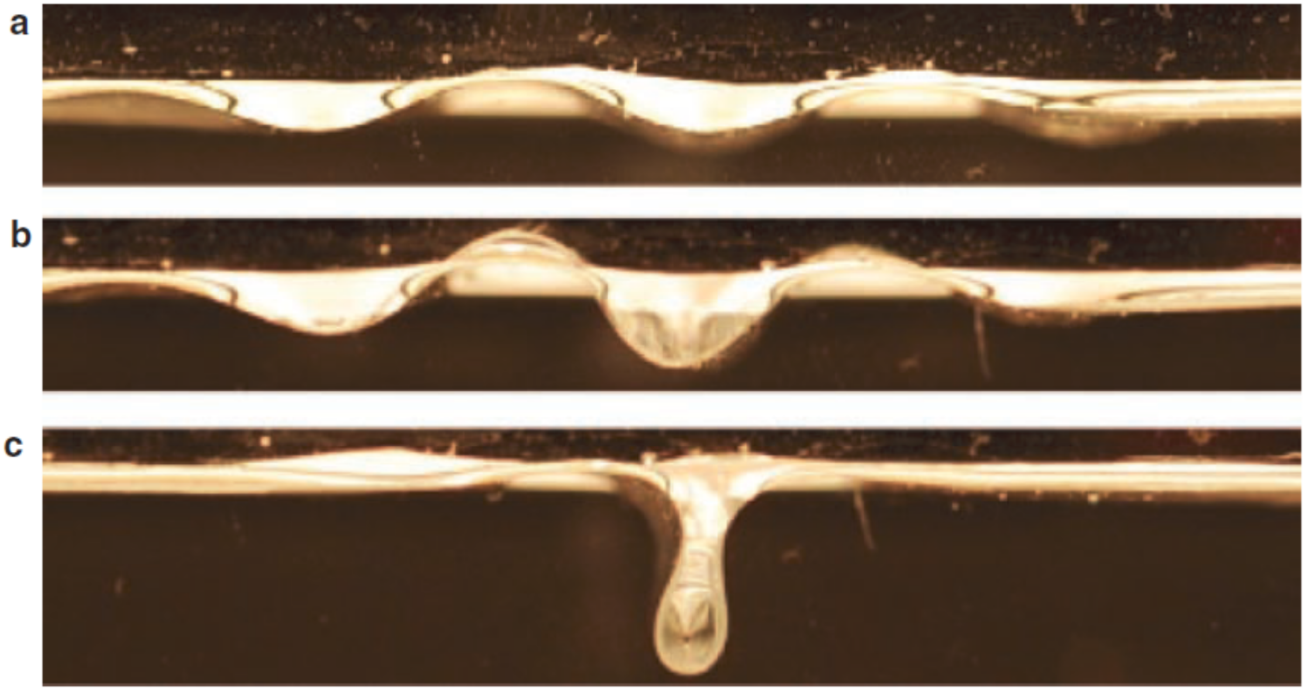
# Folding in 1D



Huang thesis 2010

# Folding in 1D

Pocivavsek Science 2008



Plastic sheet (left)

Gold nanoparticles, lung surfactant

# Folding in 1D

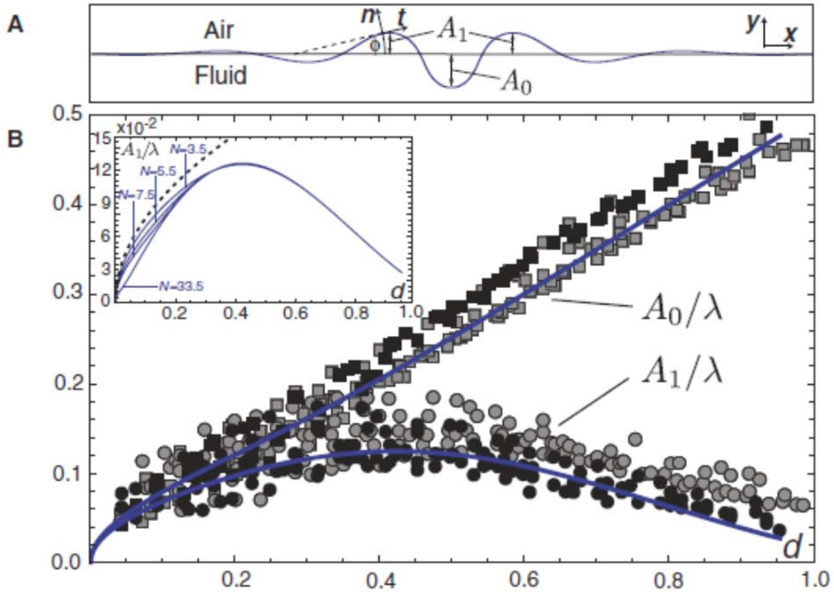


**Fig. 2.** (**A**) The figure defines $A_0$ and $A_1$ and the geometrical parameters describing a confined sheet. The deformation can be described by using a two-dimensional coordinate system. Here $t$ and $n$ are the tangent and normal to the surface, respectively. $\phi$ gives the position of the tangent with respect to the horizontal direction. (**B**) Experimental results for polyester on water for $A_0$ (squares) and $A_1$ (circles). Experimental data were taken for several membrane sizes, including when $N = 3.5$, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, and 8.0. Dark solid lines show numerical results for a sheet with $L = 3.5\lambda$. Both the physical polyester and numerical data are made dimensionless. $A_1$, $A_0$, and $\Delta$ are scaled to $\lambda$. (Inset) $A_1$ versus horizontal displacement for several numerical systems of different sizes (solid blue lines). The dashed line is the theoretical curve $A = [(\sqrt{2})/\pi]\lambda\sqrt{(d/3.5)}$ (*20*) that follows the numerical curve for $N = 3.5$ and $d << 1$. In both numerical and physical cases, the data are more scattered for $d < 0.3$ and then collapse onto more compact (perfectly so in numerical case) curves past this point. This behavior is indicative of the size-dependent behavior in the wrinkling ($d < 0.3$) regime and size-independent behavior in the folding ($d > 0.3$) regime.

- Pocivavsek 2008

- Transition to fold at around A=0.3$\lambda$

FOLDS in 1D

Energy density for the wrinkled state is intensive

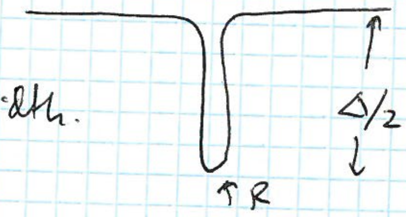
$\frac{\text{Energy}}{\text{area}} \to$ $e \simeq \sqrt{BK}\,\frac{\Delta}{W}$ for a given $\frac{\Delta}{W}$.

Folds are local, so we will write the energy E, per length
Imagine a fold whose max radius of curvature is R

$E_B \sim B\left(\frac{1}{R}\right)^2 R$

$E_K \sim K\,\Delta^2(R)$

$\Big\}$ per width.



Doing this more carefully, we including the geometric nonlinearity, get

$E_K \sim KR\Delta^2 - K\Delta^3$

Minimize $E_B + E_K \;\to\; R \sim \sqrt{\frac{B}{K}}\cdot\frac{1}{\Delta}$

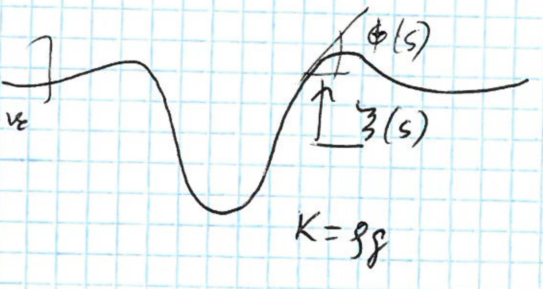
checked in numerics

expts? . affected by contact? energy

— ∘ — ∘ —

Exact solution Witten & Diamant 2012

$\dot{\zeta} = \sin\phi$ $\left[\; \dot{} = \frac{d}{ds} \text{ derivative} \;\right]$



$E_B = \frac{B}{2}\int_{-\infty}^{\infty} \dot{\phi}^2\, ds$

$K = \rho g$

$E_K = \frac{K}{2}\int_{-\infty}^{\infty} ds\, \zeta^2 \cos\phi$

$\Delta = \int_{-\infty}^{\infty} ds\,(1-\cos\phi)$

Set up as an action

$$S = \int_{-\infty}^{\infty} ds\, \mathcal{L}(\phi, \xi, \dot{\phi}, \dot{\xi})$$

$$\Big\downarrow$$

$$\dddot{\phi} + \left[\frac{3\dot{\phi}^2}{2} + \underset{\uparrow}{P}\right]\ddot{\phi} + \sin\phi = 0$$

Elastica multiplier

Borrow kink solutions to a Sine-Gordon equation.

Symmetric fold: $\phi(s) = 4\tan^{-1}\left[\frac{\kappa \sin ks}{k \cosh(\kappa s)}\right]$

[They label $\sigma_{YY} = P$]

$\kappa = \frac{1}{2}\sqrt{2-P} \quad k = \frac{1}{2}\sqrt{2+P}$

antisymm fold $\phi(s) = 4\tan^{-1}\left[\frac{\kappa \cos(ks)}{k \cosh(\kappa s)}\right]$

① $\kappa \approx \Delta/8$ Decay length

② $P = \left(2 - \frac{\Delta^2}{16}\right)$ ← becomes easier and easier to compress.

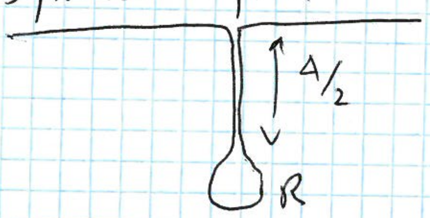
③ Exact solution confirms scaling of energy

---

Large folds Demery, 2014 $\quad U(\Delta) = 2\Delta - \Delta^3/48$

(i) Always less than wrinkles (ii) has a max.

Large symmetric fold.



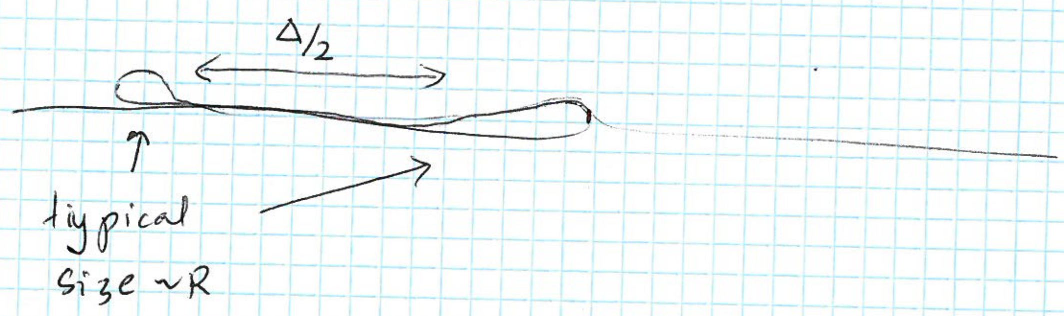
$U_{symm} = B\left(\frac{1}{R}\right)^2 R \sim B \cdot \frac{1}{R}$

$E_{grave} = \underset{\rho g}{K}\, R^2$

Self contact at $\Delta \sim 5.6$

$R \sim \Delta^{-1/3}$ ← different from what we found before self-contact

## Large antisymmetric folds



$E_{Bending} \sim B\left(\frac{1}{R}\right)^2 R$ $\qquad E_{substrate} \sim \underset{\rho g}{K}\ \underset{\text{area of each bubble}}{R^2} \cdot \underset{\text{height of bubble}}{R}$

Mininimize total energy,
get $R \sim \left(B/K\right)^{1/4}$ independent of $\Delta$!

Untill self-contact, antisymm and symm folds are degenerate, but this suggests that for large folds, antisymm wins!

* * *

Many puzzles unresolved:

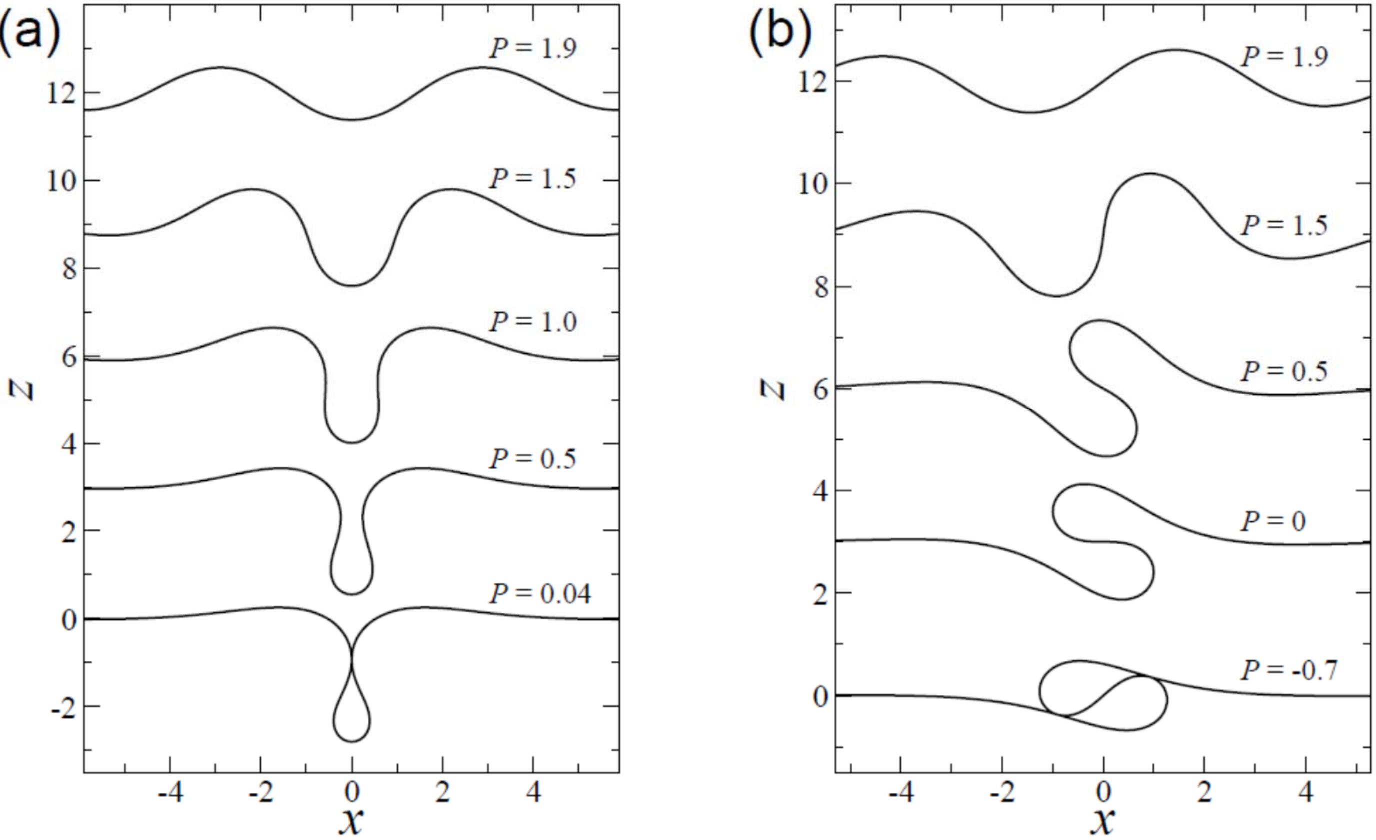
① None of these calculations predict a size scale for the fold, but expts. often see a finite scale.

② All these calculations do not have a threshold $\Delta_c$ for fold-onset. Why do we ever see wrinkles? Interesting issues of system size here.

③ Expts show both symm & antisym large folds
Is self-contact imp & adhesion important? or have we missed something?

# Exact solution

Diamant and Witten 2012



The symmetric (left) and antisymmetric (right) solutions are degenerate

Both cost less than the wrinkle solution at all $\Delta$

After self-contact, get penetration and nonphysical solutions

# Large folds

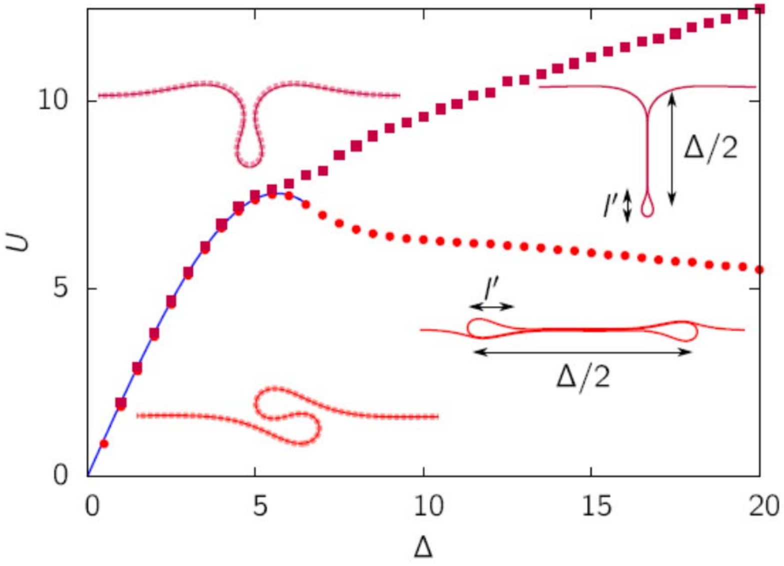


FIG. 2. (Color online) Fold energy as a function of the imposed displacement for the symmetric (squares) and antisymmetric (circle) folds. The solid blue line is the exact solution, Eq. (3), valid before self-contact. Symmetric (top) and antisymmetric (bottom) configurations are shown before self-contact (left, exact solutions from Diamant and Witten [22] are shown as thick dashed lines) and after self-contact (right). After self-contact, the size of the fold $\Delta/2$ absorbs the excess length, while bending is localized in highly curved zones of length $l'$.

Demery et al 2014

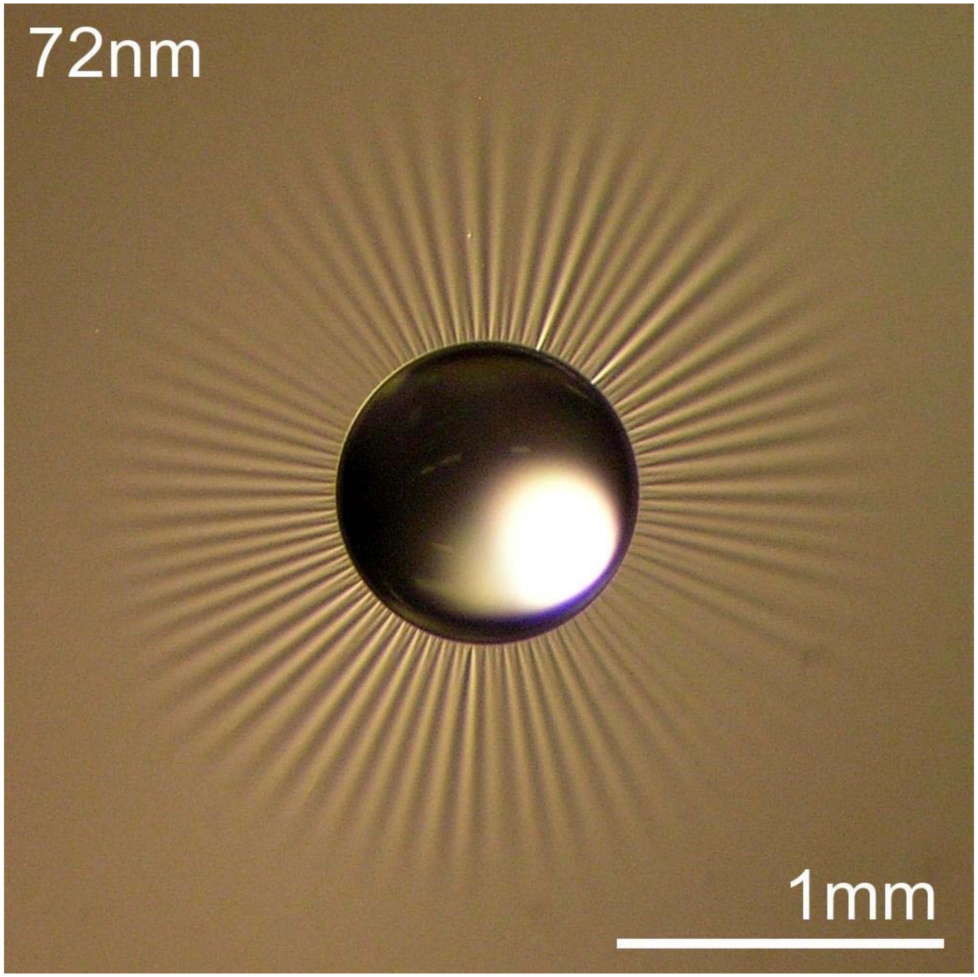
Goes beyond self-contact

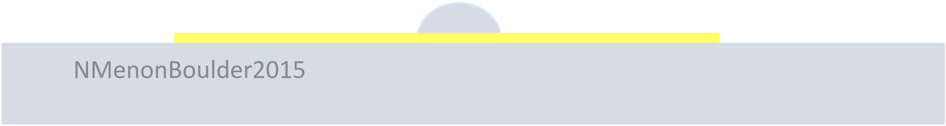
Symm and antisymm degenerate till self-contact, but anti-symmetric wins for larger folds

# 2D wrinkles

*Thin sheet of plastic (PS) floating on water with a drop of water in the middle*

Huang et al. Science 2007





# 2D wrinkles

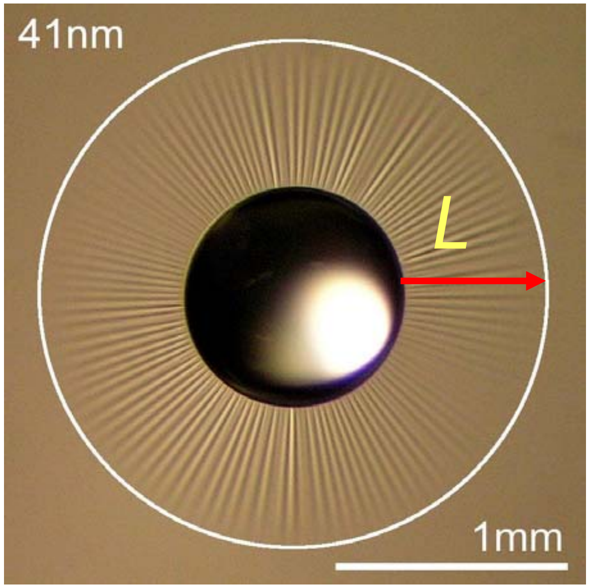
Measure:

Wavenumber, *N*

Length, *L*

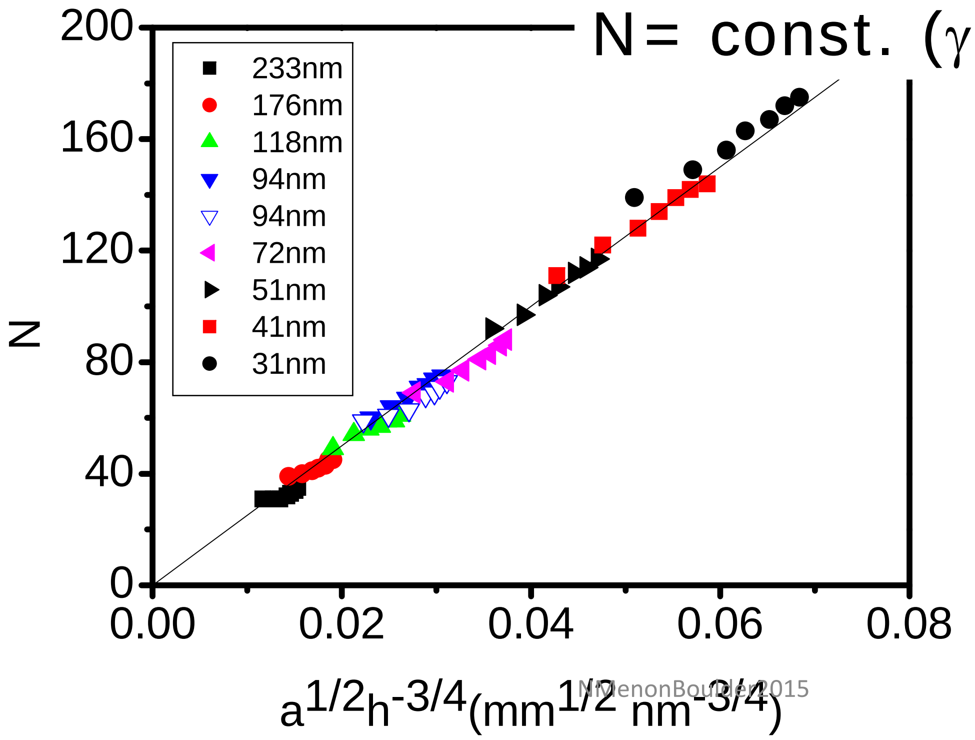
Dependence on

- elasticity of sheet
  - thickness, *t*,
  - Young's Modulus, *E*
- loading
  - radius of drop, *a*
  - surface tension, $\gamma$



# 2D wrinkling – wrinkle number

Standard (post-buckling) analysis captures dependence on drop size, film thickness

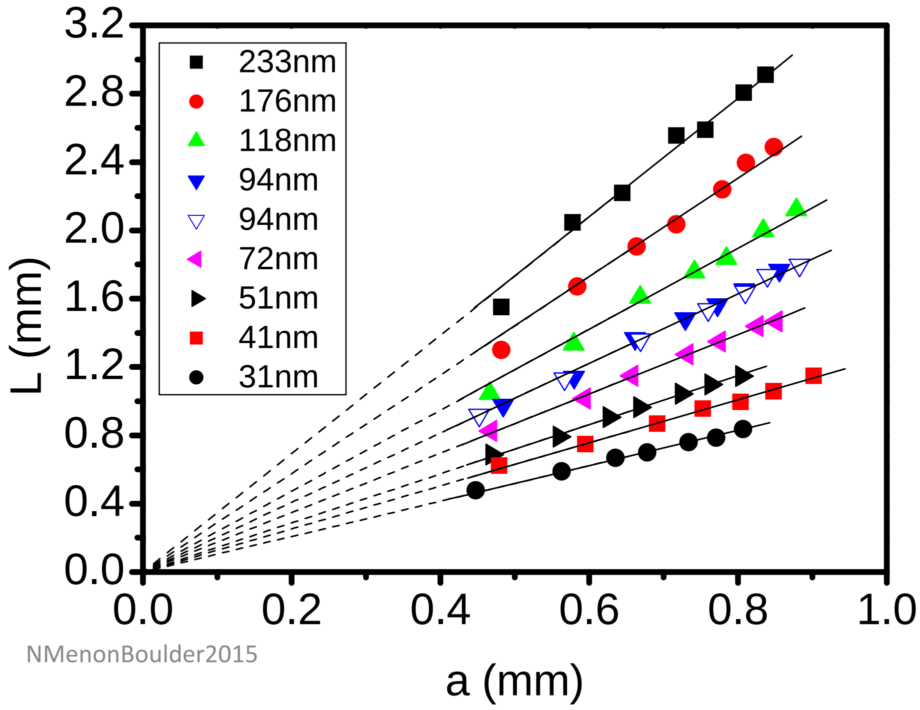


$N= const.\ (\gamma/B)^{1/4}a^{1/2}$

# Length of wrinkles

Scaling L ~ a (post-buckling) found in Cerda 2005

*L* increases with *a*, but thickness dependence, too

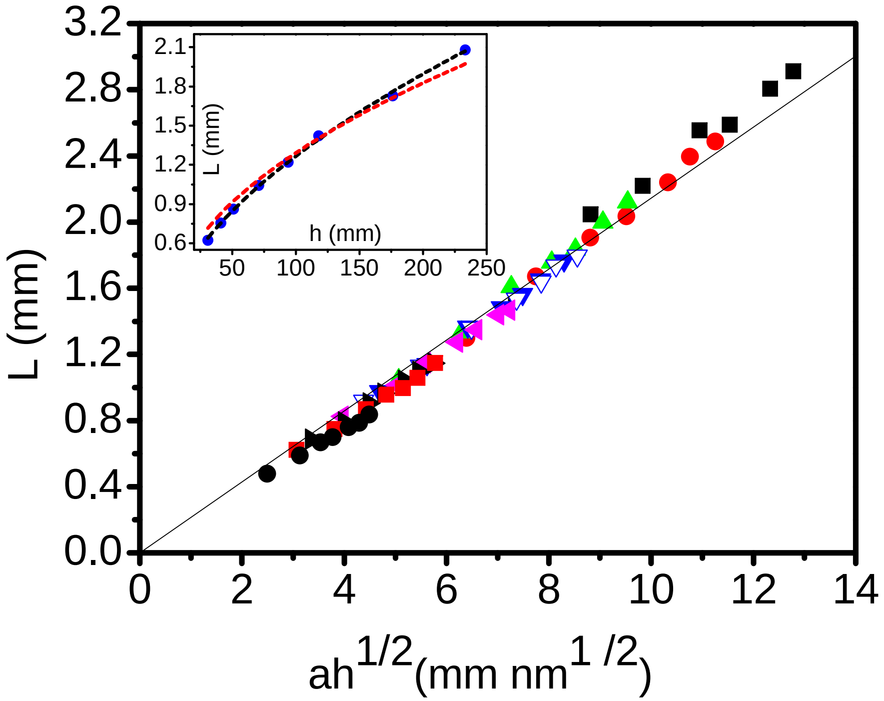


NMenonBoulder2015

# 2D wrinkling - length

Postbuckling scaling does not work $L \sim a$ *e.g. Cerda J. Biomech 2005*

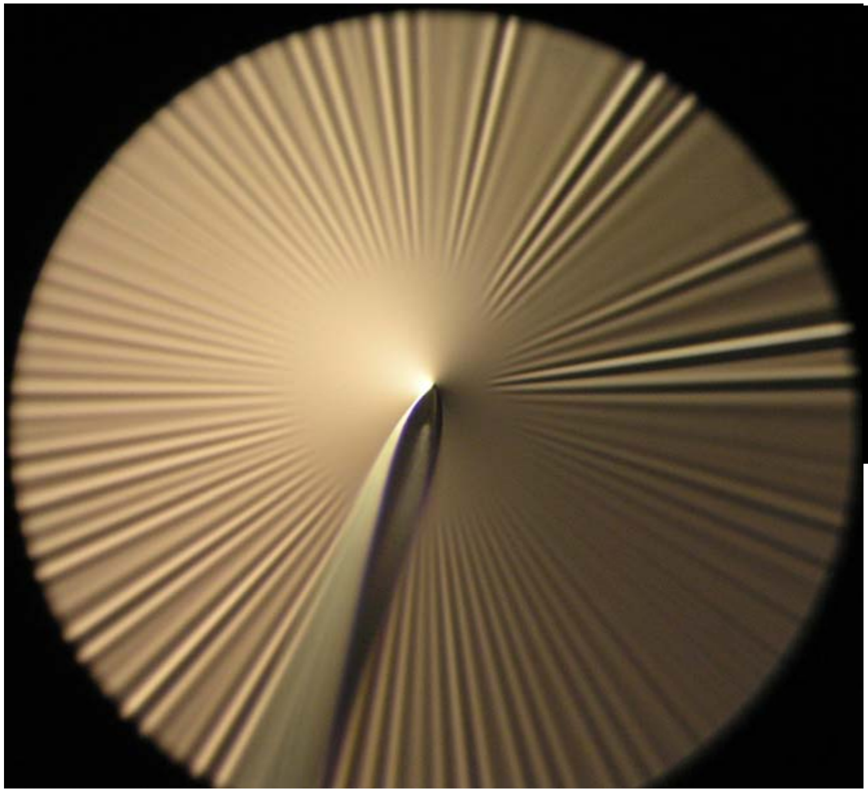
Data approximately collapsed by $L \sim a\, t^{1/2}$



Other variables available to fix dimensions: $E, \gamma$

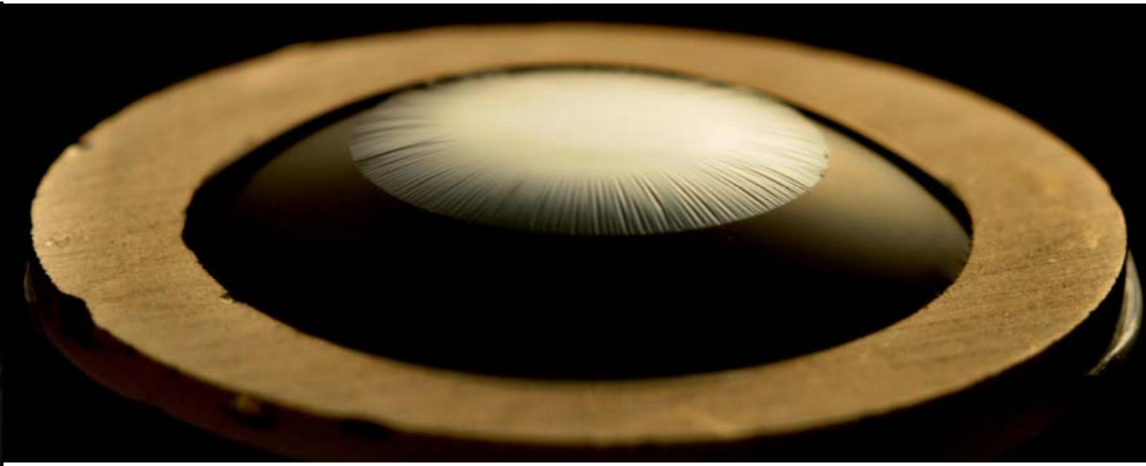
Only possible combination: $L = C\, a\, t^{1/2} (E/\gamma)^{-1/2}$

# Other axisymmetric geometries



- Poking – negative Gaussian curvature

Vella, Huang, etc 2015

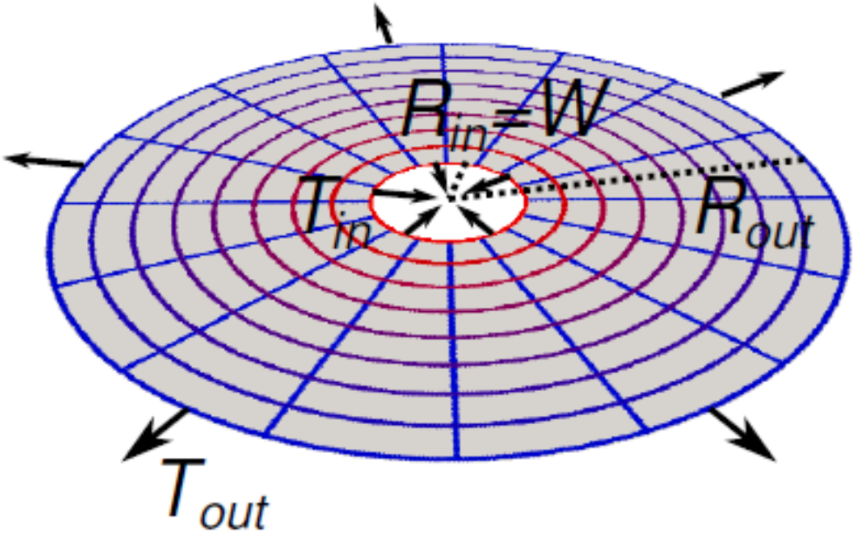


- Sheet on drop – positive Gaussian curvature

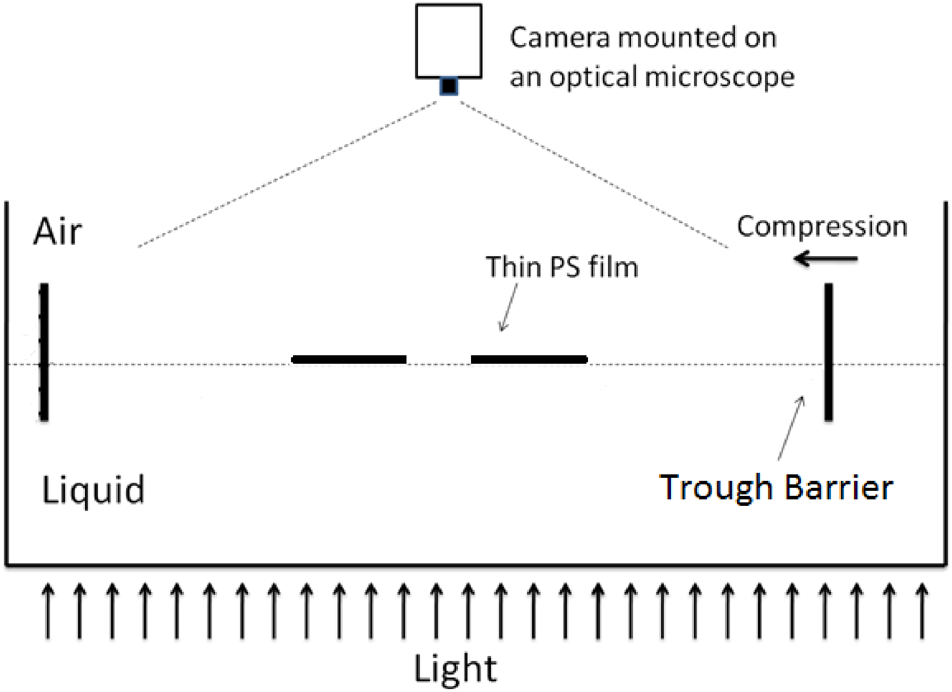
King, Schroll, etc PNAS 2012

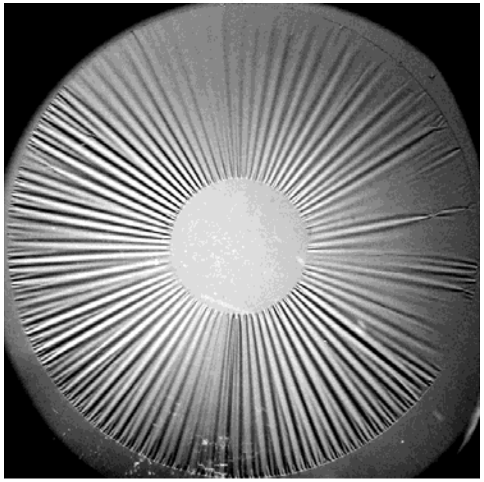
Postbuckling analysis fails to describe these situations as well

# Lamé problem



Davidovitch, et al PNAS 2011





KB Toga thesis

## 2D axisymmetric problem

Föppl - von Karman $\quad \partial_i \sigma_{ij} = 0 \quad$ in-plane.

$$B \nabla^2 (\mathrm{Tr}\, \mathcal{K}) + \sigma_{ij} \mathcal{K}_{ij} = f_{normal} \quad \text{out-of-plane.}$$

---

Hooke's Law

$$\sigma_{rr} = \frac{Y}{1-\Lambda^2} \left( \epsilon_{rr} + \Lambda \epsilon_{\theta\theta} \right)$$

$$\sigma_{\theta\theta} = \frac{Y}{1-\Lambda^2} \left( \epsilon_{\theta\theta} + A \epsilon_{rr} \right)$$

---

Strain $\quad \vec{u}(\vec{r},\theta) = u_r(r,\theta)\,\hat{r} + u_\theta(r,\theta)\,\hat{\theta} + \zeta(r,\theta)\,\hat{z}$

$$\epsilon_{rr} = \partial_r u_r + \frac{1}{2} (\partial_r \zeta)^2$$

$$\epsilon_{\theta\theta} = \frac{1}{r} u_r$$

$$\epsilon_{r\theta} = 0$$

---

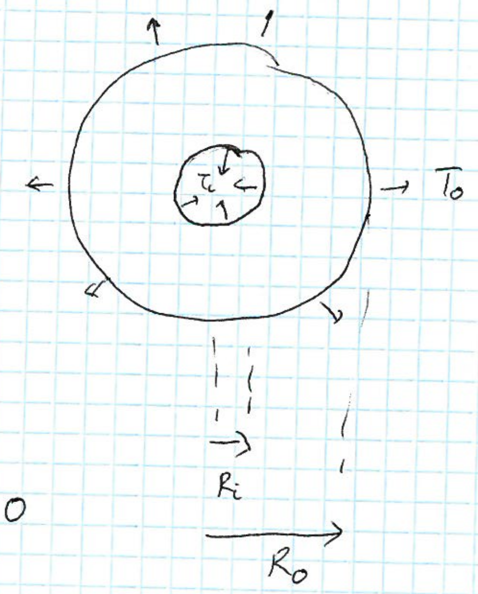
F v K

in-plane
$$\begin{cases} \partial_r \sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 & \text{radial.} \\ \partial_\theta \sigma_{\theta\theta} = 0 & \text{azimuthal.} \end{cases}$$

normal
$$B \Delta^2 \zeta - \sigma_{rr} \partial_r^2 \zeta - \frac{1}{r^2} \sigma_{\theta\theta} \left( \partial_\theta^2 \zeta + r \partial_r \zeta \right) = F_N$$

I have dropped $\epsilon_{r\theta}$ and $\sigma_{r\theta}$ in these equations as there is no shear

## Lamé Problem



$T_o = \sigma_{rr}(R_o) \qquad T_i = \sigma_{rr}(R_i)$

Planar problem first i.e.

$\xi = 0$

FvK → $\partial_r \sigma_{rr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0$

$\partial_\theta \sigma_{\theta\theta} = 0$

[Solve for H.W.]

use $R_o \to \infty$

$$\sigma_{rr} = T_o - (T_o - T_i)\frac{R_i^2}{r^2} = T_o\left[1 - (1-\tau)\frac{R_i^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = T_o + (T_o - T_i)\frac{R_i^2}{r^2} = T_o\left[1 + (1-\tau)\frac{R_i^2}{r^2}\right] \qquad \tau = \frac{T_i}{T_o}$$

For $\tau > 2$ you get azimuthal compression for $R_i < r < l$

Such that $1 + (1-\tau)\frac{R_i^2}{l^2} = 0$

$\Rightarrow l^2 = -(2-\tau) R_i^2$

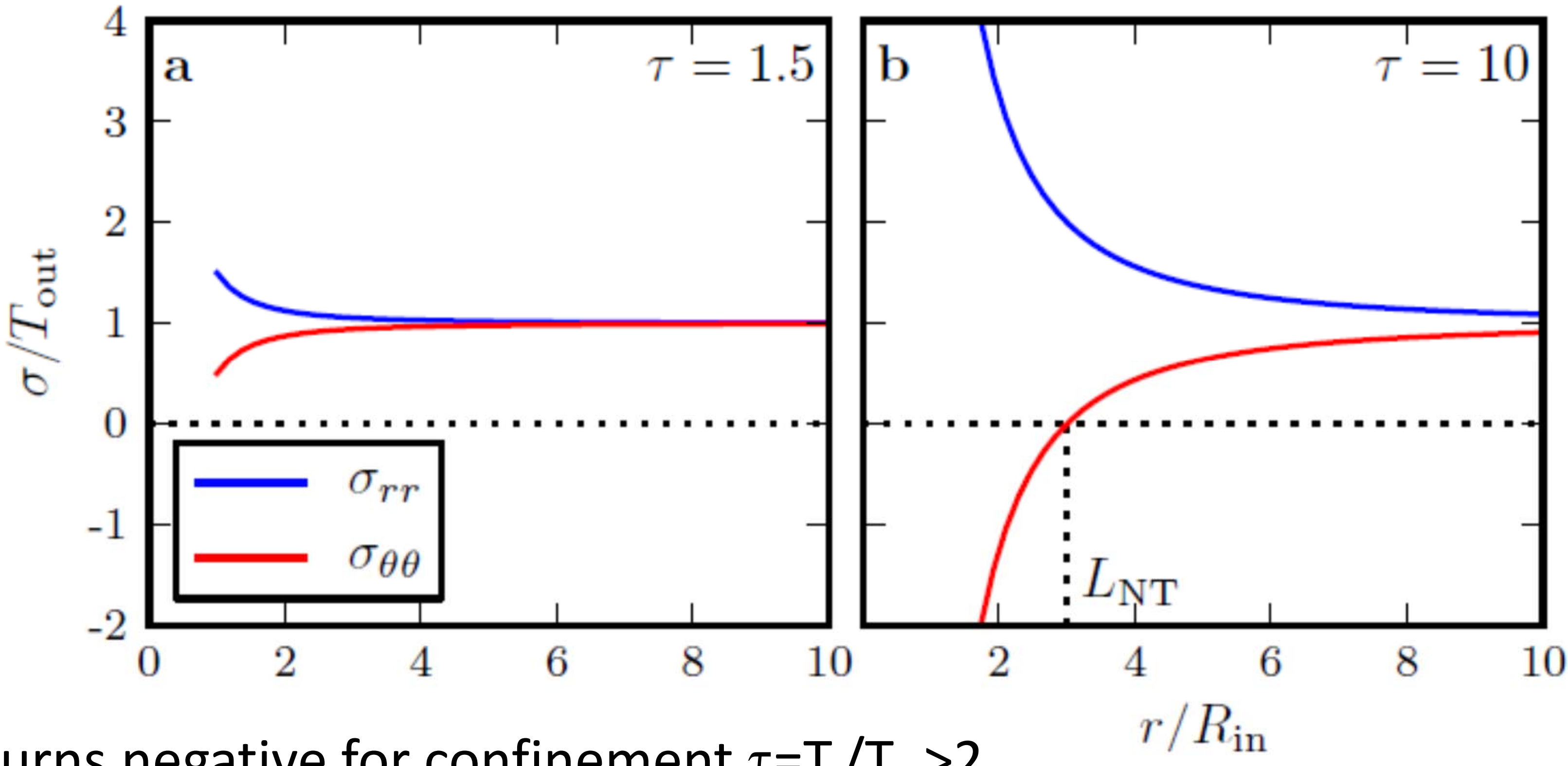
$\Rightarrow l = R_i\sqrt{\tau - 2}$

——— ∘ ——————— ∘ ———————

Compression leads to buckling only above a finite threshold that depends on $B$. $l$ is the answer only for $B = 0$

# Lamé solution

Davidovitch, et al PNAS 2011



- Azimuthal stress turns negative for confinement $\tau=T_i/T_o > 2$

Thus, you need two parameters to describe 2D wrinkling

① Confinement parmeter $\tau = \frac{T_i}{T_o}$

and you need a new dimensionless parameter that encodes when you buckle. This has been called the bendability

② $\epsilon^{-1}$ where $\epsilon = \frac{B}{R_i^2 T_o}$

[Contrast with 1D, where you needed only 1 parameter, $\frac{\Delta}{\Delta_c}$, where the mechanics info fell out]

Large bendability, $\epsilon^{-1} \gg 1$, sheet is very floppy.

Lecture 2 references:

1D folds

Pocivavsek, L., Dellsy, R., Kern, A., Johnson, S., Lin, B., Lee, K. Y. C., & Cerda, E. (2008). Stress and fold localization in thin elastic membranes. *Science*, *320*(5878), 912-916.

Diamant, H., & Witten, T. A. (2011). Compression induced folding of a sheet: An integrable system. *Physical review letters*, *107*(16), 164302.

Démery, V., Davidovitch, B., & Santangelo, C. D. (2014). Mechanics of large folds in thin interfacial films. *Physical Review E*, *90*(4), 042401.

2D wrinkling

Huang, J., Juszkiewicz, M., De Jeu, W. H., Cerda, E., Emrick, T., Menon, N., & Russell, T. P. (2007). Capillary wrinkling of floating thin polymer films. *Science*, *317*(5838), 650-653.

Davidovitch, B., Schroll, R. D., Vella, D., Adda-Bedia, M., & Cerda, E. A. (2011). Prototypical model for tensional wrinkling in thin sheets. *Proceedings of the National Academy of Sciences*, *108*(45), 18227-18232.

King, H., Schroll, R. D., Davidovitch, B., & Menon, N. (2012). Elastic sheet on a liquid drop reveals wrinkling and crumpling as distinct symmetry-breaking instabilities. *Proceedings of the National Academy of Sciences*, *109*(25), 9716-9720.

The last two papers --particularly the supplementary info of the 2012 PNAS -- are good resources to follow up my blackboard notes