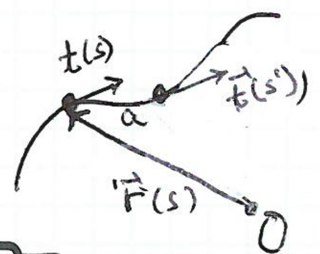
David Nelson 7/[illegible]/15

POLYMERS, MEMBRANES, & SPIN SYSTEMS

Derive Alex Levine's results by mapping onto a spin system

* Describe linear polymer by its tangent $\vec{t}(s)$ as a f'n of arclength s



$\vec{t}(s) = \frac{d\vec{r}(s)}{ds}$, discretize: $\vec{t}_j = \vec{t}(s_j)$, $s_j = ja$

$|\vec{t}(s)| = 1$; * regard $\vec{t}_j$ as a classical 3-component unit spin vector with energy

$$\boxed{\mathcal{H} = -J\sum_{j=1}^{N}(\vec{t}_j \cdot \vec{t}_{j+1}) = -JN + \frac{J}{2}\sum_{j=1}^{N}(\vec{t}_{j+1} - \vec{t}_j)^2}$$

"Heisenberg Model"

* Take continuum limit of this fictitious spin system...

$$\mathcal{H} = \text{const} + \frac{Ja^2}{2}\int_0^L \left(\frac{d\vec{t}(s)}{ds}\right)^2 \frac{ds}{a} = \text{const} + \frac{Ja}{2}\int_0 \left(\frac{d^2\vec{r}(s)}{ds^2}\right)^2 ds$$

* compare to the bending free energy of a polymer

$$\boxed{F = \frac{1}{2}\kappa\int_0^L \left(\frac{d^2\vec{r}(s)}{ds^2}\right)^2 ds} \Leftrightarrow$$

two systems same in the continuum limit with $\kappa = Ja$!

* For 1d Heisenberg model, no phase transition at any finite temperature, with $\boxed{\langle \vec{t}(s) \cdot \vec{t}(s') \rangle = e^{-|s-s'|/\xi(T)}}$

no long range order, with $\boxed{\xi(T) \sim \ell_p(T) \sim \kappa/T}$ ✓

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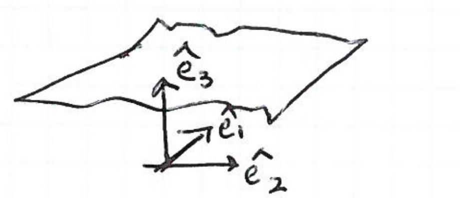
* What about liquid membranes described by a bending energy

$$F_b = \frac{1}{2}\kappa\int d^2x \left[\frac{1}{R_1(\underline{x})} + \frac{1}{R_2(\underline{x})}\right]^2, \quad \vec{r}(x_1, x_2) \approx x_1\hat{e}_1 + x_2\hat{e}_2 + f(x_1, x_2)\hat{e}_3$$

$$\Rightarrow \boxed{F_b \approx \frac{1}{2}\kappa\int d^2x \left[\nabla^2 f(\underline{x})\right]^2}$$



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NORMAL-normal correlation of a membrane

$\partial_1 f = \frac{\partial f}{\partial x_1}$

$\partial_2 f = \frac{\partial f}{\partial x_2}$

* tangents are $\vec{t}_1 = \hat{e}_1 + (\partial_1 f)\hat{e}_3$, $\vec{t}_2 = \hat{e}_2 + (\partial_2 f)\hat{e}_3$

$$\boxed{\hat{n}(\underset{\sim}{x}) = \frac{\vec{t}_1 \times \vec{t}_2}{|\vec{t}_1 \times \vec{t}_2|} = \frac{1}{\sqrt{1+|\vec{\nabla} f|^2}} \begin{pmatrix} -\partial_1 f \\ -\partial_2 f \\ 1 \end{pmatrix}}$$

$\underset{\sim}{x} = (x_1, x_2)$

* Assume <u>semi-flexible polymer like limit</u> with $\hat{n}(\underset{\sim}{x}) \approx \begin{pmatrix} -\partial_1 f \\ -\partial_2 f \\ 1 - \frac{1}{2}|\vec{\nabla} f|^2 \end{pmatrix} + \mathcal{O}(f^3)$

$$\langle \hat{n}(\underset{\sim}{x}_a) \cdot \hat{n}(\underset{\sim}{x}_b) \rangle \approx 1 - \frac{1}{2}\langle |\vec{\nabla} f(\underset{\sim}{x}_a)|^2 \rangle - \frac{1}{2}\langle |\vec{\nabla} f(\underset{\sim}{x}_b)|^2 \rangle$$
$$+ \langle \partial_1 f(\underset{\sim}{x}_a)\partial_1 f(\underset{\sim}{x}_b) \rangle + \langle \partial_2 f(\underset{\sim}{x}_a)\partial_2 f(\underset{\sim}{x}_b) \rangle$$

* Introduce Fourier modes to get, with $f(\underset{\sim}{x}) = \frac{1}{A}\sum_{\underset{\sim}{q}} f(\underset{\sim}{q}) e^{i \underset{\sim}{x} \cdot \underset{\sim}{q}}$

$$\boxed{\langle \hat{n}(\underset{\sim}{x}_a) \cdot \hat{n}(\underset{\sim}{x}_b) \rangle = 1 - \int \frac{d^2 q}{(2\pi)^2} q^2 \langle |f(\underset{\sim}{q})|^2 \rangle \left[1 - e^{i \underset{\sim}{q} \cdot (\underset{\sim}{x}_a - \underset{\sim}{x}_b)}\right]}$$

but $F_b = \frac{1}{2}\kappa \int (\nabla^2 f)^2 d^2 x \Rightarrow \langle |f(\underset{\sim}{q})|^2 \rangle = \frac{k_B T}{\kappa q^4}$, Thus

$$\langle \hat{n}(\underset{\sim}{x}_a) \cdot \hat{n}(\underset{\sim}{x}_b) \rangle = 1 - \int \frac{d^2 q}{(2\pi)^2} \frac{k_B T}{\kappa q^2} \left[1 - e^{i \underset{\sim}{q} \cdot (\underset{\sim}{x}_a - \underset{\sim}{x}_b)}\right]$$

$$\approx 1 - \frac{k_B T}{2\pi\kappa} \ln(|\underset{\sim}{x}_a - \underset{\sim}{x}_b|/a)$$

goes negative!

$\Rightarrow$ $\boxed{\text{no long range order for } |\underset{\sim}{x}_a - \underset{\sim}{x}_b| \gtrsim a\, e^{\frac{2\pi\kappa}{k_B T}}}$

Just like a Heisenberg spin system! $\boxed{\xi \sim e^{2\pi\kappa/k_B T}}$

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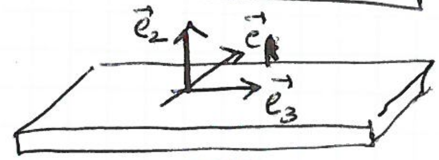
BENDING & TWISTING MODES OF A RIBBON

LIMITING CASES



$$\boxed{F = \frac{1}{2}\int ds\,\left[A_1\Omega_1^2 + A_2\Omega_2^2 + C\Omega_3^2\right]}$$

Case I $A_1 = A_2 = A$; then $A_1\Omega_1^2 + A_2\Omega_2^2 = A(\Omega_1^2 + \Omega_2^2)$

but $\left(\frac{d\vec{e}_3}{ds}\right) = \vec{\Omega}\times\vec{e}_3 \Rightarrow (\partial_s\vec{e}_3)^2 = |\vec{\Omega}|^2 - (\vec{e}_3\cdot\vec{\Omega})^2$

$= \Omega_1^2 + \Omega_2^2$, so

Integrate out

$$\boxed{F = \frac{1}{2}\int ds\left[A\left(\frac{d\vec{e}_3}{ds}\right)^2 + C\Omega_3^2\right] \equiv \text{1d Heisenberg spin model}}$$

Case II $C \to \infty$, $A_2 \to \infty$ No twist, no splay

Then $\Omega_2 = \Omega_3 = 0 \Rightarrow \frac{d\vec{e}_1}{ds} = \begin{pmatrix}\Omega_1\\0\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix} = 0$

$$\frac{d\vec{e}_2}{ds} = \begin{pmatrix}\Omega_1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\\Omega_1\end{pmatrix} \Longleftrightarrow \Omega_1^2 = \left(\frac{d\vec{e}_2}{ds}\right)^2$$

$$\boxed{F = \frac{1}{2}\int A_1\left(\frac{d\vec{e}_2}{ds}\right)^2, \quad \vec{e}_2\times\vec{e}_1 = 0, \quad \vec{e}_1 = \text{const.} \qquad \equiv \text{1d XY spin model}}$$