Filaments and Membranes

Overview

The key concepts involve the combination of elasticity and statistical mechanics. Why? The answer involves both typical energy scales and length scales in the problem. The second point really has two issues.

① We will study things composed of many atoms — a description of the system in terms of atomic coordinates is not only inelegant but needlessly confusing. Consider as an example filamentous actin. If we want to understand their networks (Fig 1) or even single filaments (Fig 2), all atom descriptions ~~and~~ (Fig 3) of the monomers are not useful, except to provide a few parameters for a higher-level, elastic description.

atomic interactions $\Longrightarrow$ elastic moduli
and coordinates $\qquad\qquad\qquad (E, \sigma)$

$\{\vec{r}_i\}\ i=1,\ldots,N \gg 1$

atomic level | elastic descriptions

Fig 1 – Filamentous Actin in cells.

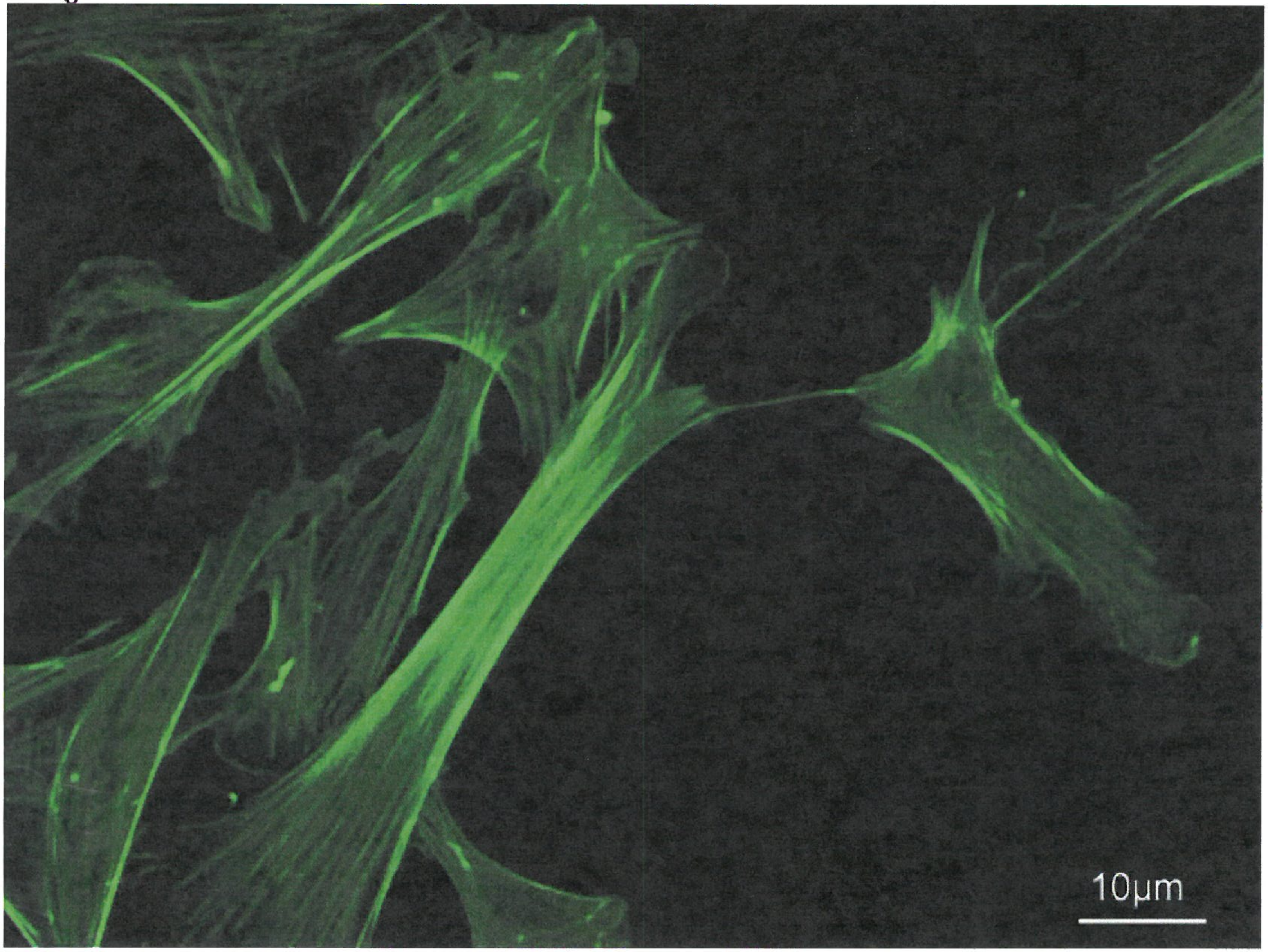


Fig 2 - all atomic
model of a single
F-actin strand.

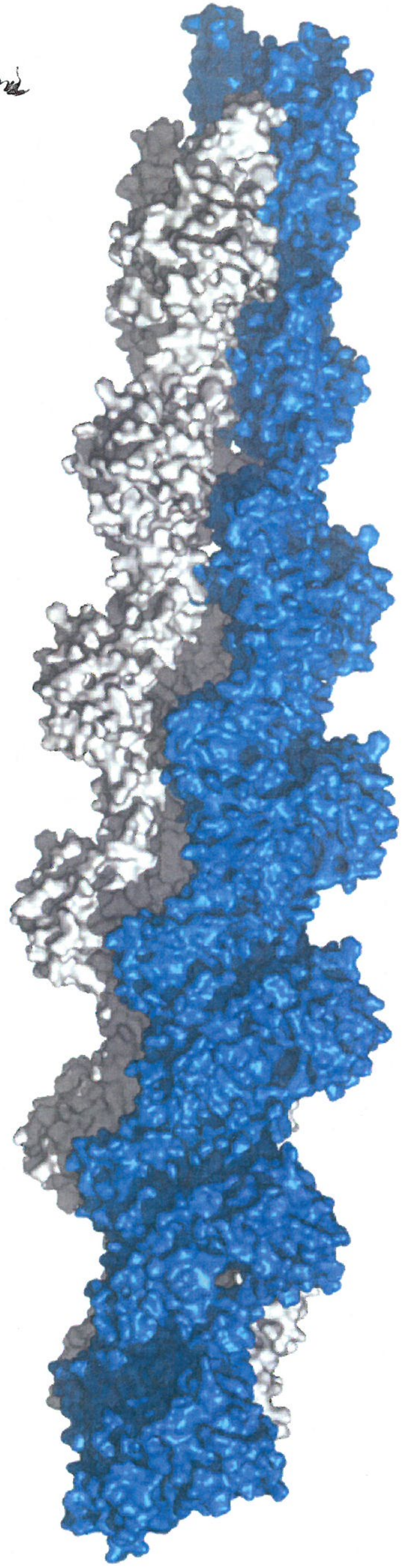
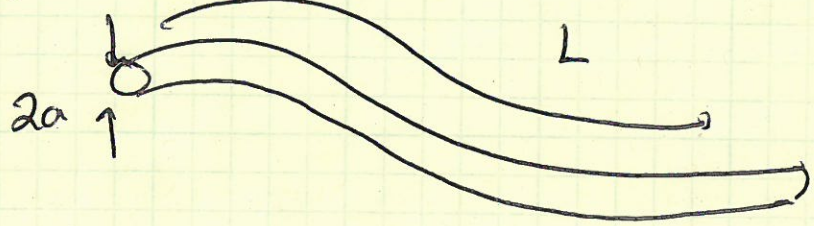
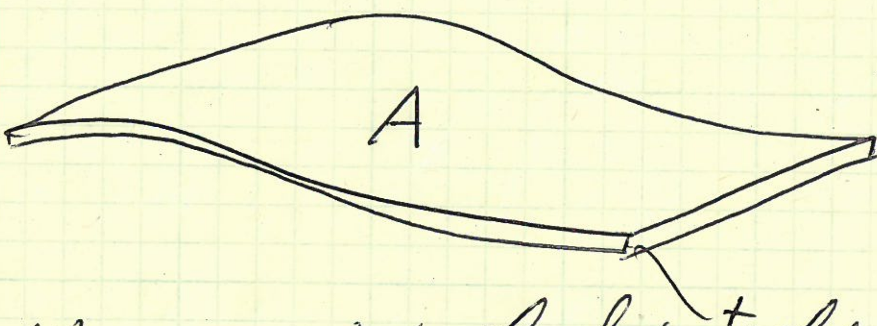


Fig 3 – A so-called "ribbon" diagram" of an actin monomer w/ bound GDP.

There is a related pt that informs our model of filament and membrane elasticity.



$L \gg a$



$t \ll \sqrt{A}$

We can exploit the fact that there are one or two (membrane and filament respectively) small dimensions

So our goal will be as follows

$$\underbrace{(\text{atomic interaction} \Longrightarrow \text{elastic constants})}_{\text{not covered here.}} \Longrightarrow \begin{pmatrix}\text{elasticity} + \\ \text{geometry}\end{pmatrix} \Longrightarrow$$

effective elastic model of a filament or membrane.

Finally, we address energy scales. For sufficiently small a or t (see above) the ~~bending energy~~ deformation energy $\sim$ thermal energy. This implies ~~what~~ that we need to account for thermal fluctuations of these structures in order to treat their mechanics / dynamics on cellular ~~and~~ (and subcellular) scales.

Outline of lecture.

I Filaments

A. Mechanics of a rod

1. Torsion
2. Bending

} Elasticity and geometry ⇒ mechanics of a "line"

3. Energy of a deformed rod and bend / twist coupling.

4. T=0 mechanics and equations of equilibrium

5. Places where local internal degrees of freedom matter — local melting and the HCWLC.

a. Using DNA to pull on proteins

B. Statistical mechanics of a filament.

1. Persistence length and diffusion on a sphere
2. Tangent vectors and the end-to-end distance

a. Flexible limit and Kuhn length
b. Stiff limit of a stick.

3. Small bending limit and stiff chains — preparation for semiflexible networks.

d. Semiflexible Networks.

1. Pulling on a WLC (Worm-Like Chain)

2. Network mechanics

II. Membranes.

A. mechanics of a thin plate

1. bending a plate and internal stresses/strains

2. energy density of a bent plate

a. The bending modulus.

b. Bulk versus surface terms.

3. T=0 mechanics and the equations of equilibrium

B. Another look:

1. A quick taste of the differential geometry of surfaces.

2. mean and Gaussian curvature.

3. The Helfrich Hamiltonian

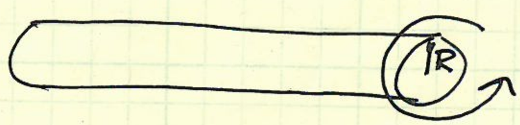
C. Statistical mechanics of surfaces.

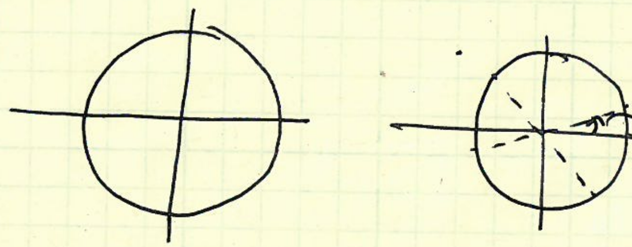
1. Undulations at $T \neq 0$ and the de Gennes Taupin Length.

I. Mechanics of Rods.

dsDNA, F-actin, and so on...

A. Torsion:





$z=0$ $\qquad$ $z=\varepsilon$

$$\frac{d\varphi}{dz}\varepsilon = \tau\varepsilon \quad \leftarrow \text{Rotation angle}$$

$$\boxed{\tau = \frac{d\varphi}{dz}}$$

We will assume that $\tau R \ll 1$ so strains are small

How do points move in a rod cross section as $z \to z + \delta z$?

$$\vec{r} \longrightarrow \vec{r} + \underbrace{\delta\vec{\varphi} \times \vec{r}}_{\delta\vec{r}}$$

$\Longrightarrow \quad u_x = -\tau z y$ and $u_y = -\tau z x$

From $\quad u_\alpha = (\delta\vec{r})_\alpha = z\tau(\hat{z}\times\vec{r})_\alpha \quad \leftarrow$ required by condition of twist.

$\alpha = x, y$

What about u_z?

$$u_z = \tau\,\psi(x,y) \quad \leftarrow \text{Torsion function}$$

must vanish as $\tau \to 0$

From $u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha)$

we have $u_{xx} = u_{yy} = u_{zz} = 0$

$u_{xy} = 0$

But $u_{xz} = \frac{1}{2}\tau\left(\frac{\partial\psi}{\partial x} - y\right)$; $u_{yz} = \frac{1}{2}\tau\left(\frac{\partial\psi}{\partial y} + x\right)$

Note: $\bar{\nabla}\cdot\bar{u} = \partial_\alpha u_\alpha = 0$ no volume change.

From $\sigma_{ik} = 2\mu\left[u_{ik} + \frac{\sigma}{1-2\sigma} u_{ll}\,\delta_{ik}\right]$

Poisson Ratio

$\sigma_{xx} = 0$ and $\sigma_{xy} = 0$
yy
zz

But $\sigma_{xz} = \mu\tau\left(\frac{\partial\psi}{\partial x} - y\right)$; $\sigma_{yz} = \mu\tau\left(\frac{\partial\psi}{\partial y} - x\right)$

Static Equilibrium $\Rightarrow$ $\partial_\alpha \sigma_{\alpha\beta} = 0 \Rightarrow$

$$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = 0 \Rightarrow \nabla_\perp^2 \psi = 0$$

where $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$

define $\chi(x,y)$ such that $\sigma_{xz} = 2\mu\tau\,\partial_y \chi$

$\sigma_{yz} = -2\mu\tau\,\partial_x \chi$

so $\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = 0$ is taken care of automatically.

Now $\partial_x \psi = y + 2\,\partial_y \chi$; $\partial_y \psi = -x - 2\,\partial_x \chi$

Using $\partial_x \partial_y \psi = \partial_y \partial_x \psi$

$\Rightarrow$

$$1 + 2\,\partial_y^2 \chi = -1 - 2\,\partial_x^2 \chi$$

$$\Rightarrow \quad \nabla_\perp^2 \chi = -1$$

What are the boundary conditions on the surface of the rod?

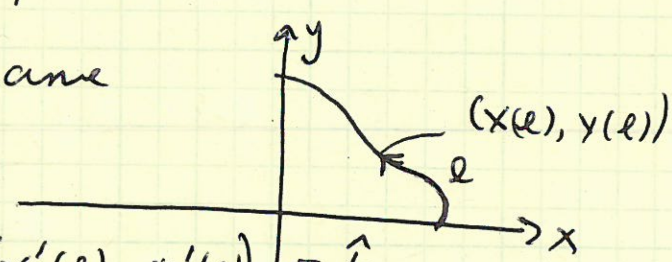
$$\sigma_{ik}\,\hat{n}_k = 0$$

So $\sigma_{zx}\,\hat{n}_x + \sigma_{zy}\,\hat{n}_y = 0$ Note ① $i = x, y$ are simple. Recall $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$. ② $\hat{n}_z = 0$

Thus:

$$\partial_y \chi\,\hat{n}_x - \partial_x \chi\,\hat{n}_y = 0$$

Consider the boundary curve



The local tangent is $(x'(l), y'(l)) = \hat{t}$

The local (outward) normal is $\hat{n} = (+y'(l), -x'(l))$

so $$(\partial_x \chi)\frac{dx}{dl} + (\partial_y \chi)\frac{dy}{dl} = d\chi = 0$$

$\Rightarrow$ χ = constant on the boundary. As long as there is only one boundary we can take it to be

$\chi = 0$ on the boundry.

Now what is the energy density (per length) of the twisted rod

over the cross section

$$F = \frac{1}{2}\int \sigma_{ik}\, u_{ik} = \int \left(\sigma_{xz}^2 + \sigma_{yz}^2\right)\frac{1}{2\mu}\, d^2x$$

In terms of χ this is

$$F = 2\mu\tau^2\left(\vec{\nabla}_\perp \chi\right)^2$$

$$\Longrightarrow \quad \underset{\text{Twist}}{F} = \frac{1}{2}\int_0^L C\,\tau^2\, dz \quad \overset{\text{where}}{\Longleftrightarrow} \quad C = 4\mu\int d^2x\, \left(\vec{\nabla}_\perp \chi\right)^2$$

$$\text{But} \quad \left(\vec{\nabla}_\perp \chi\right)^2 = \vec{\nabla}_\perp \cdot \left(\chi \vec{\nabla}_\perp \chi\right) - \chi\left(\vec{\nabla}_\perp^2 \chi\right)$$

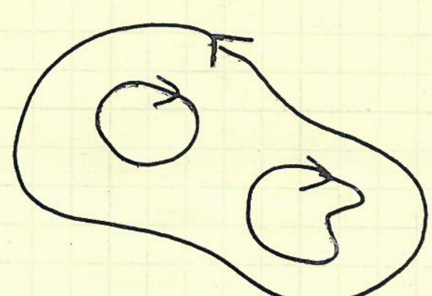
equal to -1!

$$= \vec{\nabla}_\perp \cdot \left(\chi \vec{\nabla}_\perp \chi\right) + \chi$$

$$C = 4\mu\oint \chi\, \hat{n}\cdot\vec{\nabla}_\perp \chi\, dl + 4\mu\int d^2x\, \chi$$

boundry term

w/ holes we get

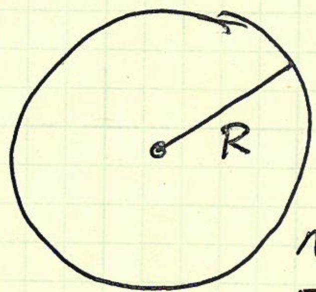


$$C = 4\mu\sum_k \chi_k S_k + 4\mu\int d^2x\, \chi.$$

So we know the following: (assuming C is independent of z)

$$F_{\text{Twist}} = \frac{1}{2} C \int \tau^2 \, dz$$

and for a circular rod cross section.



$$\chi = (-x^2 - y^2 + R^2) A$$

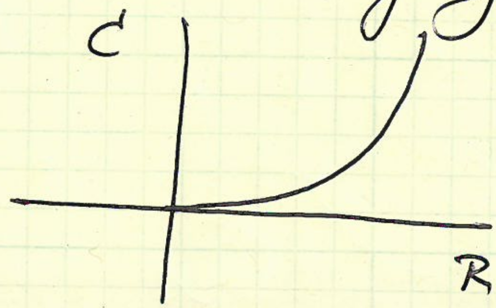
need $\nabla_\perp^2 \chi = -1 \Rightarrow A = 1/4$

$$C = 4\mu \int_0^R r\,dr \int_0^{2\pi} d\varphi \, \frac{(R^2 - r^2)}{4}$$

$$C = \frac{4\mu}{4} 2\pi \cdot \left[R^2 \frac{r^2}{2} - \frac{1}{4} r^4 \right]_0^R = 2\pi\mu \left(\frac{R^4}{2} - \frac{R^4}{4} \right)$$

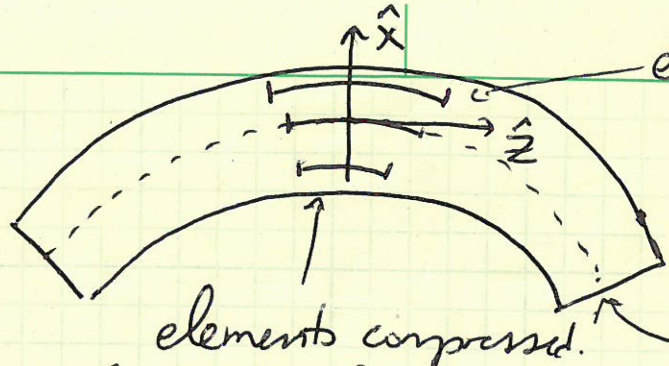
$$C = \frac{1}{\cancel{4}2} \pi \mu R^4$$

torsional rigidity $\propto R^4$



B. Bending

We work in the same way. To start consider a planar bend...



R = radius of curvature of the neutral surface

Original (undeformed) length dz

dz' is the arc of a circle

$$dz' = \frac{R+x}{R} dz = \left(1 + \frac{x}{R}\right) dz$$

$\Longrightarrow$ recall

$$dl'^2 = dl^2 + 2u_{ik}\, dx_i\, dx_k$$

$$\Longrightarrow \quad 2u_{zz} = 2\frac{x}{R} \Longrightarrow \boxed{u_{zz} = x/R} \text{ for bending}$$

Now $\quad \sigma_{zz} = \dfrac{Ex}{R} \qquad E = \dfrac{9K\mu}{3K+\mu}$ or $E = \dfrac{2\mu}{1+\sigma}$

Young's modulus

Now over the cross section of the rod

$$\int \sigma_{zz}\, d^2x = 0$$ no tension being applied. pure bending

$$\Longrightarrow \int x\, d^2x = 0.$$ $\Longleftarrow$ implies the neutral surface has coordinate of the center of mass.

Now all $u_{\alpha\beta}=0$ except

$u_{zz} = x/R$ and $u_{xx} = u_{yy} = -\sigma u_{zz}$

What about the displacement field?

$$u_z = \frac{zx}{R} + g(x,y) \qquad u_x = -\frac{\sigma x^2}{2R^2} + h(y,z)$$

$$u_y = \frac{-xy\sigma}{R} + k(x,z)$$

must set to zero!

and $u_{xz}=0 \Rightarrow \partial_x g + \frac{z}{R} + \partial_z h = 0 \qquad \mathrm{I}$

$$u_{yz} = 0 \Rightarrow \partial_y g + \partial_z k = 0 \qquad \mathrm{II}$$

$$u_{xy} = 0 \Rightarrow \frac{-\sigma y}{R} + \partial_x k + \partial_y h = 0 \qquad \mathrm{III}$$

u_y cannot depend on $z \longleftrightarrow$ each slice has the same deformation in y. so $\partial_y g = 0$ and $\partial_z k = 0$ from II

from I $\quad h = -\frac{z^2}{2R} + \tilde{h}(y)$ and $\partial_x g = 0$

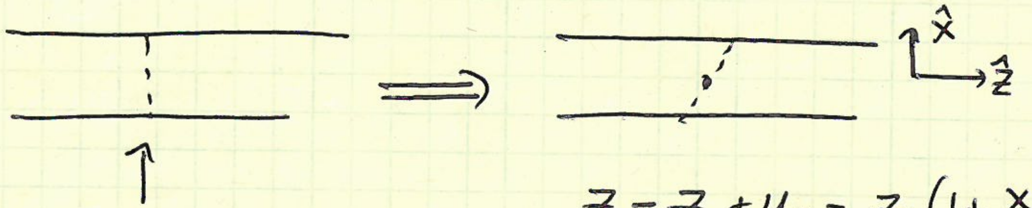
III $\Rightarrow \partial_y \tilde{h} = \frac{y\sigma}{R} \Rightarrow \tilde{h} = \frac{\sigma y^2}{2R}$ and $\partial_x k = 0$

$\Rightarrow$

$$u_x = -\frac{1}{2R}\left[z^2 + \sigma(x^2 - y^2)\right]$$

$$u_y = -\frac{xy\sigma}{R}\,; \quad u_z = xz/R$$

Notes ① each undeformed z-slice tilts under bend.

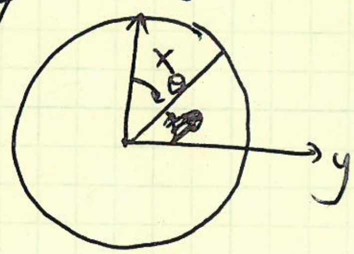


$z = z_0$

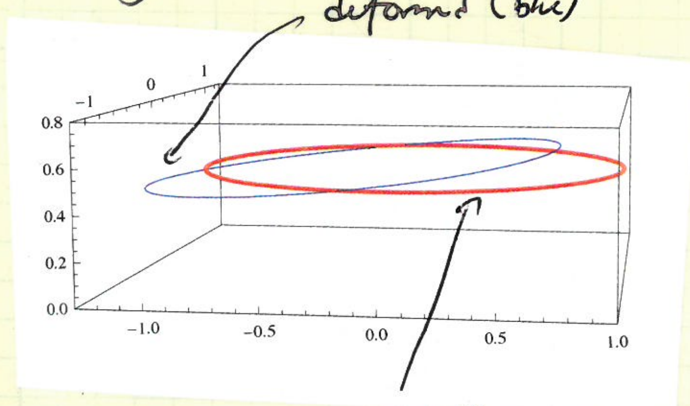
$z = z_0 + u_z = z_0(1 + x/R)$

② Cross sections typically deform under bending. Here is an example.

Consider a slice at $z = .5$ of an undeformed circular rod of radius 1. Its boundary curve is $(\cos\theta, \sin\theta)$



deformed (blue)



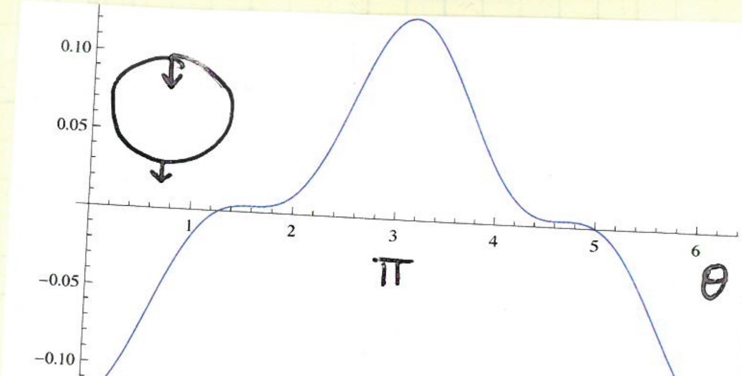
undeformed (red)

Using $\sigma = 1/2$ incompressible.

Now consider the $z=0$ slice. No tilt.

Look at $h = \dfrac{(x+u_x)^2 + (y+u_y)^2 - a^2}{a^2}$ on the boundary

$h \Rightarrow$



More importantly for our purposes, we get the energy per unit length by integrating over a cross section.

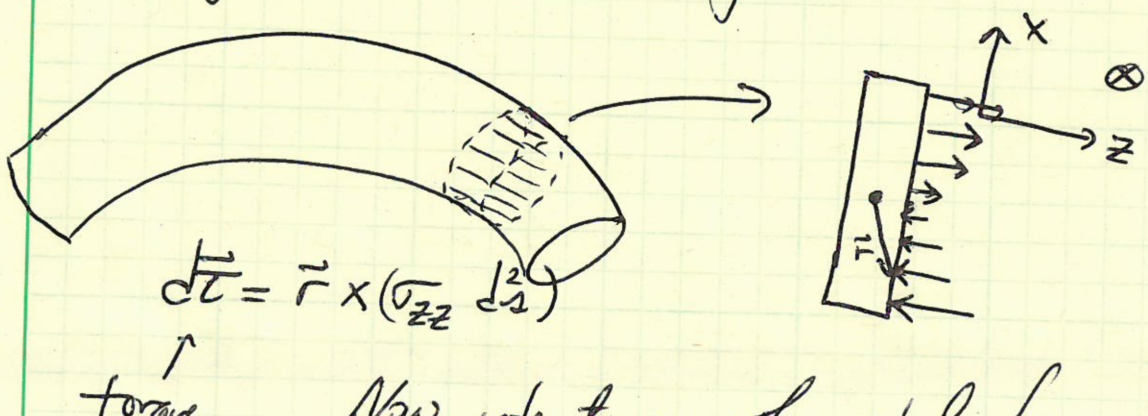
$$\tilde{F} = \frac{1}{2}\int d^2x\, \sigma_{ik} u_{ik} = \frac{1}{2} R^{-2} \int x^2 d^2x$$

Just the moment of inertia tensor (component) in 2d (without the mass density)

$$I_{\alpha\beta} = \int d^2x \left\{ \delta_{\alpha\beta} r^2 - x_\alpha x_\beta \right\} \Rightarrow I_{yy} = \int x^2 d^2x$$

so $$\boxed{\tilde{F} = \frac{1}{2} E I_{yy} R^{-2}}$$

What is the bending moment of the internal stresses on a given cross section of the curved rod?



$$d\vec{\tau} = \vec{r} \times (\sigma_{zz}\, d^2s)$$

torque

Now integrate over the whole face.

$$\tau_y = -\frac{E}{R}\int x^2 d^2x = -\frac{E I_{yy}}{R}$$

torque on the element

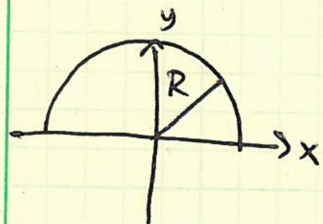
switch sign for moment of force by element

$\tau_x = 0$

What is $I_{yy} = I_{xx} = I_{zz}$ for a circular cross section?

integral over only one quadrant.

$$I_y = \int x^2 \, dA = 4\int_0^R x^2\sqrt{R^2-x^2}\,dx = 4R^4\int_0^1 x^2\sqrt{1-x^2}\,dx$$



$$= \frac{\pi}{16}4R^4 = \frac{1}{4}\pi R^4$$

What does this say about biopolymers?

They are all made of essentially the same stuff so

$$\frac{K_I}{K_{II}} \sim \left(\frac{R_I}{R_{II}}\right)^4$$

bending moduli

In terms of persistence lengths (more on that soon)

$l_p \propto K$

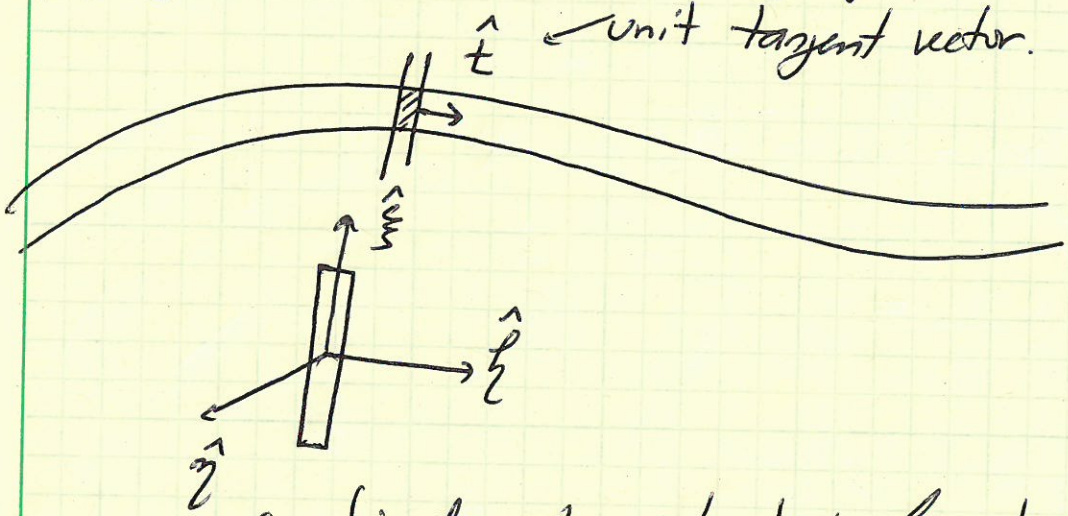
F-actin $\quad l_p = 17\,\mu m \quad$ <u>And</u> $\quad R \sim 7\,nm$

microtubules $\quad l_p \cong 3\,mm$!! $\quad R \sim 25\,nm$

$$\frac{R_{MT}}{R_{FA}} \sim \left(\frac{25}{7}\right)^4 = 163 \qquad \frac{l_p^{MT}}{l_p^{FA}} \sim \frac{3\times10^3}{17} = 176$$

Better than we have a right to expect!

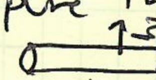
Putting it all together: The energy of a deformed rod.



coordinate system bound to the slice.

Deformation $\Longrightarrow$ series of rotations

$$\vec\Omega = \frac{d\vec\varphi}{dl} \quad ; \quad \vec\Omega \parallel \hat\xi \ \text{ pure twist}$$



$$\vec\Omega \perp \hat\xi \ \text{ pure bending.}$$

How does $\hat t$ move as we go from $l \to l+\delta l$?

$$\frac{d\hat t}{dl} = \vec\Omega \times \hat t \quad \text{or}$$

$$\hat t \times \frac{d\hat t}{dl} = \hat t \times (\vec\Omega \times \hat t) = \vec\Omega - \hat t\,(\hat t \cdot \vec\Omega)$$

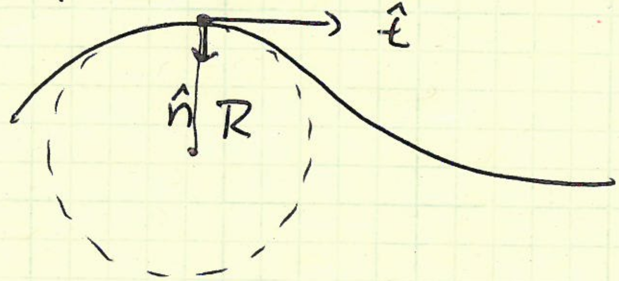
or

$$\vec\Omega = \underbrace{\hat t \times \frac{d\hat t}{dl}}_{\text{bending part.}} + \hat t\,(\hat t \cdot \vec\Omega)$$

This is Ω_ξ the twist angle

Now $\frac{d\hat{t}}{dl} = \frac{1}{R}\hat{n}$ ← principal normal

↗ radius of curvature



$R^{-1} = |d\hat{t}/dl|$

F_{rod}

We can write the energy of the deformed rod as a quadratic function of $\vec{\Omega}$

There can be no terms of the form $\Omega_\xi \Omega_\zeta$
$\Omega_\eta \Omega_\zeta$

Because F_{rod} invariant under $\hat{\zeta} \to -\hat{\zeta}$

$\Rightarrow$ We must have $\frac{1}{2} C \Omega_\zeta^2$ for torsion but in the $\hat{\eta}\hat{\xi}$ plane we have a general bilinear form:

$$\frac{1}{2} E \left\{ I_{\eta\eta} \Omega_\eta^2 + I_{\xi\xi} \Omega_\xi^2 + 2 I_{\eta\xi} \Omega_\eta \Omega_\xi \right\}$$

$$= \frac{1}{2} E \begin{pmatrix} \Omega_\eta & \Omega_\xi \end{pmatrix} \begin{pmatrix} \overleftrightarrow{I} \end{pmatrix} \begin{pmatrix} \Omega_\eta \\ \Omega_\xi \end{pmatrix}$$

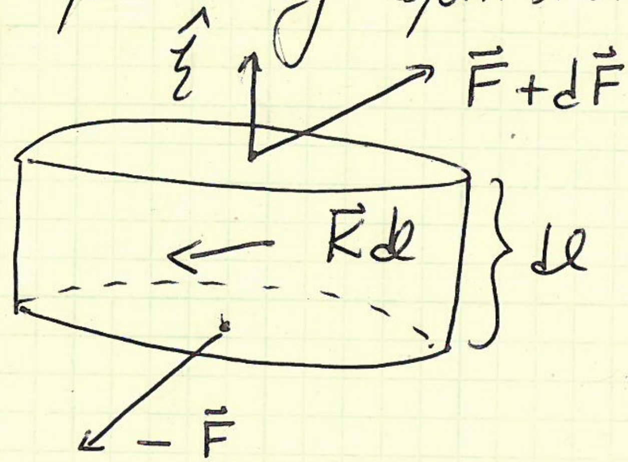
We can diagonalize by working in the principal axis frame of the I.

(13)

$$F_{rod} = \int ds \left\{ \frac{1}{2} I_1 E \Omega_\xi^2 + \frac{1}{2} I_2 E \Omega_\eta^2 + \frac{1}{2} C \Omega_\zeta^2 \right\}$$

Energy of a deformed rod.

The equations of equilibrium for rods.



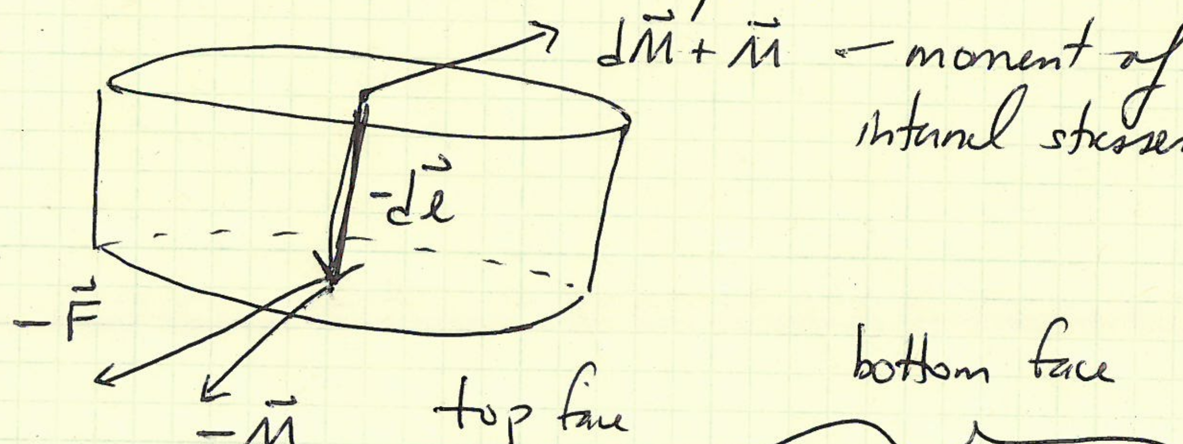
Body force $\vec K$ = force/volume on the slice.

$$F_\alpha = \int \sigma_{\alpha\xi}\, d^2 s$$

Force balance $\Rightarrow$ $\vec F + d\vec F + \vec K dl - \vec F = 0$

$$\Rightarrow \boxed{\frac{d\vec F}{dl} = -\vec K}$$

We also have to balance torques.



Torque balance: $\overbrace{\vec M + d\vec M}^{\text{top face}} + \overbrace{\left[-\vec M + -d\vec l \times -\vec F\right]}^{\text{bottom face}} = 0$

$$d\vec{M} + d\vec{\ell} \times \vec{F} = 0$$

$$\frac{d\vec{M}}{d\ell} = -\hat{t} \times \vec{F} \Rightarrow \boxed{\frac{d\vec{M}}{d\ell} = \vec{F} \times \hat{t}}$$

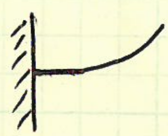
So with no body forces $\vec{F} = \text{const}$ and

$$\frac{d\vec{M}}{d\ell} = \vec{F} \times \frac{d\vec{r}}{d\ell} = \frac{d}{d\ell}(\vec{F} \times \vec{r}) \Rightarrow$$

$$\vec{M} = \vec{F} \times \vec{r}(\ell) = \text{const.}$$

A short discussion on boundary conditions:

① "Clamped"



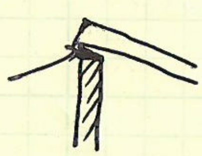
$\vec{r}(0)$ and $\hat{t}(0) = \left.\frac{d\vec{r}}{d\ell}\right|_0$ fixed.

② "hinged"

$\vec{r}(0)$ fixed $\hat{t}(0)$ free

$\vec{M} = 0$

③ "supported" friction free



$\vec{F} \perp \hat{t}$ at $\ell = 0$

$\vec{M} = 0$ at $\ell = 0$

④ "free"

$\vec{F} = 0$, $\vec{M} = 0$ at the end ($\ell = 0$)

Consider a circular cross section so we can eliminate twist bend coupling.

To see this:

$$\Omega_\xi = \hat{t}\cdot\vec{\Omega}$$

Using $\Omega_\xi = \frac{1}{C} M_\xi$

$$\frac{d}{dl}(\vec{M}\cdot\hat{t}) = C\frac{d\Omega_\xi}{dl} = \frac{d\vec{M}}{dl}\cdot\hat{t} + \vec{M}\cdot\frac{d\hat{t}}{dl}$$

Using $\frac{d\vec{M}}{dl} = \vec{F}\times\hat{t}$ we see that vanishes.

$$\Rightarrow \quad C\frac{d\Omega_\xi}{dl} = \vec{M}\cdot d\hat{t}/dl \quad \text{and} \quad \frac{d\hat{t}}{dl} = \vec{\Omega}\times\hat{t}$$

$$C\frac{d\Omega_\xi}{dl} = \vec{M}\cdot(\vec{\Omega}\times\hat{t})$$

and from $M_\xi = EI_1\Omega_\xi$, $M_\eta = EI_2\Omega_\eta$

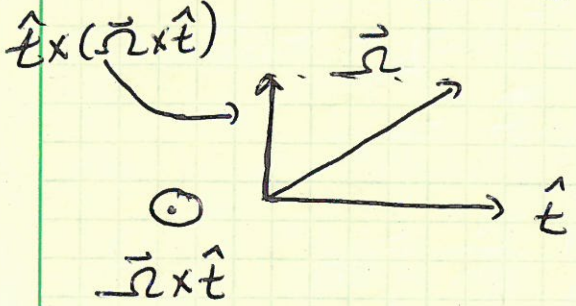
we see that if $I_1 = I_2$ then

$$\vec{M}_\perp = EI\,\vec{\Omega}_\perp$$

part $\perp$ to $\hat{t}$.

In other words.

$$\vec{M} = EI\,\hat{t}\times(\vec{\Omega}\times\hat{t}) + \hat{t}\,C\,\Omega_\xi$$



so $$\boxed{\vec{M} = EI\,\hat{t}\times\frac{d\hat{t}}{dl} + \hat{t}\,C\,\Omega_\xi}$$

Thus (finally)

$$C\frac{d\Omega_\xi}{dl} = EI\,\cancelto{0}{\left(\hat{t}\times\frac{d\hat{t}}{dl}\right)\times\frac{d\hat{t}}{dl}} +$$

$$+\ C\,\Omega_\xi\,\cancelto{0}{\hat{t}\cdot\frac{d\hat{t}}{dl}} = 0$$

since $\frac{d}{dl}(\hat{t}\cdot\hat{t}) = 0$

$\Longrightarrow$ There is no twist bend coupling.

We also have a way to look at the shape of a rod.

$$\vec M = EI\,\hat t \times \frac{d\hat t}{dl} = EI\,\frac{d\vec r}{dl} \times \frac{d^2\vec r}{dl^2} \quad \text{and}$$

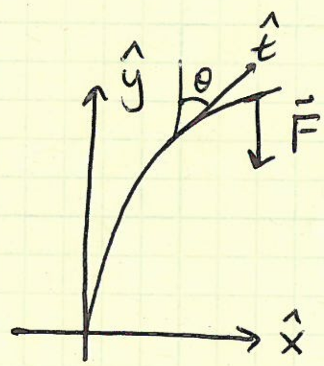
$$\frac{d\vec M}{dl} = \vec F \times \hat t = \vec F \times \frac{d\vec r}{dl}$$

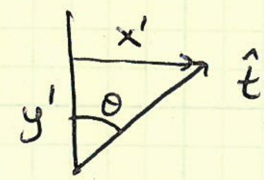
so

$$\boxed{EI\,\frac{d\vec r}{dl} \times \frac{d^3\vec r}{dl} = \vec F \times \frac{d\vec r}{dl}}$$

Eqn of equilibrium for a rod.

Let's try it at:





$$\downarrow \vec F = (0, -F)$$

$$\vec r = (x(l), y(l)) \qquad \hat t = \frac{d\vec r}{dl} = (x', y') = (\sin\theta, \cos\theta)$$

$$\frac{d^3\vec r}{dl} = (\theta''\cos\theta - \theta'^2\sin\theta)\,\hat x + \hat y\,(-\theta''\sin\theta - \theta'^2\cos\theta)$$

$$\hat z \cdot \left(\frac{d\vec r}{dl} \times \frac{d^3\vec r}{dl^3}\right) = -\theta''\sin^2\theta - \theta'^2\cos\theta\sin\theta - (\theta''\cos^2\theta - \theta'^2\cos\theta\sin\theta)$$

$$= -\theta''(\sin^2\theta + \cos^2\theta) = -\theta'' \;!$$

So the equation of equilibrium becomes.

$$-EI\,\theta'' = \left[-\hat{y}F \times (\sin\theta\,\hat{x} + \cos\theta\,\hat{y})\right]\cdot\hat{z}$$

$$= F\sin\theta$$

or

$$EI\,\theta'' + F\sin\theta = 0$$

$$\theta'' + \frac{F}{EI}\sin\theta = 0$$

$$\theta'' + f\sin\theta = 0 \qquad f = \frac{F}{EI}$$

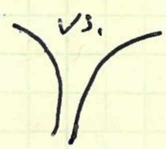
There is a first integral $\qquad [f] = \dfrac{E/L}{E/L^3\ L^4} = 1/L^2$ ✓

$$\frac{d}{dl}\left[\tfrac{1}{2}\theta'^2 - f\cos\theta\right] = 0$$ as long as $\theta' \neq 0$ everywhere.

$$\Rightarrow \tfrac{1}{2}\theta'^2 - f\cos\theta = c_1$$

and at $l = L$ $\theta' = 0$; (torque free) $\theta = \theta_0$ so $c_1 = -f\cos\theta_0$

$$\tfrac{1}{2}\theta'^2 - f\cos\theta = -f\cos\theta_0 \Rightarrow \theta' = \pm\sqrt{-f\cos\theta_0 + f\cos\theta}\,\sqrt{2}$$

Choose the + side 

$$\int_0^{\theta} \frac{d\bar{\theta}}{\sqrt{\cos\bar{\theta} - \cos\theta_0}}\sqrt{\frac{1}{2f}} = l$$

so $$\ell = \sqrt{\frac{EI}{2F}} \int_0^{\theta(\ell)} \frac{dz}{\sqrt{\cos z - \cos\theta_0}}$$

where we get θ_0 from the condition that

$\ell(\theta_0) = L \leftarrow$ length of the rod.

We recover $$x = \int \sin\theta \, d\ell$$
$$y = \int \cos\theta \, d\ell$$

from this solution.

Finally note that for small bends

$$L \cong \sqrt{\frac{EI}{F}} \int_0^{\theta_0} \frac{dz}{\sqrt{-z^2 + \theta_0^2}} \qquad z = \theta_0 \alpha$$

$$L \cong \sqrt{\frac{EI}{F}} \int_0^1 \frac{\theta_0 \, d\alpha}{\theta_0 \sqrt{-\alpha^2 + 1}} = \frac{\pi}{2}\sqrt{\frac{EI}{F}}$$

so if $\quad F \leq \dfrac{\pi^2 EI}{4L^2}$ there will be no bent solution $\Rightarrow$ this is an example of <u>Euler buckling</u>.

Finally $\boxed{EI = K \text{ bending modulus}}$

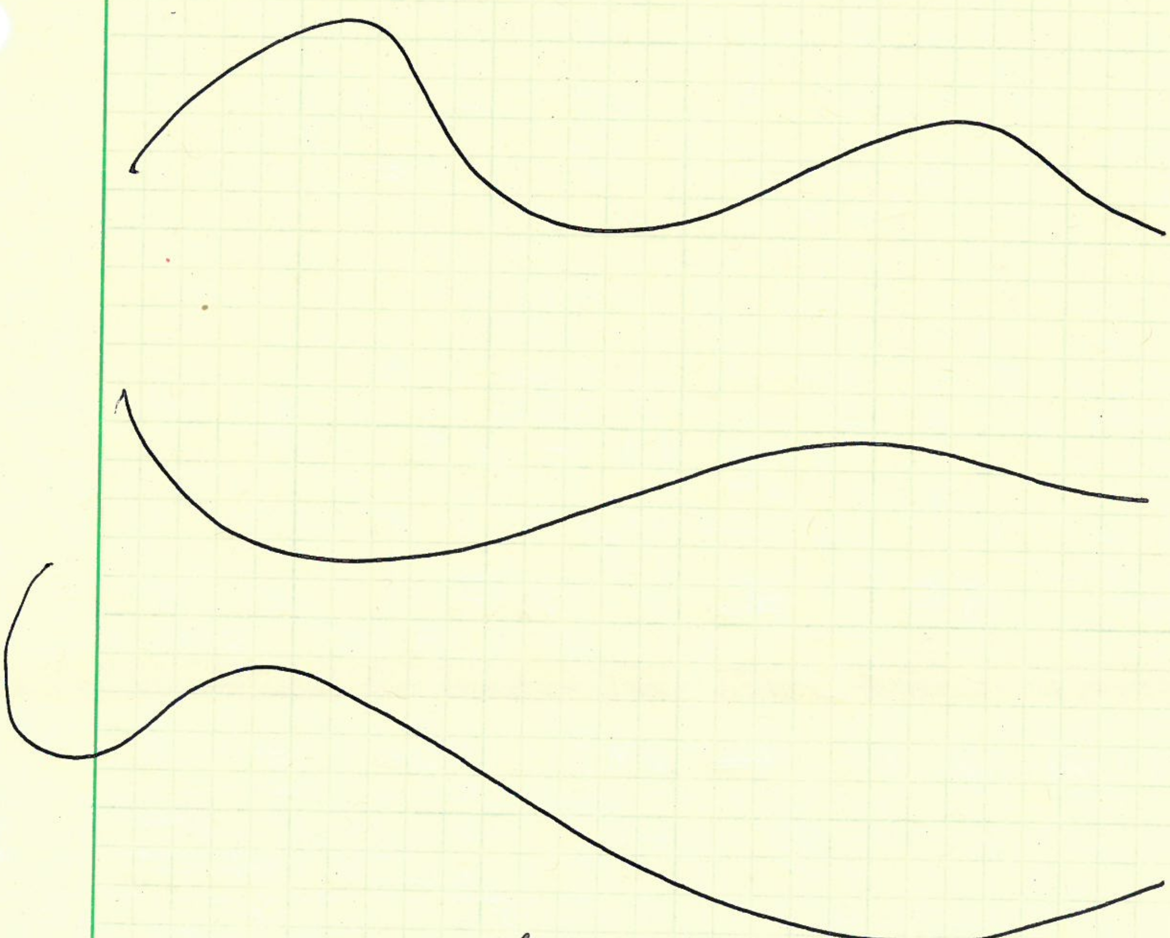
Note $[K] = \frac{E}{L^3} L^4 = E \cdot L$

We typically write it as $K = k_B T \, l_p$

Thermal persistence length.

To see what that means, we should look at the statistical mechanics of a wiggly line or filament

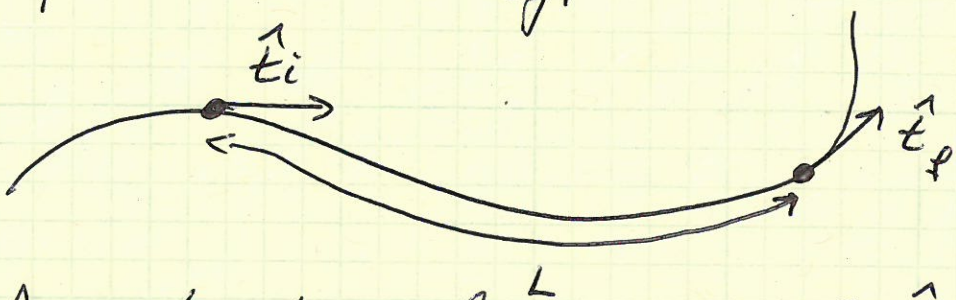
At temperature T we might see



What sets the tangent vector correlations?

Persistence length and diffusion on a sphere.

$G(\hat{t}_f, \hat{t}_i | L)$ = probability of having a tangent of $\hat{t}_f$ given that the filament has a tangent $\hat{t}_i$ at a point L to the left.

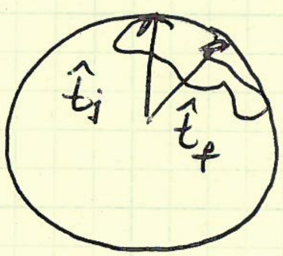


A couple of general pts: ① $G(\hat{t}_f, \hat{t}_i | L)$

$= G(\hat{t}_f - \hat{t}_i | L)$

no special directions.

③ Rotational diffusion of $\hat{t}$



② $G(\hat{t}_f, \hat{t}_i | L)$

$= G(\hat{t}_f, \hat{t}_i | -L)$

can go either direction

We may not need to know in general what G looks like, but we do need it for infinitesimal distances (arclengths)

$$G(\hat{t}_b, \hat{t}_a | \varepsilon) \longrightarrow N e^{-\frac{K}{2T}\int_0^{\varepsilon} D\hat{t}(s)\, (\partial_s \hat{t})^2} \xrightarrow[\varepsilon \to 0]{}$$

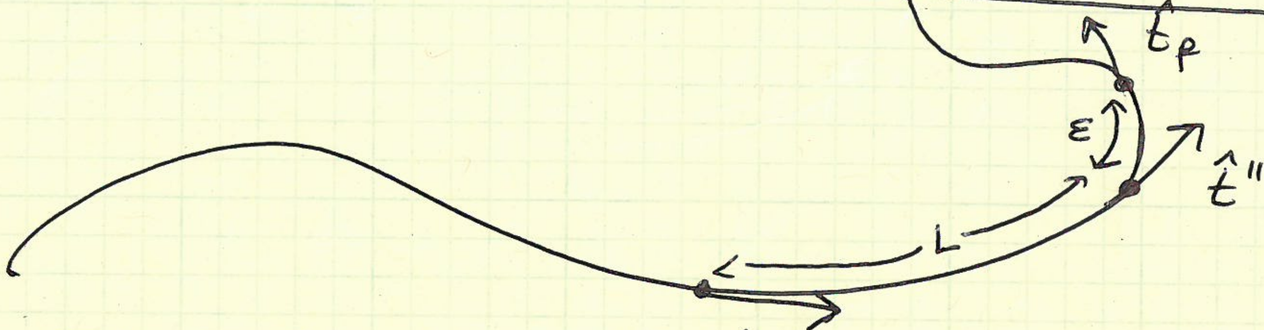
normalization

$$G(\hat{t}_b, \hat{t}_a | \varepsilon) \xrightarrow[\varepsilon \to 0]{} N \exp\left\{ -\frac{K}{2T} \frac{(\hat{t}_b - \hat{t}_a)^2}{\varepsilon^2} \varepsilon \right\}$$

in the limit $\varepsilon \to 0$ we get $\delta^2(\hat{t}_b - \hat{t}_a) = G(\hat{t}_b, \hat{t}_a | 0)$

Composition of probability:

$$\boxed{G(\hat{t}_f, \hat{t}_i | L+\varepsilon) = \int d\hat{t}'' \, G(\hat{t}_f, \hat{t}'' | \varepsilon) \, G(\hat{t}'', \hat{t}_i | L)}$$



Taking small ε (in the $\hat{t}_i$ process of looking at the limit)

$$\hat{t}'' = \hat{t}_f + \vec{\eta} \quad \text{where} \quad \vec{\eta} = \vec{\delta\theta} \times \hat{t}_f$$

$$\text{where } |\vec{\delta\theta}| \xrightarrow[\varepsilon \to 0]{} 0$$

Our plan is to expand both sides order-by-order in ε and match terms.

RHS is easy...

$$G(\hat{t}_f, \hat{t}_i | L+\varepsilon) = G(\hat{t}_f, \hat{t}_i | L) + \varepsilon \, \partial_s G(\hat{t}_f, \hat{t}_i | s)\Big|_{s=L} + \mathcal{O}(\varepsilon)$$

Now

$$G(\hat{t}'', \hat{t}_i | L) = G(\hat{t}_f, \hat{t}_i | L)\, \delta^2(\hat{t}'' - \hat{t}_f) +$$

$$(\vec{\delta\theta} \times \hat{t})_\alpha\, \partial_{\hat{t}_\alpha} G(\hat{t}, \hat{t}_i | L)\Big|_{\hat{t}=\hat{t}_f} + \frac{1}{2} (\vec{\delta\theta} \times \hat{t})_\alpha (\vec{\delta\theta} \times \hat{t})_\beta \,\cdot$$

$$\cdot\, \partial_{\hat{t}_\alpha} \partial_{\hat{t}_\beta} G(\hat{t}, \hat{t}_i | L)\Big|_{\hat{t}=\hat{t}_f} + \mathcal{O}(\delta\theta^3)$$

and

$$\exp\left[-\frac{K}{2T\varepsilon} (\hat{t}_f - \hat{t}'')^2\right] = \exp\left[-\frac{K}{2T\varepsilon} (\vec{\delta\theta} \times \hat{t})^2\right]$$

$$= \exp\left[-\frac{K}{2T\varepsilon} \left\{ (\vec{\delta\theta})^2 - \cancel{(\vec{\delta\theta} \cdot \hat{t})^2}^{\,0} \right\}\right]$$

The terms linear in $\delta\theta_\alpha$ must integrate to zero. There is a $\delta\theta_\alpha \leftrightarrow -\delta\theta_\alpha$ symmetry in the probability distribution

So we are left with:

$$\varepsilon\, \partial_s G(\hat{t}_f, \hat{t}_i | s)\Big|_{s=L} = \frac{1}{2} \int d^2\vec{\delta\theta}\, (\vec{\delta\theta} \times \hat{t})_\alpha (\vec{\delta\theta} \times \hat{t})_\beta \,\cdot$$

$$\cdot\, \partial_{\hat{t}_\alpha} \partial_{\hat{t}_\beta} G(\hat{t}, \hat{t}_i | L)\, N e^{-\frac{K}{2T\varepsilon} (\delta\theta)^2}$$

In other words, we need to compute:

$$I = \langle \varepsilon_{\alpha i j}\, \varepsilon_{\beta k l}\, \hat{t}_j \hat{t}_l\, \delta\theta_\beta\, \delta\theta_\alpha \rangle\, \partial_{\hat{t}_\alpha} \partial_{\hat{t}_\beta}$$

$$I = \langle (\delta\theta_i)^2 \rangle \left(\partial^2_{\hat{t}_\alpha} - \hat{t}_\ell \hat{t}_j \partial_{\hat{t}_\ell} \partial_{\hat{t}_j} \right)$$

just one of them

$$I = \langle (\delta\theta)^2 \rangle (\hat{t} \times \vec{\partial}_t)^2$$

$$I = \frac{2T\varepsilon}{2K} \Longrightarrow \boxed{\partial_s G = \frac{T}{2K} (\hat{t} \times \vec{\partial}_t)^2 G}$$

Now we can compute the tangent-tangent correlation function as follows:

Consider the s-derivative

$$\partial_s \langle \hat{t}(s) \cdot \hat{t}(0) \rangle = \int d\hat{t}_f \, d\hat{t}_i \; \hat{t}_f \cdot \hat{t}_i \, \partial_s G(\hat{t}_f, \hat{t}_i | s)$$

$$= D \int d\hat{t}_f \, d\hat{t}_i \; \hat{t}_f \cdot \hat{t}_i \, (\hat{t} \times \vec{\partial}_t)^2 G(\hat{t}, \hat{t}_i | s) \Big|_{\hat{t} = \hat{t}_f}$$

Integrate by parts twice

$$= D \int d\hat{t}_f \, d\hat{t}_i \left[(\hat{t} \times \vec{\partial}_t)^2 \hat{t} \cdot \hat{t}_i \right] G(\hat{t}, \hat{t}_i | s) \Big|_{\hat{t} = \hat{t}_f}$$

and $(\hat{t} \times \vec{\partial}_t)_\alpha \hat{t}_\beta = - \varepsilon_{\alpha\beta\gamma} \hat{t}_\gamma$ so

$$(\hat{t} \times \partial_{\hat{t}})^2_\alpha \hat{t}_\beta = -2 \hat{t}_{\alpha\beta}$$

This is related to an old friend "bac - cab" rule...

So

$$\partial_s \langle \hat{t}(s) \cdot \hat{t}(0) \rangle = -2D \int d\hat{t}_f \, d\hat{t}_i \; \hat{t}_f \cdot \hat{t}_i \; G(\hat{t}_f, \hat{t}_i | s)$$

$$= -2D \langle \hat{t}(s) \cdot \hat{t}(0) \rangle$$

$\Longrightarrow$

$$\langle \hat{t}(s) \cdot \hat{t}(0) \rangle = \exp\left[-2D|s|\right]$$

Since $\langle \hat{t}^2 \rangle = 1$, we know the initial condition

$\Longrightarrow$

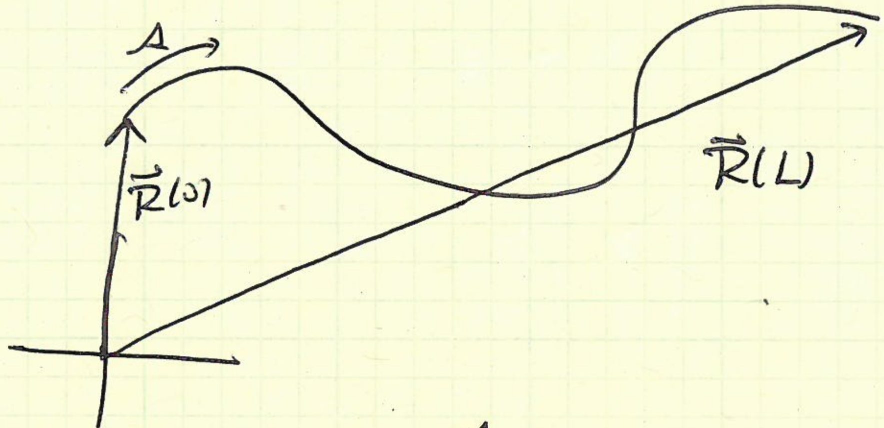
$$\boxed{\langle \hat{t}(s) \cdot \hat{t}(0) \rangle = e^{-|s|/\ell_p}}$$

where $\ell_p = K/T$ is the persistence length.

What do the tangent correlations tell you? (26)

end-to-end length.

$$\langle R^2 \rangle = \langle [\vec{R}(L) - \vec{R}(0)]^2 \rangle$$



Note $\vec{R}(s) = \int_0^s dz\, \hat{t}(z)$ so putting the origin at $\vec{R}(0)$

$$\langle R^2 \rangle = \left\langle \int_0^L ds\, \hat{t}(s) \cdot \int_0^L ds'\, \hat{t}(s') \right\rangle$$

$$= \int_0^L ds\, ds' \, \underbrace{\langle \hat{t}(s) \cdot \hat{t}(s') \rangle}_{e^{-|s-s'|/\ell_p}}$$

$$= 2 \int_0^L ds \int_0^s ds'\, e^{-(s-s')/\ell_p} = 2 \int_0^L ds\, e^{-s/\ell_p}\, \ell_p e^{s'/\ell_p} \Big|_0^s$$

$$\langle R^2 \rangle = 2\ell_p \int_0^L ds \left[1 - e^{-s/\ell_p}\right] = 2\ell_p \left[L + \ell_p e^{-s/\ell_p} \Big|_0^L \right]$$

$$\langle R^2\rangle = 2l_pL\left[1+\frac{l_p}{L}\left(e^{-L/l_p}-1\right)\right]$$

Does this make sense? Check the limits.

$L \ll l_p$ rod-like filament

$$\langle R^2\rangle \cong 2l_pL\left[1+\frac{l_p}{L}\left(\cancel{1}-\frac{L}{l_p}+\frac{1}{2}\frac{L^2}{l_p^2}\cancel{-1}\right)\right]$$

$$\cong 2l_pL\left[1-1+\frac{1}{2}\frac{L}{l_p}\right] = L^2 + \mathcal{O}(L/l_p)$$

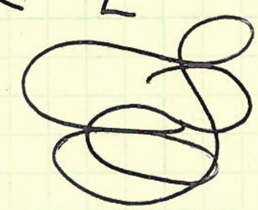
nearly straight filament

essentially straight + small shortening from wiggles

$L \gg l_p$ flexible filaments

$$\langle R^2\rangle \cong 2l_pL\left[1-\frac{l_p}{L}\right] \cong 2l_pL$$

$$\langle R^2\rangle = (2l_p)L \ll L^2$$



coiled filament.

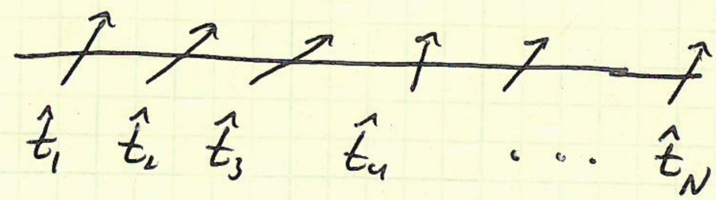
we can write this as

$\langle R^2\rangle = b^2N$ where $N = L/2l_p \Leftarrow$ # segments

$b = 2l_p \Leftarrow$ Kuhn length.

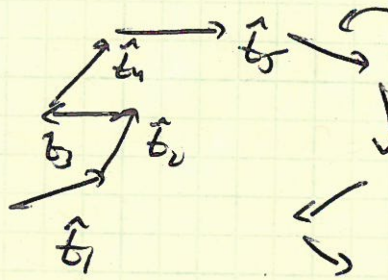
A short side bar on random coils. There is an easy way to see this result.

A lattice model



with no correlation

$$\vec{R}^2 = \sum_{n,m=1}^{N} (\hat{t}_n b)\cdot(\hat{t}_m b) \quad \text{so}$$



length of step $= b$ Kuhn length

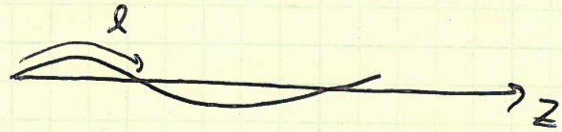
and so on.

$$\langle \vec{R}^2 \rangle = b^2 \sum_{n,m=1}^{N} \underbrace{\langle \hat{t}_n\cdot\hat{t}_m \rangle}_{\delta_{nm}} = b^2 \sum_{n=1}^{N} \langle \hat{t}_n^2 \rangle = N b^2 .$$

Looking at the small bending limit allows you to make reasonable simplifying approximations.

Small bending limit

$$\frac{d\hat{t}}{dl} = \frac{d^2\vec{r}}{dl^2} \cong \frac{d^2\vec{r}}{dz^2}$$



so $\Omega_\xi = -\dfrac{d^2 u_y}{dz^2}$ $\quad \Omega_\eta = \dfrac{d^2 u_x}{dz^2}$

$$\Rightarrow \quad F_{rod} = \frac{1}{2} E \int_0^L \left\{ I_1 \left(\frac{d^2 u_x}{dz^2}\right)^2 + I_2 \left(\frac{d^2 u_y}{dz^2}\right)^2 \right\} dz$$

To get the equation of ~~motion~~ equilibrium

$$\frac{\delta F}{\delta u_x} = 0 \quad \Rightarrow \quad \frac{1}{2}\int_0^L K \left[\partial_z^2 (u_x + \delta u_z)\right]^2 dz - \ldots$$

Just look at u_x. u_y is the same.

Integrate by parts.

$$\frac{1}{2}\int_0^L K \cdot 2\, \partial_z^2 u_x \partial_z^2 \delta u_x \, dz$$

$$= K \int_0^L \left\{ \partial_z \left[\partial_z^2 u_x \, \partial_z \delta u_x \right] - \partial_z^3 u_x (\partial_z \delta u_x) \right\} dz$$

$$= K \partial_z^2 u_x \, \delta u_x' \Big|_0^L - K \int \partial_z^3 u_x \, \partial_z (\delta u_x) \, dz$$

$$= K \partial_z^2 u_x \, \delta u_x' \Big|_0^L - K (\partial_z^3 u_x) \, \delta u_x \Big|_0^L +$$

$$+ K \int \partial_z^4 u_x \, \delta u_x \, dz$$

TOPS

We could include localized forces acting on the length of the rod by incorporating a term like

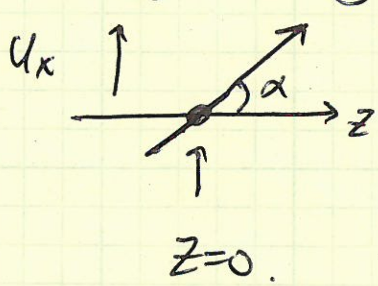
$$\int_0^L -f_x(z)\, u_x(z)\, dz \quad \text{in the energy.}$$

We get

$$\frac{\delta F_{rod}}{\delta u_x} = 0 \Rightarrow \quad 0 = K\,\partial_z^2 u_x\, \delta u_x'\Big|_0^L - K(\partial_z^3 u_x)\,\delta u_x\Big|_0^L$$
$$+ K\,\partial_z^4 u_x - f_x(z)$$

It is easy to identify the boundary terms:

① $K\,\partial_z^2 u_x\, \delta u_x'\Big|_0^L$ = energy proportional to local angle.



$\tan\alpha \cong \alpha \cong \delta u_x'\Big|_0$

$\Rightarrow \quad K\,\partial_z^2 u_x$ = torque at the boundary.

② $-K\,\partial_z^3 u_x\, \delta u_x\Big|_0^L$ = energy proportional to local displacement

$\Longrightarrow \quad -K\,\partial_z^3 u_x$ = force at the boundary.

example clamped end at 0 free end at L.

$$\delta U_x\Big|_0 = 0 \quad \delta U_x'\Big|_0 = 0 \quad \text{and} \quad \partial_z^2 U_x\Big|_L = 0$$

$$\partial^3 U_x\Big|_L = 0$$

Soln $U_x^{(iv)} = 0 \Rightarrow$

$$U_x(x) = A + Bz + Cz^2 + Dz^3$$

b.c. at $z=0$ $\quad A=0 \quad B=0$

$$U_x''(L) = 2C + 6DL = 0$$

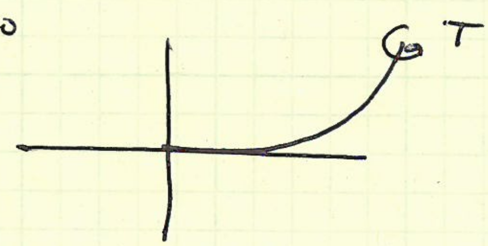
$$U_x'''(L) = D = 0 \Rightarrow \quad U_x(z) = 0 \quad \text{of course.}$$

Now put a bending moment at the end.

$$U_x'' = T \neq 0 \quad \Rightarrow \quad 2C = T$$

and $D=0$

$$U_x(z) = \frac{T}{2} z^2$$



Now try $U_x'(0) = \alpha \quad U_x'(L) = -\alpha$

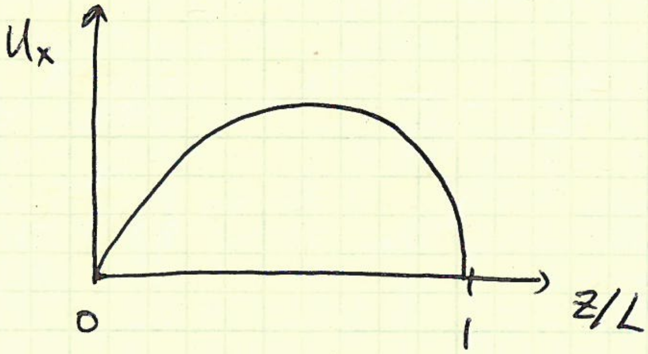
$$U_x(0) = U_x(L) = 0$$

$\Rightarrow A=0$ and $BL + CL^2 + DL^3 = 0$

$B = \alpha$ and $B + 2CL + 3DL^2 = -\alpha$

$C = -\alpha/L$ and $D = 0$

$$u_x(z) = \alpha z - \frac{\alpha}{L} z^2 = \alpha z\left[1 - z/L\right]$$

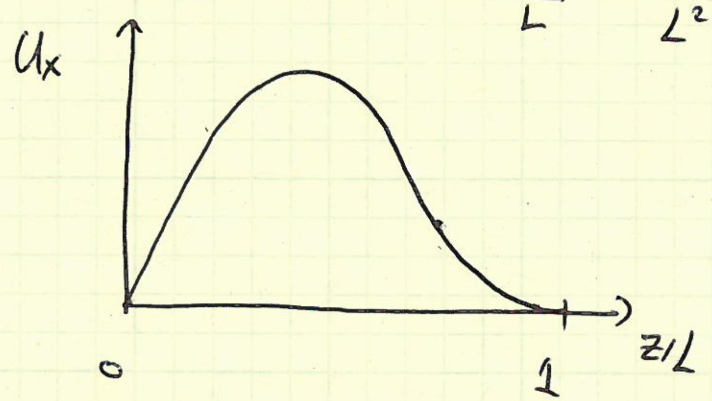


and finally $u_x'(0) = \alpha$ $u_x'(L) = 0$

$u_x(0) = u_x(L) = 0$

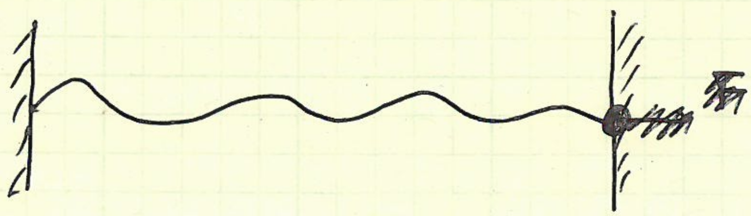
$C = -2\alpha/L$ $D = \alpha/L^2$

$$u_x(z) = \alpha z - \frac{2\alpha z^2}{L} + \frac{\alpha z^3}{L^2} = \alpha z\left[1 - 2\frac{z}{L} + \left(\frac{z}{L}\right)^2\right]$$



Try it w/ a force applied to the end.

Now consider a thermalized filament.



$$F_{rod} = \frac{1}{2}K\int_0^L u_x''(z)^2\,dz$$ expand in sine series.

$$u_x(z) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}z\right)$$

$\bar{z} = z/L$

$\Rightarrow$

$$F_{rod} = \frac{1}{2}K\sum_{nm}\int_0^L A_n^* A_m \sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{m\pi}{L}z\right)dz\left(\frac{n\pi}{L}\right)^2\left(\frac{m\pi}{L}\right)^2$$

$$F_{rod} = \frac{1}{2}K\sum_{n,m=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(\frac{m\pi}{L}\right)^2 A_n A_m L\int_0^1 d\bar{z}\,\sin(n\pi\bar{z})\sin(m\pi\bar{z})$$

$$F_{rod} = \frac{1}{2}K\sum_{n,m=1}^{\infty}\left(\frac{n\pi}{L}\right)\left(\frac{m\pi}{L}\right)A_n A_m L\frac{1}{2}\delta_{n,m}$$

$$\boxed{F_{rod} = \frac{1}{4}KL\sum_{n=1}^{\infty}\frac{(n\pi)^4}{L^4}A_n^2}$$

The partition sum $Z = \prod_{n=1}^{\infty}\int_{-\infty}^{\infty} dA_n\, e^{-\frac{KL}{4T}\sum_{n=1}^{\infty}\frac{n^4\pi^4}{L^4}A_n^2}$

We see that (through the magic of Gaussian integrals)

$$\langle A_n\rangle = 0 \qquad \langle A_n A_m\rangle = \delta_{n,m}\frac{1}{2\,\frac{KL}{4T}\,\frac{n^4\pi^4}{L^4}}$$

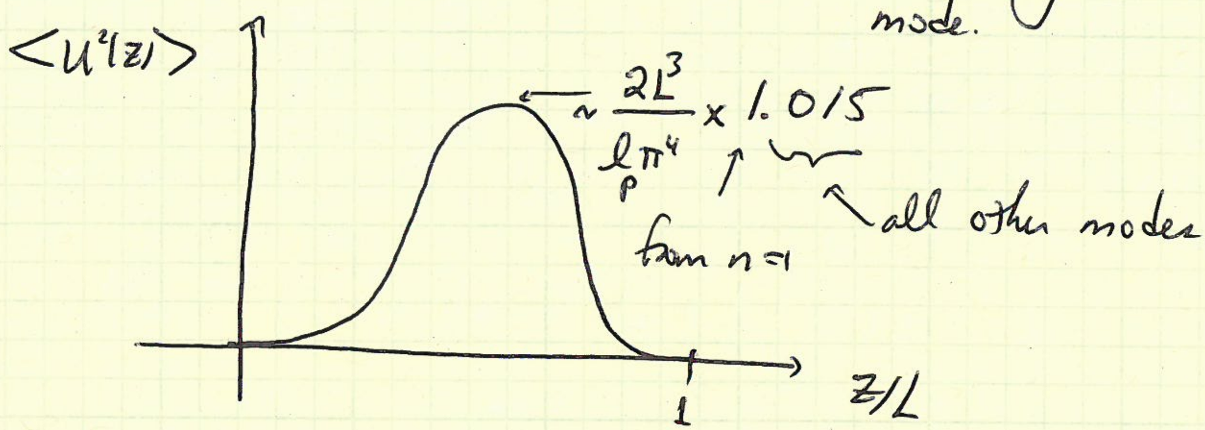
so $$\langle A_n^2\rangle = \frac{L^{\cancel{4}\,3}}{n^4\pi^4}\,\frac{2T}{K} = \frac{2L^3}{\ell_p}\,\frac{1}{n^4\pi^4}$$

So what is $\langle u^2(z) \rangle$?

$$\langle u^2(z) \rangle = \sum_{\substack{n=1 \\ m=1}}^{\infty} \left\langle A_n \sin\left(\frac{n\pi}{L} z\right) A_m \sin\left(\frac{m\pi}{L} z\right) \right\rangle$$

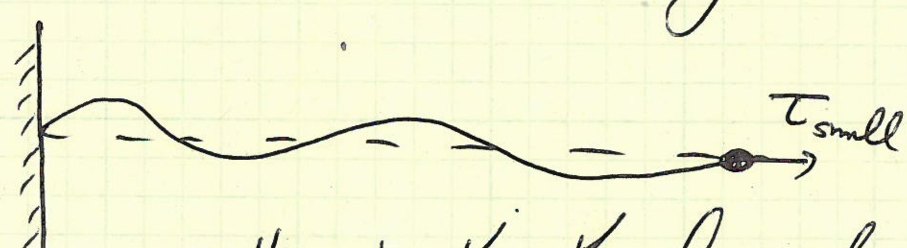
$$= \sum_{n=1}^{\infty} \frac{2L^3}{l_p \pi^4} \frac{1}{n^4} \sin^2\left(\frac{n\pi}{L} z\right)$$

dominated by the $n=1$ mode.



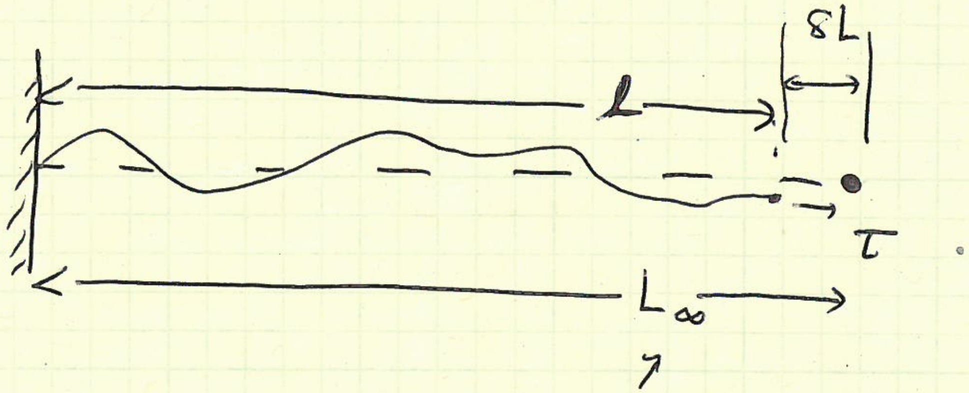
What about pulling on a filament.

[From Käs Mackintosh Janmey 1995]



pulling at the thermal wiggles w/ tension τ.

How much length is "used up" by transverse thermal fluctuations?



infinite tension result.

For a configuration $u_x(z)$ we have arclength L in distance

$$L_\infty = \int_0^L \sqrt{1+u'^2}\,dz \cong L + \frac{1}{2}\int_0^L u'^2\,dz \Rightarrow$$

$$\delta L = L_\infty - L = \frac{1}{2}\int_0^L u'^2\,dz$$

We add a term to the energy of the form: $\tau\,\delta L$

$$\Rightarrow F_{rod} = \frac{1}{2}\int_0^L dz\,\left\{K u''^2 + \tau u'^2\right\}$$

using our Fourier sine series again.

$$F_{rod} = \frac{1}{4}\sum_{n=1}^{\infty}\left\{K\frac{n^4\pi^4}{L^4} + \tau\frac{n^2\pi^2}{L^2}\right\}(A_n)^2$$

Compute $\langle \delta L \rangle_\tau$ ← ensemble w/ tension τ.

$$\langle \delta L \rangle = \sum_{n=1}^{\infty} \frac{1}{4} \left\langle A_n^2 \frac{n^2\pi^2}{L^2} \right\rangle = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} \langle A_n^2 \rangle$$

and $$\langle A_n^2 \rangle = \frac{1}{2} T \frac{1}{\frac{1}{4}\left[\frac{n^4\pi^2}{L^2} K + \tau \frac{n^2\pi^2}{L^2} \right]}$$

$$\Rightarrow \langle \delta L \rangle = \sum_{n=1}^{\infty} \frac{T}{2} \frac{n^2\pi^2/L^2}{\frac{n^4\pi^2}{L^4} K + \tau \frac{n^2\pi^2}{L^2}} \times 2$$ ← for x, y polarization

$$\underset{\langle L_\infty - L \rangle}{\langle \delta L \rangle} \cong T \sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{L^2} K + \tau}$$

so $$\langle L \rangle \cong L_\infty - \underbrace{\sum_{n=1}^{\infty} \frac{T}{\tau + K n^2\pi^2/L^2}}_{\text{length used up in undulations}}$$

as τ gets bigger, more and more terms are of the form $\sim \frac{1}{\tau}$ instead of $\frac{1}{n^2}$ so the sum

<u>decreases</u> w/ increasing τ. ← As it should!

Now for small τ we can write

$$\sum_{n=1}^{\infty} \frac{1}{\tau + \frac{K n^2 \pi^2}{L^2}} \cong \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \frac{L^2}{K\pi^2} - \tau \sum_{n=1}^{\infty} \frac{1}{\left(\frac{K n^2 \pi^2}{L^2} \right)^2}$$

$$+ \tau^2 \sum_{n=1}^{\infty} \frac{1}{\left(\frac{K n^2 \pi^2}{L^2} \right)^3} \cong \frac{\pi^2}{6} \frac{L^2}{K\pi^2} - \frac{\tau L^4}{K^2 \pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} +$$

$$+ \frac{\tau^2 L^6}{K^3 \pi^6} \sum_{n=1}^{\infty} \frac{1}{n^6} \cong \frac{L^2}{6K} - \frac{\tau L^4}{K^2 \pi^4} \frac{\pi^4}{90} + \frac{\tau^2 L^6}{K^3 \pi^6} \frac{\pi^6}{945} + \cdots$$

or

$$\langle L \rangle \cong L_\infty - \frac{T}{6K} L^2 + \frac{\tau T L^4}{K^2\, 90} + \frac{\tau^2 L^6 T}{K^3\, 945}$$

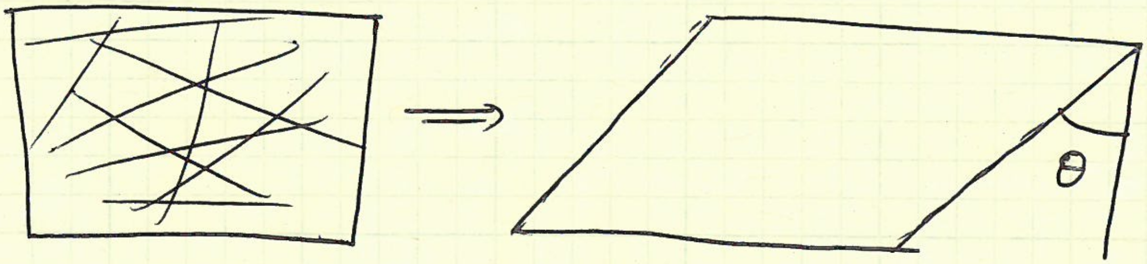
$$\langle L \rangle \cong L_\infty \underbrace{- \frac{L^2}{6 l_p}}_{\substack{\text{equilibrium shortening} \\ \text{at } \tau = 0}} + \underbrace{\frac{\tau L^4}{K l_p\, 90}}_{\substack{\text{linear} \\ \text{response}}} + \underbrace{\frac{\tau^2 L^6}{K^2 l_p\, 945}}_{\text{first nonlinear correction}}$$

In particular $\tau \sim \delta L \dfrac{K l_p\, 90}{L^4}$ or

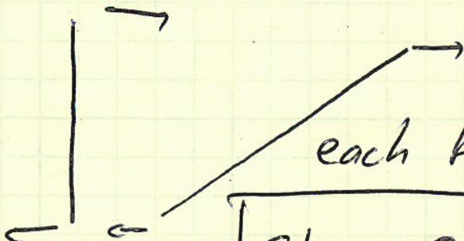
$$\boxed{\tau \sim 90 \frac{K^2}{T L^4} \delta L}$$

thermal spring-const of the filament

What is the modulus of a filament network?



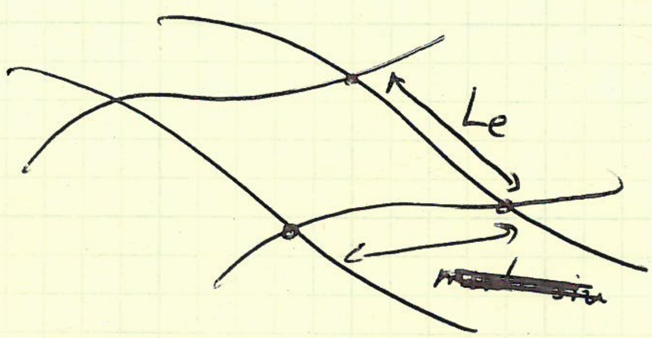
①



each filament gets stretched.

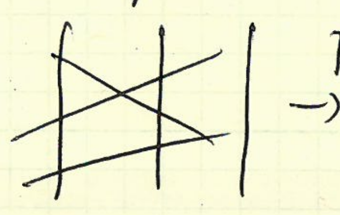
$$\boxed{\delta L \sim \theta L_e}$$

Length between entanglements or crosslinks

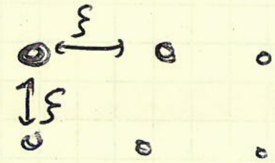


② There are $\frac{1}{\xi^2}$ filaments per unit area.

mesh size



→ put on a lattice



$$\sigma = \frac{\text{Force}}{\text{area}} \sim \frac{1}{\xi^2} \times \tau \sim \frac{K^2}{T L_e^4} \theta L_e \frac{1}{\xi^2}$$

G modulus (shear) is given by

$\theta G \sim \sigma$ so

$$G \sim \frac{K^2}{T L_e^3 \xi^2}$$ flexible chains T/ξ^3

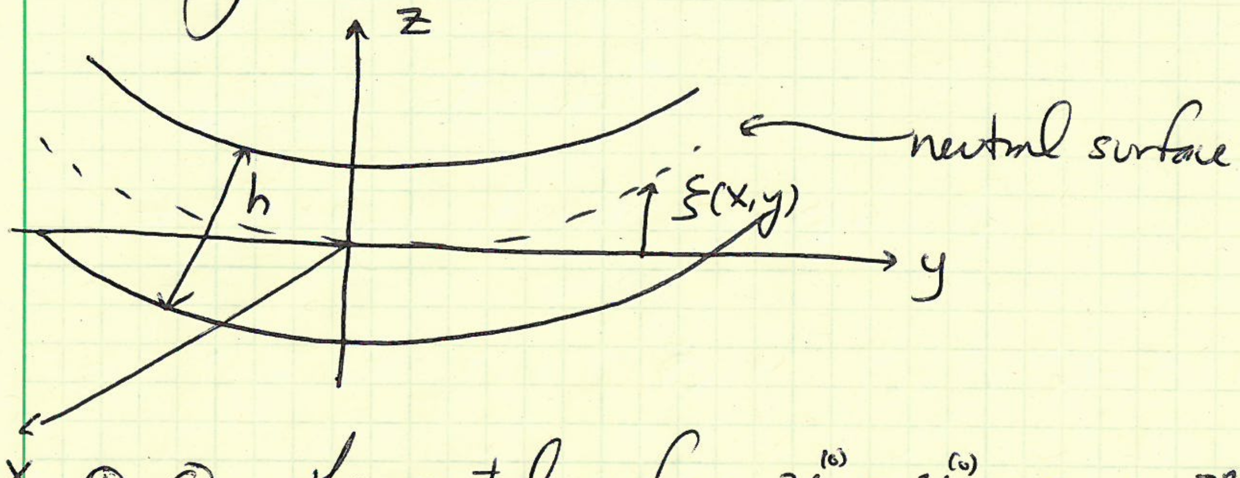
if the network is densely crosslinked $L_e \approx \xi$

$$G \sim \frac{K^2}{T} \xi^{-5} \sim \frac{K^2}{T} c^{5/2}$$ for densely cross linked gel.

since $\xi \sim \frac{1}{(ac)^{1/2}}$

number density of filaments

Bending a plate: Mechanics [LL Chapter II] ①



① On the neutral surface $u_x^{(0)} = u_y^{(0)} = 0$ ← 2^{nd} order in $\zeta(x,y)$

$$u_z^{(0)} = \zeta(x,y)$$

② Take $\hat{n} \approx \hat{z}$ small bending

$$\sigma_{ik}\hat{n}_k = 0 \text{ on surface} \Longrightarrow \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$$

③ For linear elasticity

$$\sigma_{\substack{xz \\ y}} = \frac{E}{1+\sigma} u_{\substack{zx \\ y}} \; ; \; \sigma_{zz} = \frac{E}{(1+\sigma)(1-2\sigma)}\left\{(1-\sigma)u_{zz} + \sigma(u_{xx}+u_{yy})\right\}$$

These must vanish

④ Thus
$$\partial_z u_x = -\partial_x u_z$$
$$\partial_z u_y = -\partial_y u_z$$

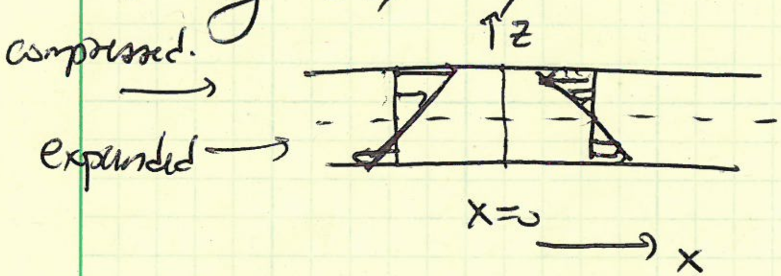
and
$$u_{zz} = -\frac{\sigma}{(1-\sigma)}\left[u_{xx} + u_{yy}\right]$$

Replace u_z by $\zeta(x,y)$

$$\Longrightarrow \boxed{\begin{aligned} \partial_z u_x &= -\partial_x \zeta \\ \partial_z u_y &= -\partial_y \zeta \end{aligned}}$$

$\Longrightarrow$ $u_x = -z\,\partial_x \zeta$; $u_y = -z\,\partial_y \zeta$

setting $u_x = u_y = 0$ at the neutral surface.



if $\partial_x \zeta > 0$ $x > 0$

$\partial_x \zeta < 0$ $x < 0$

say $\zeta \sim \frac{1}{2} x^2$

Then $u_x = -zx$

Now, we can directly compute u_{xx} u_{yy} $\Longrightarrow$

$u_{xx} = -z\,\partial_x^2 \zeta$, $u_{yy} = -z\,\partial_y^2 \zeta$, $u_{xy} = -z\,\partial_x \partial_y \zeta$

$u_{xz} = u_{yz} = 0$ and $u_{zz} = \frac{\sigma}{1-\sigma} z \left(\nabla_\perp^2 \zeta\right)$

$\partial_x^2 + \partial_y^2$

Now we can write the energy of deformation

③

$$F = \frac{E}{2(1+\sigma)}\left[u_{ik}^2 + \frac{\sigma}{1-2\sigma}u_{ll}^2\right]$$

$\Longrightarrow$

$$F = z^2 \frac{E}{1+\sigma}\left[\frac{1}{2(1-\sigma)}\left(\nabla_\perp^2 \zeta\right)^2 + \left[\left(\partial_{xy}^2 \zeta\right)^2 - \partial_x^2\zeta\, \partial_y^2\zeta\right]\right]$$

all z dependence is here. We can get an effective energy for the plate by integrating over the thickness.

$$\int_{-h/2}^{h/2} dz\, z^2 = \frac{1}{3} z^3 \Big|_{-h/2}^{h/2} = \frac{1}{24} \times 2h^3$$

Plate elastic energy $\Longrightarrow$

four derivatives like rod bending.

$$F_{plate} = \frac{Eh^3}{24(1-\sigma^2)}\int dx\,dy \left[\left(\nabla_\perp^2 \zeta\right)^2 + 2(1-\sigma)\left\{\left(\partial_{xy}^2\zeta\right)^2 - \partial_x^2\zeta\,\partial_y^2\zeta\right\}\right]$$

area integral

The first variation and the equation of equilibrium for plates

note we can add forces too.

We need to minimize F wrt $\zeta(x,y)$

$$\frac{\delta F_{plate}}{\delta \zeta(x,y)} = 0 \Longrightarrow \text{A P.D.E.}$$

The surface terms will tell us about the boundary conditions on the field $\xi(x,y)$.

The bending modulus $D \rightarrow K_{2D} = \dfrac{Eh^3}{24(1-\sigma^2)} \sim h^3$ and

- has dimensions of energy.
- has max for incompressible material $\sigma = 1/2$.

1^{st} variation : First term $(\nabla_\perp^2 \xi)^2$

$$\frac{1}{2}\,\delta \int (\vec{\nabla}_\perp^2 \xi)^2 d^2x = \cancel{\tfrac{1}{2}} \int \nabla_\perp^2(\xi + \delta\xi)\, \nabla_\perp^2 \xi \; d^2x - (\nabla_\perp^2 \xi)^2$$

so we need to look at

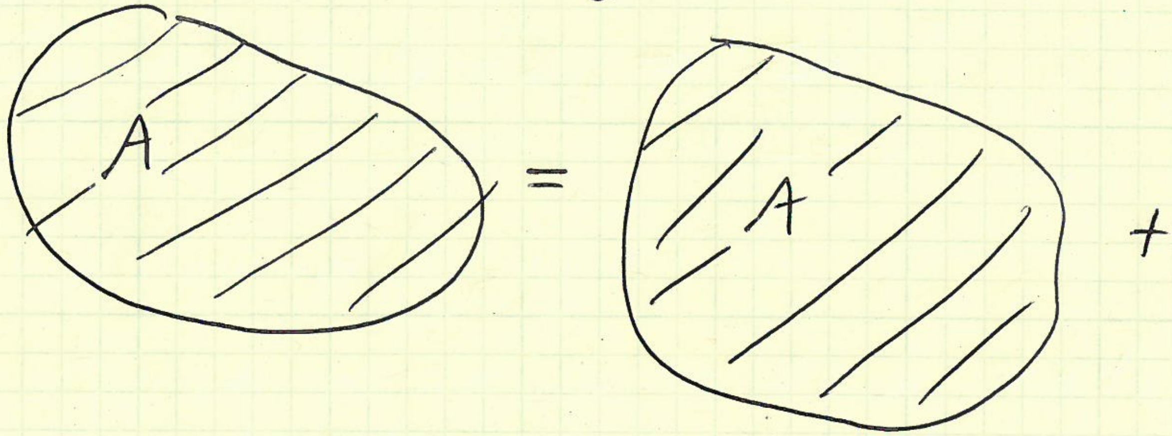
$$I = \int_A (\nabla_\perp^2 \xi)\, \nabla_\perp^2(\delta\xi)\; d^2x = \int_A \left\{ \nabla_\perp \cdot \left[\nabla_\perp^2 \xi \, \vec{\nabla}_\perp \delta\xi \right] - \vec{\nabla}_\perp (\nabla_\perp^2 \xi) \cdot \vec{\nabla}_\perp \delta\xi \right\}$$

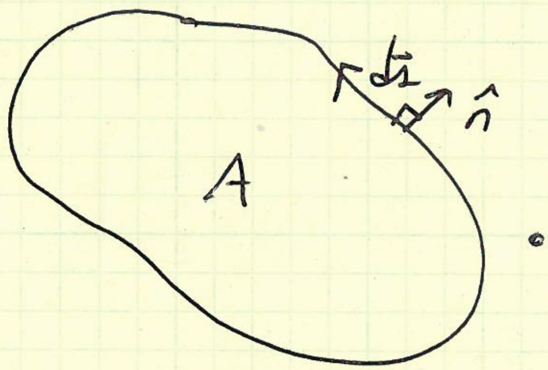
area A $\quad \nabla_\perp^2 = \nabla_\perp \cdot \nabla_\perp$

$$I = \oint_{\partial A} (\hat{n} \cdot \vec{\nabla}_\perp \delta\xi)\, \nabla_\perp^2 \xi \; ds - \int_A \vec{\nabla}_\perp (\nabla_\perp^2 \xi) \cdot \vec{\nabla}_\perp \delta\xi \; d^2x$$

boundary of area A

We have converted a surface (2d) integral into another one plus a boundary line integral.





. Now do it again!

$$-\int \vec{\nabla}_\perp(\nabla_\perp^2\zeta)\cdot\vec{\nabla}_\perp\delta\zeta\, d^2x = -\int \vec{\nabla}_\perp\cdot\left[\vec{\nabla}_\perp(\nabla_\perp^2\zeta)\,\delta\zeta\right] +$$

$$+\int(\nabla_\perp^4\zeta)\,\delta\zeta\, d^2x = -\oint_{\partial A}\left[\hat{n}\cdot\vec{\nabla}_\perp(\nabla_\perp^2\zeta)\right]\delta\zeta \; +$$

$\int(\nabla_\perp^4\zeta)\,\delta\zeta\, d^2x$, so collecting all the terms we get

$$\frac{1}{2}\,\delta \int_A (\nabla_\perp^2 \zeta)^2 d^2x = \int_A (\nabla_\perp^4 \zeta)\, \delta\zeta\, d^2x - \oint_{\partial A} \delta\zeta\, [\hat{n}\cdot\vec{\nabla}_\perp(\nabla_\perp^2 \zeta)]\, ds \; +$$

$$+ \oint_{\partial A} [(\hat{n}\cdot\vec{\nabla}_\perp)\delta\zeta]\, \nabla_\perp^2 \zeta\, ds$$

If we had added a normal force to the plate

$$\delta F_{plate} \rightarrow \delta F_{plate} - \int_A P(x,y)\, \delta\zeta\, d^2x \implies$$

equation of equilibrium - $\boxed{K_{2D} \nabla_\perp^4 \zeta - P = 0}$

What about the other term?

It turns out that it is all surface terms in other words it is a total derivative!

why? $\delta \int [(\partial_{xy}^2 \zeta)^2 - (\partial_x^2 \zeta)(\partial_y^2 \zeta)]\, d^2x$

$$= \int 2\, (\partial_{xy}^2 \zeta)\, \partial_{xy}^2 (\delta\zeta) - \partial_x^2 \zeta\, \partial_y^2 (\delta\zeta) - \partial_y^2 \zeta\, \partial_x^2 (\delta\zeta)$$

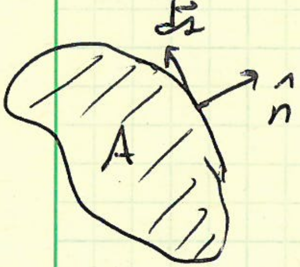
cancels.

$$= \int \{ \partial_x (\partial_y \delta\zeta\, \partial_{xy}^2 \zeta - \partial_x \delta\zeta\, \partial_y^2 \zeta) + \partial_y (\partial_x \delta\zeta\, \partial_{xy}^2 \zeta - \partial_y \delta\zeta\, \partial_x^2 \zeta)$$

cancels.

So
$$\delta \int_A \left[\left(\partial^2_{xy} \xi\right)^2 - \left(\partial^2_x \xi\right)\left(\partial^2_y \xi\right) \right] d^2x$$
$$= \int_A \vec{\nabla}_\perp \cdot \vec{V} \, d^2x = \oint_{\partial A} \hat{n} \cdot \vec{V} \, ds$$

of course $\vec{V}$ is a mess! But it is proportional to $\hat{n} \cdot \vec{\nabla}_\perp \delta\xi$ and $\delta\xi$ evaluated on the bound



If we take $\xi = 0$ $\hat{n} \cdot \nabla_\perp \xi = 0$ on the boundy then all the surface terms vanish.

 clamped b.c.

also the terms $\propto \delta\xi$ give us the force exerted on the boundy.

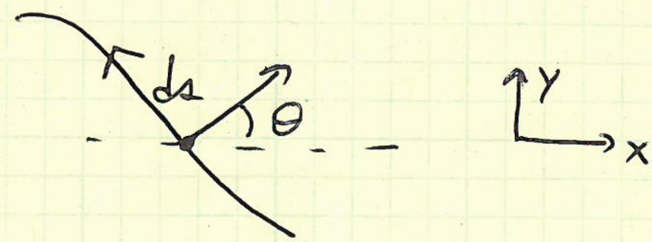
and the terms $\propto \hat{n} \cdot \vec{\nabla}_\perp \delta\xi$ give us the torques exerted by the boundy.

We get $$F = -D\left[\partial_n^3 \zeta + \frac{d\theta}{ds}\partial_n^2 \zeta\right] \quad \underline{\underline{\text{Force}}}$$

$\hat{n}\cdot\vec{\nabla}_\perp$

$$M = D\,\partial_n^2 \zeta \qquad \underline{\underline{\text{Torque}}}$$

where



so $$\partial_x = \cos\theta\,\partial_n - \sin\theta\,\partial_s$$
$$\partial_y = \sin\,\partial_n + \cos\theta\,\partial_s$$

So, why did the answer come out this way?

Why these two terms and why does one contribute only a surface term?