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# I. MICHAEL LEVIN

## I.1. Exactly solvable models for 2D topological phases

A topological phase is a quantum phase (i.e., a Hamiltonian) with two properties:

1. An energy gap in the bulk.

2. Quasiparticle excitations with non-trivial braiding statistics.

We will construct a large class of exactly solvable models for topological phases consisting out of lattices of spins. Such models allow us to build intuition, derive universal properties (e.g., properties on boundaries or topological entanglement entropy), and study tensor networks on them.

### *I.1.1. Simple example: Toric code on a honeycomb lattice*

$$H = -\sum_{\text{vertices } v} \underbrace{\prod_{\ell\in v}\sigma^x_\ell}_{Q_v \text{ (3-body)}} - \sum_{\text{plaquettes } p} \underbrace{\prod_{\ell\in p}\sigma^z_\ell}_{B_p \text{ (6-body)}} \,.$$

Terms of opposite types give *three* minus signs (different from square lattice), so everything commutes (frustration-free):

$$[Q_v, Q_{v'}] = [B_p, B_{p'}] = [Q_v, B_p] = 0\,.$$

One can simultaneously diagonalize $B_p, Q_v$ to get states $|\{b_p, q_v\}\rangle$ with $b_p, q_v = \pm 1$. The energy of such a state is then

$$E = -\sum_v q_v - \sum_p b_p\,.$$

For the ground state, we want to minimize the energy, so $b_p = q_v = 1$ for all $p, v$ (although we didn't show that this state exists or is unique). There are two types of excitations, both of which cost an energy $\Delta E = E_{exc} - E_{gnd} = 2$:

1. Charge excitations with $q_v = -1$ on some vertex.

2. Flux excitations with $b_p = -1$ on some plaquette.

Therefore, we have satisfied property 1 — the system is gapped. Moving a charge (excitation) at vertex $v_0$ around a flux at plaquette $p_0$ gives a statistical Berry phase of -1. Moving charges can be done by applying $\sigma^z$'s on vertices around a curve $C$:

$$|p_0, v_0\rangle \longrightarrow \prod_{\ell\in C}\sigma^z_\ell|p_0, v_0\rangle = \prod_{p\in\mathrm{Int}(C)} B_p|p_0, v_0\rangle = -|p_0, v_0\rangle\,.$$

This obtains the full Berry phase, but does not extract its statistical component. To do that, we need to do a "placebo" experiment in which we move the charge around $C$ without the flux being in the center:

$$|p_0', v_0\rangle \longrightarrow \prod_{\ell\in C}\sigma^z_\ell|p_0', v_0\rangle = \prod_{p\in\mathrm{Int}(C)} B_p|p_0', v_0\rangle = +|p_0, v_0\rangle\,.$$

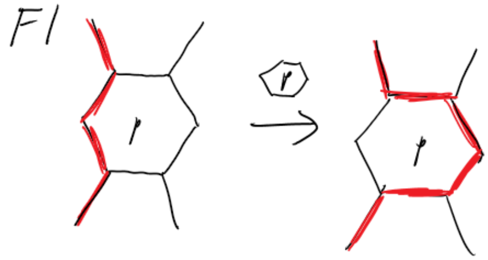
Now we can see that the statistical Berry phase is indeed $-1$.

### I.2. String picture

In this different interpretation, whenever we see a $\sigma_\ell^x = -1$ on a link $\ell$, we say "there is a string on link $\ell$". In the subspace for which $\prod_v \sigma_\ell^x = +1$, there must be an even number of strings on the three links of the vertex, so there are either 0 or 2 strings. Thus, strings form closed loops in this subspace. The $B_p$ term provides a way for the strings to hop (F1). The ground state then consists of all possible closed loops of strings — a **string condensate**:

$$|\Phi\rangle = \sum_{X \text{ closed}} |X\rangle \,.$$

$\Phi$ is an "ideal" condensate because correlations between spins vanish after the spins are separated by only a finite number of lattice spacings. This has to do with the Hamiltonian being a sum of commuting projections (for the toric code, one simply needs to add 1 and divide by 2 to make projections out of $Q_v, B_p$) and is likely to be true in general.





### I.3. String-net models: generalizations of the toric code

Now we consider **string-nets** — closed networks of different types of strings which can branch, but only in a three-fold way (the toric code cannot do that). Just like the toric code describing quantum dynamics of strings, these generalized models describe the dynamics of string nets (F2). To specify a model (at least for the special case we consider), you need to provide

1. The number of string types $N$: $-----i------$.

2. What types of branchings are allowed, i.e., which sets of triplets $(i, j, k)$ are allowed to meet at a point. Not all branching rules lead to exactly solvable models.

   - For example, for $N = 2$, let's allow $(1, 2, 2)$ and $(2, 2, 2)$ to branch.

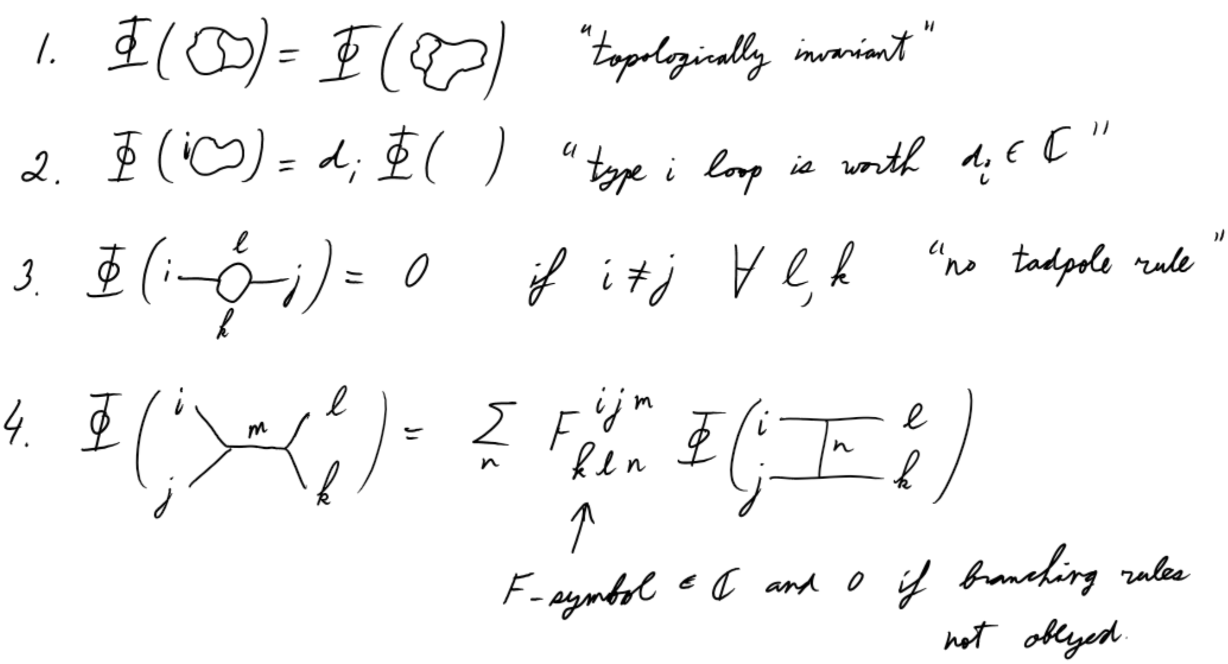
The above rules then set the low-energy Hilbert space for the model — the sets of all possible string-net configurations that obey the branching rules on the honeycomb lattice.

To specify the model, first let's construct the ground state wavefunction, then the corresponding Hamiltonian. We want our wavefunctions to have no length scale (i.e., zero correlations), similar to fixed point wavefunctions in RG. We construct it out of the Hilbert space of all string-net configurations,

$$|\Phi\rangle = \sum_{\text{configuration } c} \Phi(c) \, |c\rangle \,,$$

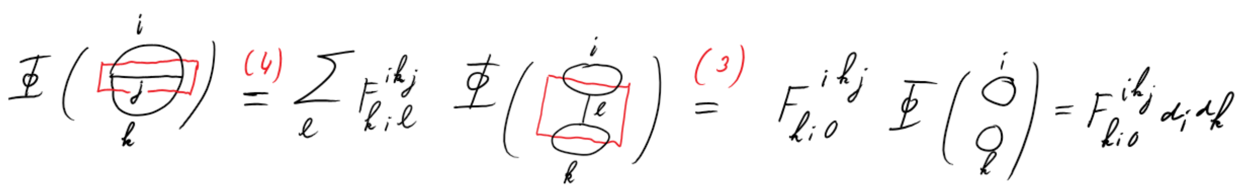
so thus we need to specify the amplitudes $\Phi(c)$. We will do so implicitly by specifying rules the $\Phi(c)$'s have to follow (similar to Chetan Nayak's talk), shown in F3.

F3




The indices $i, j, k, l, n, m$ now run from 0 to $N$ (instead of 1 to $N$), where 0 is the **vacuum string**. By extending the index, we see that more rules can be obtained by removing lines whenever the index is 0. For example, rule 3 with $i, j = 0$ gives a rule about no tadpoles (hence the name) and rule 4 with either $i, l, j, k = 0$ gives the three-fold branching rules. In the example below, we simplify a diagram using rules 4 and 3 and setting $\Phi\,(\ \ ) = 1$ in the last step.

F4

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The branching rules are defined to be

$$\delta_{ijk} = \begin{cases} 1 & \text{branching allowed} \\ 0 & \text{otherwise} \end{cases}$$

The above rules define consistency conditions

1. $\sum_n F^{mlq}_{k\rho n} F^{ji\rho}_{mns} F^{jsn}_{lkr} = F^{jip}_{qkr} F^{riq}_{mls}$
2. $F^{ijm}_{kln} = F^{lkm}_{jin} = F^{jim}_{lkn} = F^{imj}_{knl} \sqrt{\frac{d_m d_n}{d_j d_l}}$
3. $F^{ijk}_{ij\rho} = \sqrt{\frac{d_k}{d_i d_j}} \delta_{ijk}$

Each solution $\left(F^{ijm}_{kln},\, d,\, \delta_{ijk}\right)$ ($F$'s,[1] dimensions, and branching rules) defines a string net wavefunction. We list example 2 below, while example 1 can be found in Michael Levin's notes.

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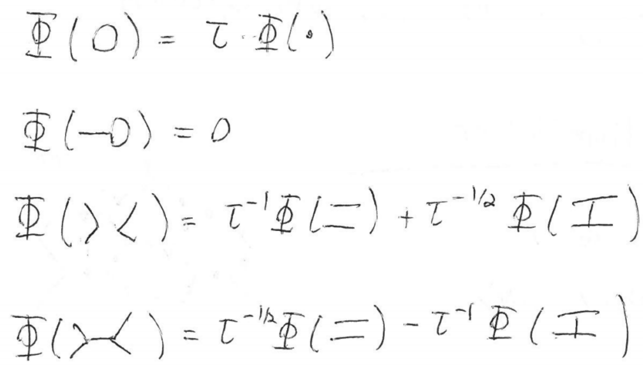
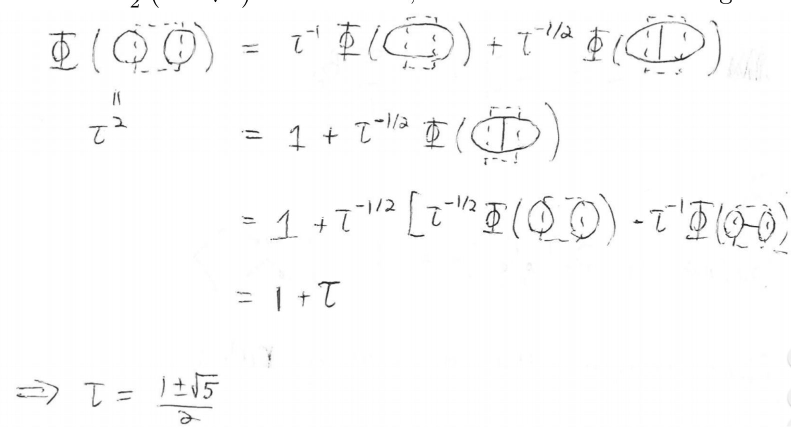
[1] These are not quite the $F$-symbols of a fusion category. In fact, they can encode both fusion and braiding properties.

1. Example 2: One string type ($N = 1$) and branching rule $\{1, 1, 1\}$. There are two solutions:




where $\tau = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. To obtain $\tau$, we can work with the diagram




Despite $\Phi$ being specified explicitly, there is no closed form expression for it.

### II.1. Hamiltonians

Consider a honeycomb lattice with $N+1$ dimensional spins on its links. Each state $|N\rangle$ of the spin corresponding to a string occupying the link, and $|0\rangle$ means there is no string there. The Hilbert space here then consists of all possible configurations, with or without branching. The Hamiltonian is

$$H = -\sum_v Q_v - \sum_p B_p \, .$$

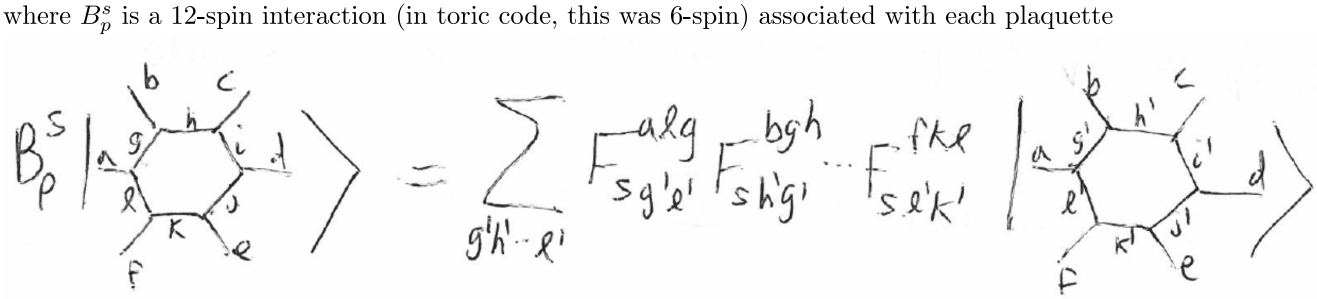
The first, $Q_v$, is a three-spin interaction defined by

$$Q_v \left| {}^{i}_{j} > k \right\rangle = \delta_{ijk} \left| {}^{i}_{j} > k \right\rangle \, ,$$

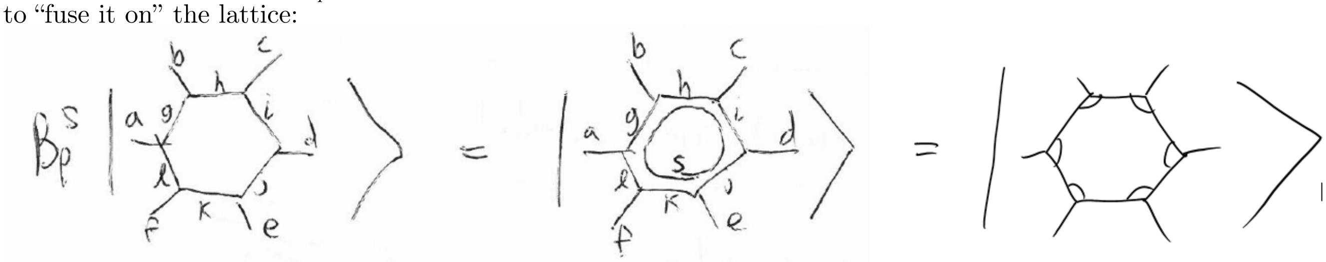
thereby favoring the string-nets which obey the branching rules. The second,

$$B_p = \frac{1}{D} \sum_{s=0}^{N} d_s B_p^s \qquad\qquad D = \sum_{s=0}^{N} d_s^2 \, ,$$

where $B_p^s$ is a 12-spin interaction (in toric code, this was 6-spin) associated with each plaquette



Another way to act with $B_p^s$ is just to add a type-$s$ string to the boundary of the plaquette and then use $F$-moves to "fuse it on" the lattice:



### II.2. Properties of Hamiltonian

Suppose we have data $\left(F^{ijm}_{kln},\, d,\, \delta_{ijk}\right)$ which satisfies the above conditions and also the **unitarity condition**

$$F^{ijm}_{kln} = \left(F^{inl}_{kmj}\right)^\star .$$

This guarantees that $\left(B_p^s\right)^\dagger = B_p^s$. Then

1. $\{B_p, Q_v\}$ commute with each other.

   - The $Q_v$'s diagonal, $[Q_v, B_p] = 0$ because $B_p$'s obey branching rules, and $B_p$'s commute because you can fuse strings on different plaquettes in different order (if the rules are self-consistent). Note that $B_p$'s project onto the subspace of strings which obey the branching rules.

2. $\{B_p, Q_v\}$ are Hermitian.

3. $\{B_p, Q_v\}$ can be made into projection operators.

4. The ground state is the above $\Phi$.

   (a) To prove this, one can prove the equality of certain amplitudes of the true ground state and remember that $\Phi$ exactly follows those rules.

For example, consider the **doubled semion model**: $N = 1$, $d_1 = F^{110}_{110} = -1$ with local spin states $|0\rangle = |+1_X\rangle$ and $|1\rangle = |-1_X\rangle$. Then

$$Q_v = \frac{1}{2}\left(1 + \prod_{\ell \in v} \sigma^x_\ell\right)$$

is like the $Q_v$ term in the toric code and

$$B_p = \frac{1}{2} P_p \left(1 + \prod_p \sigma^z_\ell \cdot \prod_{\text{near } p} i^{\frac{1}{2}(1-\sigma^x_\ell)}\right) P_p\,,$$

where $P_p = \prod_{v \in p} Q_v$ makes sure branching rules are obeyed and the product over the external legs ("near $p$") gives a $\pm$ when projected using $P_p$. This model realizes the doubled semion phase with 4 qparticles

$$\{1, s\} \times \{1, \bar{s}\} = \{1, s, \bar{s}, s\bar{s}\}$$

with $s$, $\bar{s}$ having exchange statistics $\pm i$ and trivial mutual statistics. This is equivalent to having a bilayer of $\nu = \pm\frac{1}{2}$ states.

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Two more examples of generalized toric code models, the doubled semion model and the doubled Fibonacci model, are in pgs. 15-18 in the notes. Excitations in these models are very similar to those of the toric code, namely pairs of string and plaquette excitations.

There are however more general models:

1. Strings can carry an orientation (e.g., $\mathbb{Z}_3$ gauge theory).
2. Vertices can have additional degrees of freedom: $\delta_{ijk}$ can be more than one in general.
3. There are more rules for erasing a vacuum string and reversing string orientation (e.g., putting a dot near a vertex). This has to do with Frobenius-Schur indicators.

Among other phases, these models provide a recipe for

1. Realizing any $T \times \bar{T}$ topological phase:

| Top. phase | String-net |
|---|---|
| qparticles | string types |
| fusion rules | branching rules |

2. They can also realize any gauge theory with a finite gauge group (quantum doubles). String types can be group elements and branching can be group multiplication. Alternatively, string types can correspond to group representations and branching rules to tensor products of group representations. So there are actually two recipes to make the same phase.
3. Finally, they can realize any Dijkraaf-Witten theory with finite gauge group $G$ (twisted quantum double models).

Mathematicians would say you can realize any Drinfield/monoidal center. The input data $\left(F^{ijm}_{kln}, d_i, \delta_{ijk}\right)$ make up a **unitary finite spherical fusion category** $\mathcal{C}$ (Kitaev/Kong). The output is a topological phase with some set of anyons $\mathcal{A} = \{a, b, c, ...\}$ and braiding data $N^c_{ab}, R^c_{ab}, F^d_{abc}$. How is the input data dependent on the output data? It turns out that $\mathcal{A}$ is exactly the **Drinfield center** $Z(\mathcal{C})$.

### III.1. Physical characterization

**Conjecture:** a bosonic topological phase can be realized by a string-net model iff it supports a gapped edge.

The backward direction is easy, but forward direction has only been proven in Abelian case. A corollary of this is that string-net models cannot realize topological phases with nonzero central charge because those have gapless edges. In particular, cannot realize Laughlin states.

**Conjecture 2:** string-nets can realize any bosonic topological phase that can be described by a commuting projector Hamiltonian.

### III.2. Applications: Boundaries

These models can be used to study gapped boundaries of topological phases. Let's show this with the toric code on a square lattice with a boundary with bulk Hamiltonian (Bravyi/Kitaev 1998)

$$H_{bulk} = -\sum_{\text{vertices } v} \underbrace{\prod_{\ell \in v} \sigma^x_\ell}_{Q_v \text{ (4-body)}} - \sum_{\text{plaquettes } p} \underbrace{\prod_{\ell \in p} \sigma^z_\ell}_{B_p \text{ (4-body)}} .$$

Can we find $H_{edge}$ such that $H = H_{bulk} + H_{edge}$ is gapped? It turns out that **topological phases can support gapped boundaries**. The argument for this is that the boundary itself provides a sharp phase transition. In the limit that the boundary becomes infinitely smooth, a gap will emerge.

- For example, consider a smooth boundary and add

$$H_{edge} = -\sum_{\text{vertices } v} \underbrace{\prod_{\ell\in v} \sigma^x_\ell}_{Q_v^{edge}\text{ (3-body)}} .$$

  All terms still commute, so we can use the usual machinery to see that $q_v = q_v^{edge} = b_p = +1$ for the ground state. The ground state is unique in a disk geometry.

- As another example, we can consider a rough boundary (i.e., with vertices sticking out of it). Then, we modify the other term on the edge:

$$H_{edge} = -\sum_{\text{plaquettes } p} \underbrace{\prod_{\ell\in p} \sigma^z_\ell}_{B_p^{edge}\text{ (3-body)}} .$$

There is a difference between smooth and rough boundaries in that plaquette excitations (fluxes) can be annihilated at a smooth boundary, but charges cannot. Conversely, charges can be annihilated at a rough boundary, but fluxes cannot. At gapped boundaries, it turns out you can always annihilate some of the quasiparticles.

We can study the presence of two boundaries: a strip with one rough and one smooth. The ground state degeneracy is

$$\begin{aligned} D &= \mathrm{Tr}\left\{P_{GS}\right\} \\ &= \mathrm{Tr}\left\{\prod_{p\in bulk}\frac{1+B_p}{2}\cdot\prod_{v\in bulk}\frac{1+Q_v}{2}\cdot\prod_{p\in edge}\frac{1+B_p}{2}\cdot\prod_{v\in edge}\frac{1+Q_v}{2}\right\} . \end{aligned}$$

Let $N_p^{bulk}$ and $N_v^{bulk}$ be the number of plaquettes and vertices in the bulk (and similarly, on boundary). The only term with nonzero trace is then the one consisting of no Pauli matrices, since those are traceless. So we only need to count how many two's we have:

$$D = \frac{\mathrm{Tr}\left\{I\right\}}{2^{N_p^{bulk}+N_v^{bulk}+N_p^{edge}+N_v^{edge}}} .$$

The answer is two, and we can obtain it by counting the number of faces $F$, vertices $V$, and edges $E$:

$$N_p^{bulk} + N_v^{bulk} = F \qquad\qquad N_p^{edge} + N_v^{edge} = V - 2$$

Then, using Euler's characteristic $V - E + F = 1$, we get

$$D = 2^{E-F-(V-2)} = 2 .$$

The two-fold degeneracy can be understood by observing a system with two rough and two smooth boundaries, creating two fluxes and two charges, and annihilating them at the boundary. Both of those strings $W_{x,z}$ commute with $H$ and anticommute with each other. This means the ground state degeneracy is at least 2.

This effect is similar to what happens when you take a TI and gap it out using two ferromagnets and two superconductors (with those being the analogies of smooth and rough boundaries). Then you will get Majoranas, and the model can be thought of as the ungauged version of the toric code (cf. Xie Chen's lectures).

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### IV.1. Exactly solvable models for 3D topological phases

Consider a toric code analogue on a 3D cubic lattice:

$$H = -\sum_{\text{vertices } v} \underbrace{\prod_{\ell\in v} \sigma^x_\ell}_{Q_v\text{ (6-body)}} - \sum_{\text{plaquettes } p} \underbrace{\prod_{\ell\in p} \sigma^z_\ell}_{B_p\text{ (4-body)}} .$$

The ground state is once again $|\{b_p, q_v = +1\}\rangle$. The ground state degeneracy can be computed the same way as before — by taking the trace of the ground state projection. There are two types of excitations: point-like charges $q_v = -1$ (which are ends of open strings) with $E = 2$ and closed 3D flux loops $b_p = -1$ with $E = 2 \times \text{length}$. The flux loops are closed because the product of plaquettes over a cube has to be 1. The excitations are expected to have trivial braiding statistics since we are in 3D. Moreover, they have trivial exchange statistics, i.e., they are bosons. But moving a charge through a flux loop gives you nontrivial braiding statistics.

Now let's visualize the ground state, which is the sum over all closed string configurations — a 3D string condensate — where a string on link $\ell$ means $\sigma^x_\ell = -1$. An alternative picture can be drawn using the dual lattice in which the links are nodes. There, strings are represented by membranes, and closed strings correspond to closed membranes. The ground state is a 3D membrane condensate (in 2D, looking at the dual lattice doesn't give you a different picture).

### IV.2. Known 3D topological phases (not layered states like or Haah code)

Two types of phases are 3D gauge theories with boson/fermion charges and 3D Dijkgraaf/Witten theories with boson charges. The former are amenable to a string-net picture while the latter with a membrane-net theory 3D gauge theories coupled to bosonic charges like the toric code are elements of both.

Let's focus on Dijkgraaf/Witten (**quantum double model twisted** by a non-trivial $\alpha$), which can be defined in any spatial dimensions, but we focus on 2D for simplicity. The input data is a finite group $G$, a 3-cocycle

$$\alpha : G \times G \times G \to U(1)$$

obeying

$$\frac{\alpha(g_2, g_3, g_4)\,\alpha(g_1, g_2, g_3)\,\alpha(g_1, g_2 g_3, g_4)}{\alpha(g_1 g_2, g_3, g_4)\,\alpha(g_1, g_2, g_3 g_4)} = 1$$
$$\alpha(1, g, h) = \alpha(g, 1, h) = \alpha(g, h, 1) = 1$$

Each link can be in states $|g\rangle$, where $g \in G$ (e.g., in the toric code, you have two states which correspond to the two elements of $\mathbb{Z}_2$). The lattice is triangular (with orientations going left to right by the convention picked) and the Hamiltonian is

$$H = -\sum_{\text{vertices } v} Q_v - \sum_{\text{plaquettes } p} B_p\,.$$

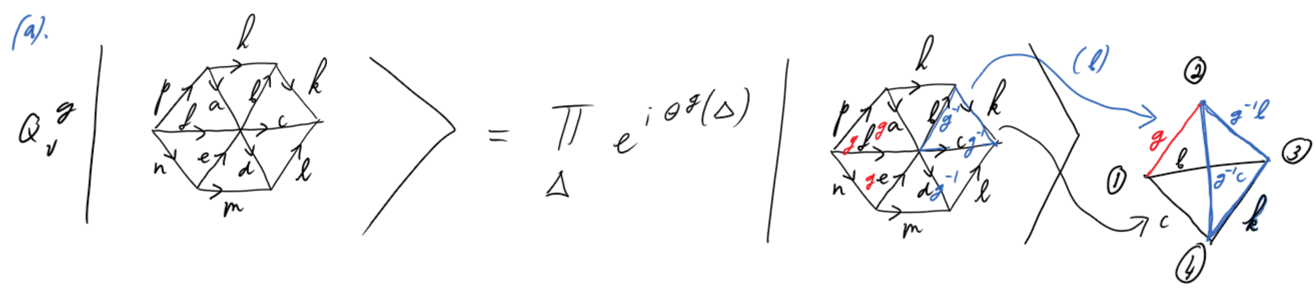
The latter term is easier, with branching rules determined by the group $G$:

$$B_p \left| {}^{g}\underset{k}{\triangle}{}^{h} \right\rangle = \delta_{ghk} \left| {}^{g}\underset{k}{\triangle}{}^{h} \right\rangle \qquad\qquad \delta_{ghk} = \begin{cases} 1 & ghk^{-1} = 1 \\ 0 & \text{otherwise} \end{cases}\,.$$

The $Q_v$ term is a 12-spin interaction:

$$Q_v = \frac{1}{|G|} \sum_{g \in G} Q_v^g$$

with




To compute the phase, complete the initial triangle into a tetrahedron using the final triangle [see (b) above], put $g$ on the remaining vertex, and go from 1 to 4:

$$\exp\left\{ i\theta^g \left( {}^{g}\underset{k}{\triangle}{}^{h} \right) \right\} = \alpha\left(g, g^{-1}b, k\right)^{-1}\,,$$

where inverse is because the tetrahedron is left-handed.

Once again, both $B_p$ and $Q_v$ are projections, model is exactly solvable, etc. If we didn't have any phases ($\alpha = 1$), this would be a generalization of the toric code to a gauge group $G$ — the usual quantum double model. When $G = \mathbb{Z}_2$, there are two possibilities for $\alpha$:

- $\alpha = 1$, getting the toric code.
- $\alpha(g, g, g) = -1$ and 1 otherwise, getting the doubled semion model. The map can be seen by mapping the triangular lattice to its dual, the honeycomb.
- All Dijkgraad/Witten models are equivalent to gauged group cohomology models with symmetry group $G$ — models for SPT phases with unitary symmetry group $G$.

To generalize to 3D, we triangulate 3D space, filling it with tetrahedra. The 3-cocycle extends to a 4-cocycle applied to tetrahedra, and the product of phases would be over 10 tetrahedra.

What is the effect of the twist, i.e., the difference between usual 3D gauge theory and D/W models? Let's focus on $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and compare the two types of models.

For the (doubled) 3D toric code, excitations can either be point-line $\{1, r, b, rb\}$ (on either copy) and flux-like $\{1, r, b, rb\}$. The statistics are exactly like for two copies of the toric code (remembering that we are braiding charges through loops.

The number of possible cocycles $\alpha$ for D/W models is $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$, leading to 3 non-trivial models (related to each other by re-naming). Their excitations and statistics are the same. Braiding two loops doesn't distinguish the models either.

The only distinction lies in braiding a loop through another loop while they are linked by a third "base" loop. Looking down on loops linked to a base loop, the other loops puncture the 2D plane of the base loop, leading to an almost 2D problem in which the punctures have braiding statistics. You need a base loop to get a non-trivial statistical phase; otherwise, you only get an AB phase.

- For the toric code, the phases obtained by this three-loop move are $\pm 1$.
- For D/W models, the phases are $\pm i$ (Wan/Levin 2014).