Boulder notes by Victor V. Albert

I. ARI TURNER

I.1. Hasting's theorem

Hastings proved that if a system **with local interactions** has long-range correlations in the ground state, then there is no gap (meaning there is a thermal conductivity and power-law heat capacity at low temperatures):

$$\langle\phi\left(x\right)\phi\left(y\right)\rangle=\frac{1}{\left|x-y\right|}\Rightarrow\text{gapless excitations}\,.$$

The contrapositive means that if there is a gap, then the correlations decay exponentially:

$$\text{gap}\Rightarrow\langle\phi\left(x\right)\phi\left(y\right)\rangle=e^{-\left|x-y\right|/\xi}\,.$$

The converse is not generally true (e.g., a ferromagnet has no correlations, but is gapless). This theorem relates to metals and insulators, which have power-law (Friedel oscillators divided by a power of r) and exponentially decaying correlations $\langle n\left(r\right)n\left(0\right)\rangle$, respectively. This theorem also reflects zero-point motion, which can give a more precise meaning of entanglement. We discuss this now.

EPR noticed that if a two-spin system is in an entangled state

$$|\psi\rangle=\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)$$

and if Alice measures σ_z, then Bob has to also have σ_z be down. Similarly, if Alice measures σ_x to be up, then Bob has σ_x to be down. But if Alice measures σ_z to be up and Bob decides to measure σ_x, then he has a 50/50 chance of getting up or down. This leads to Bell's inequalities...

A generalization of this is for a three-spin GHz state $\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle+|\downarrow\downarrow\downarrow\rangle\right)$. If two people measure σ_y and one measures σ_x, then either one of them or three of them will get -1. On the other hand, if all three measure σ_x, then an even number of them will get $+1$. If $\sigma_{x,y}$ expectation values are pre-defined and not random, this is impossible.

I.2. Entanglement

A composite state in a system with two parts A and B is **entangled** if it cannot be written as a product of a state in A and a state is B. In other words, the state is entangled if the sum

$$|\psi\rangle=\sum_{i}\sqrt{p_i}|i\rangle|i\rangle$$

has more than one term (given that both $|i\rangle$'s are an orthonormal set). This includes cases where A and B are different sizes because the $|i\rangle$'s do not have to form a basis. This is known as the **Schmidt decomposition**, and we can show it always exists by starting with a generic representation

$$|\psi\rangle=\sum_{a,b}r_{ab}|a\rangle|b\rangle$$

and performing a singular value decomposition of the matrix r_{ab}. The degree of entanglement of a composite state is related to the degree of mixedness of the individual reduced density matrices

$$\rho_{A,B}\equiv\text{Tr}_{B,A}\left\{|\psi\rangle\langle\psi|\right\}=\sum_{i}p_i|i\rangle\langle i|\,.$$

This matrix will not be in a definite (i.e., pure) state due to the state's correlations with the region B. Local observables can then be evaluated using only the reduced density matrix:

$$\langle\psi|O_A|\psi\rangle=\text{Tr}\left\{\rho_A O_A\right\}\,. \tag{1.1}$$

For an entangled state, the entropy $S_{AB} = \mathrm{Tr}\left\{\rho \log \rho\right\} = 0$, but entropies for the parts are equal:

$$S_A = -\sum_i p_i \ln p_i = S_B \,.$$

Then, given a subsystem A of a system, the entropy of the ground state in that part is related to the fluctuations of the state in A. Then the **mutual information**

$$I\left(A, B\right) = S\left(A\right) + S\left(B\right) - S\left(AB\right) = 2S\left(A\right)$$

which quantifies the correlations between the two systems. In a classical system, you expect a bigger system to have a bigger entropy, but than doesn't have to be true in QM. This can be generalized to the conditional mutual information

$$I\left(A, B \,|\, C\right) = S\left(AC\right) + S\left(BC\right) - S\left(C\right) - S\left(ABC\right) \,.$$

Thermalization means that the reduced density matrix follows a Gibbs distribution

$$\rho_A = \sum_i e^{-\beta E_i} |i\rangle\langle i| \,.$$

For generic p_i, one can then define the **entanglement Hamiltonian**

$$H = -\frac{1}{\beta}\sum_i \ln p_i |i\rangle\langle i| \qquad\qquad \rho_A = e^{-\beta H} \,.$$

It turns out this Hamiltonian typically has local interactions and its low-energy degrees of freedom are typically localized near the boundary. The entanglement entropy turns out to be proportional to the area separating A from the rest of the system:

$$S_A \propto L^{d-1} \,.$$

The topological entanglement entropy can help determine whether a system is topologically ordered.

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Eigenstate thermalization hypothesis states that if you have an ergodic system, then the expectation value of any operator in a high-energy state $|\psi\rangle$ of Hamiltonian H (with energy related to a temperature T) is

$$\langle\psi|O|\psi\rangle = \frac{\mathrm{Tr}\left\{e^{-\beta H} O\right\}}{\mathrm{Tr}\left\{e^{-\beta H}\right\}} \,.$$

This doesn't work for all operators (e.g., $O = |\psi\rangle\langle\psi|$), but may work for local ones. By eq. (1.1), we then see that

$$\mathrm{Tr}\left\{\rho_A O\right\} = \langle\psi|O|\psi\rangle = \frac{\mathrm{Tr}\left\{e^{-\beta H} O\right\}}{\mathrm{Tr}\left\{e^{-\beta H}\right\}} \,,$$

where ρ_A is the reduced density matrix on the region A of the support of the operator.

II.1. Entanglement Hamiltonian

When you accelerate at a constant rate, there is light that will never reach you. In such an accelerating reference frame, you will observe thermal radiation (Unruh effect). The Hamiltonian describing the thermal state of the radiation is related to the entanglement Hamiltonian defined above. We can calculate this thermal state using imaginary-time path integrals using the (Wick-rotated) 1+1D action

$$S_E = \int dx d\tau \, \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial \tau}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2} m^2 \phi^2\right] .$$

The wavefunction is then

$$\psi\left(\phi\right)=\int D\phi e^{-S[\phi(x,t)]}=e^{-H\infty}\,,$$

where the integral is over all paths of light up to present time. We split space into two, and calculate the state on the right half by tracing out the left half. In other words, the state on the right half is

$$\rho\left(\phi_R,\phi_R\right)=\sum_{\phi_L}\left|\psi\left(\phi_L,\phi_R\right)\right|^2\,.$$

To perform the integral, we use angular quantization of ϕ_R, with the angular direction being time and radial being space. The (Euclidean) action S_E in that change of coordinates is

$$\tilde{S}_{Euc}=\int rdrdt\,\frac{1}{2}\left[\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{2}m^2\phi^2\right]\,.$$

The evolution operator which evolves in a small angle $\phi_R\to\phi_R'$ (i.e., segment in time) is the action integrated over the radial direction:

$$\rho_{\phi_R\phi_R'}=e^{-H_E}=\left(1-\frac{H_E}{N}\right)^N\,.$$

The resulting entanglement Hamiltonian (after canonical transformation; recall that $S=\int dtH$) is

$$H_E=2\pi\int_0^\infty dr\frac{r}{2}\left[\left(\Pi_\phi\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+m^2\phi^2\right]\,.$$

Changing coordinates to $r=e^u$ and placing a cutoff $\log a$:

$$H_E=2\pi\int_{\log a}^\infty\frac{1}{2}du\left[\left(\Pi_\phi\right)^2+\left(\frac{\partial\phi}{\partial r}\right)^2+e^{2u}m^2\phi^2\right]\,,$$

there is an effective potential $v\left(u\right)=m^2e^{2u}$. The potential $v\left(u\right)$ is sharply increasing above $u=\ln\frac{1}{m}$ and sharply decreasing below $u=\ln\frac{1}{m}$. So we can approximate it with a box of length $\ln\frac{1}{ma}$. The entanglement entropy is just the thermal entropy of e^{-H_E}

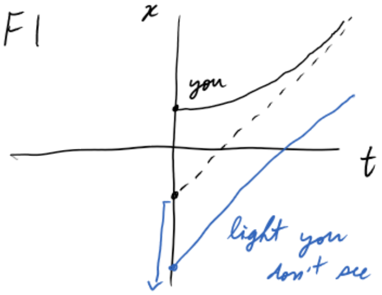
$$S_E=S\left(\ln\frac{1}{ma},\frac{1}{2\pi}\right)=\frac{c}{6}\ln\frac{1}{ma}\,,$$

where c is the central charge.

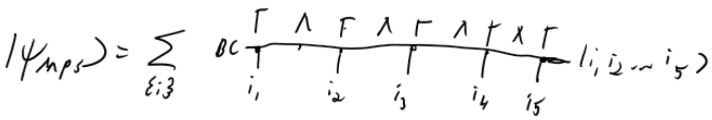
II.2. Tensor networks

Tensor network states of multiple spins s,t,m can be interpreted as coupling those spins to ancillary particles α,β,γ and tracing out those particles. More specifically, a matrix product state can be defined as shown in F2. Expectation value of quantities in the thermodynamic limit are equivalent to the projection on the left and right eigenvectors of the transfer matrix (also, double tensor).

F1



F2



F3