Disordered elastic systems

$\vec{u}(\vec{x})$ — $\vec{u} \in \mathbb{R}^N$, $\vec{x} \in \mathbb{R}^d$ internal

$$H_V[u] = \frac{1}{2}\int d^dx\, c(\nabla_x u)^2 + V(x,u(x))$$

$$\frac{1}{2}\int_q c q^2 u_{-q} u_q$$

$$\overline{V(x,u)\,V(x',u')} = \delta^d(x-x')\,R_0(u-u')$$

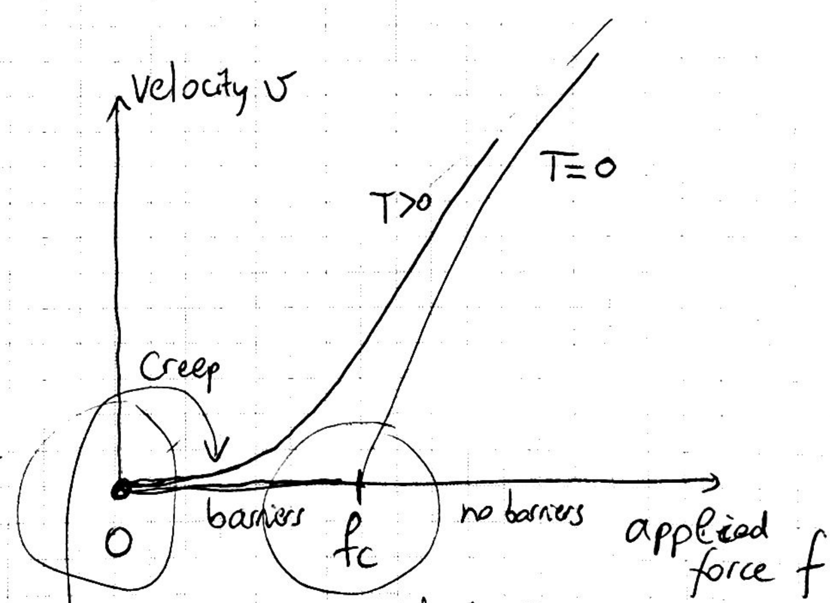
R_0: smooth, varies scale r_f

$V(x,u)$, $\in \mathbb{R}^D$, $D = d+N$

- $R_0(u)$ SR random bond
- $R_0(u)$ LR $\sim |u|^{-\gamma}$, random field $\sim \sigma |u|$

$\Rightarrow$ interfaces $N=1$, $u(x) \in \mathbb{R}$

Magnetic DW
$D = 1+1$
film



statics $T=0$ min
$T>0$ equil

$$\overline{(u(x)-u(0))^2} \sim x^{2\zeta_{eq}}$$

$d=1$ $\zeta_{eq} = 2/3$

critical $T=0$ FP

depinning
non-eq critical pt FP

$$v \sim e^{-\frac{U_c}{T}\left(\frac{f_c}{f}\right)^{\mu}}$$

highly non linear

$\mu \simeq 1/4$

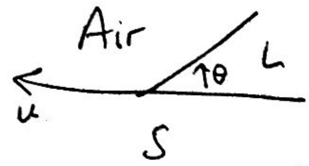
$\lim_{f \to 0} = 0$, $\mu = 0$

diverging barriers slow

TODAY

- eq m_0
- f_c estimate it? where from?
- exp. statics $\rightarrow$ creep law
- depinning critical pt FRG..

Contact line depinning partial wetting

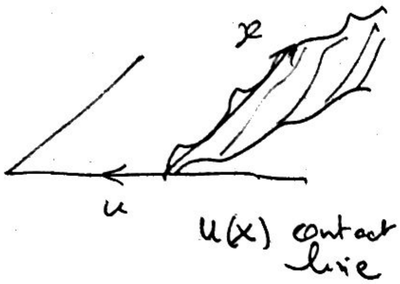


$$E = \gamma A_{LA} - \gamma' A'_{LS}$$

area

$\downarrow$

$\gamma_{SA} - \gamma_{LS}$



$f = -\gamma\cos\theta + \gamma'$ pushes CL

$\theta > \theta_{eq}$ $f > 0$ advances

$\cos\theta_{eq} = \gamma'/\gamma$

$<$ $<$ recedes

fill reservoir ~~flux~~ $\Downarrow$ $\theta \uparrow$

pinning glass plate

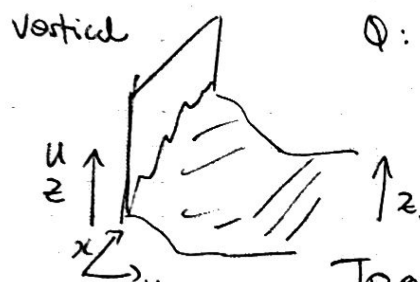
$\gamma'(x, u)$

what type of disorder SR, LR ?

Q: random bond or RF disorder ?

pinning + $\theta_a > \theta_{eq} > \theta_r$ hysteresis

– RF disorder –



Q: elastic energy of CL ?

γ area surface (minimize)

B.C $z(x, y=0) = u(x)$

LR non-local elast.

Joanny de gennes

(we extended it arbitrary angle)

$$E \simeq \int_q \frac{1}{2}\gamma |q| u_{-q} u_q \simeq \frac{\gamma}{4\pi}\int_{xx'} \frac{(u(x)-u(x'))^2}{(x-x')^2}$$

Capillary length

effect of gravity $q_{cap} = \frac{1}{L_{cap}} \sim \sqrt{\frac{\rho g}{\gamma}}$

$$\int dx\, \gamma \frac{q_{cap}}{2}(u(x) - z_0)^2$$

LR elasticity

+ anisotropy $cq^2 \to q^\alpha$

$z_0 \uparrow$ pulled by a spring

$z_0 = vt$ driven fixed velocity

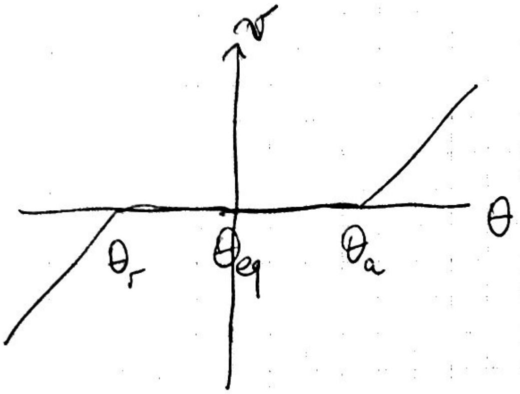
→ dipolar forces DW + anisotropy q_x, q_y

→ ferroelectrics

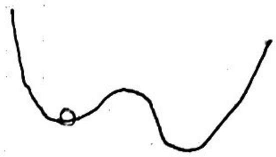
→ crack tip in fracture $\sim q$ $\Rightarrow \zeta_{theory} \simeq 0.39$

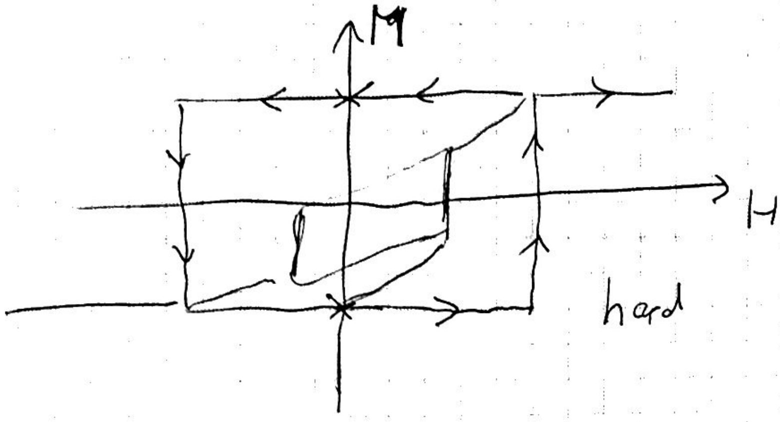
→ dry frict. 2 plates

LR fields (E, B) or ~~medium elastic~~

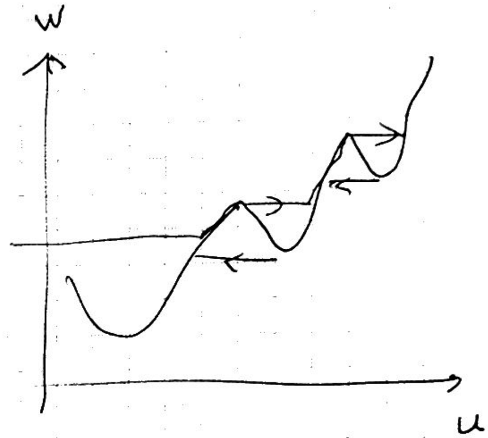


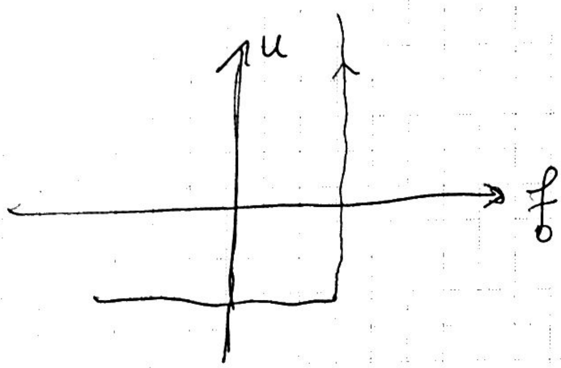
hysteresis
in u (not in v)



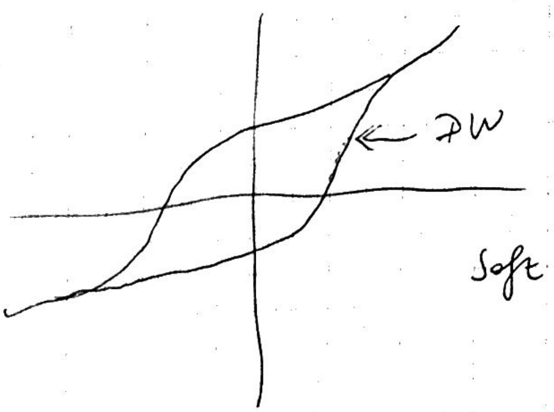


hard





$$w = u - \frac{F(u)}{m^2}$$



soft

$$\eta \dot{u} = F(u) + m^2(w - u)$$

fluctuation of 1d edge of 2d medium

$$\langle u_{q_x, q_y}\, u_{-q_x', q_y'} \rangle = \frac{T}{q_x^2 + q_y^2}\, \delta_{\vec{q}', \vec{q}}$$

$$\langle u(y=0, q_x)\, u(y=0, q_x') \rangle$$

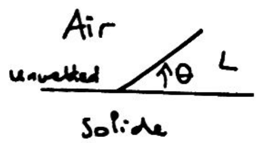
$$= \int \frac{dq_y}{2\pi} \frac{T}{q_x^2 + q_y^2}\, \delta_{q_x', q_x}$$

$$\Longleftarrow \quad |q_x|\, |u_{q_x}|^2 \sim T/q_x$$

$$\int \frac{dq_y}{q_y^2 + q_x^2} \longrightarrow \frac{1}{|q_x|}$$

Contact line depinning

Gaetan Borot

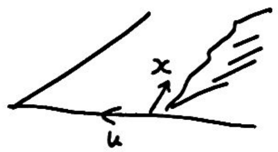


wants to be minimal surface ↑

$$E = \gamma A_{LA} - \gamma' A'_{LS} \qquad \gamma' = \gamma_{SA} - \gamma_{LS}$$

≠ surf. tension between SA and SL interfaces

$f = -\gamma \cos\theta + \gamma'$ pushes CL to unwetted ($\gamma' > 0$ increase A'_{LS})

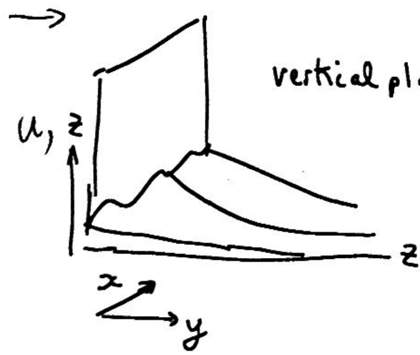


$\cos\theta_{eq} = \gamma'/\gamma$ $\quad \theta > \theta_{eq}$ $f > 0$ advancing

$\theta < \theta_{eq}$ receding

$\gamma'(u, x)$ $\quad \theta_a > \theta_{eq} > \theta_r$ pinning hysteresis $\qquad$ elastic line elasticity? (GLASS champaign)

exert > 0 advance $\quad$ < 0 recede



vertical plate $\quad$ fluid surface $z(x, y)$

$$z(x, y = 0) = u(x)$$

$$E = \gamma \int_0^\infty dx\, dy \, \left(1 + (\partial_x z)^2 + (\partial_y z)^2\right)^{1/2} + \frac{1}{2}\rho g (z - z_0)^2 \Big)$$

LS

$$\simeq 1 + \tfrac{1}{2}(\partial_x z)^2 + \tfrac{1}{2}(\partial_y z)^2$$

local force $\gamma'(x, u(x))$

wetted area

$$- \gamma' \int dx\, u(x)$$

$$- \int dx \int^{u(x)} \gamma'(x, u')\, du'$$

Solve $z(x, y) \big|_{u(x)}$ $\qquad$ RANDOM FIELD disorder (area swept) ⟵

$$K = \frac{1}{L_{cap}} = \sqrt{\frac{\rho g}{\gamma}}$$

elastic energy (Joanny de Gennes)

$$\frac{E}{\gamma} = \frac{1}{2} \int \frac{dq}{2\pi} \left(\sqrt{K^2 + q^2} - K\right) u_{-q} u_q + \int dy\, \frac{K}{2} \left(u(y) - z'_0\right)^2$$

$$\int_q \equiv \int \frac{dq}{2\pi}$$

$g \to 0$, $x \ll L_c$, $q \gg K$: $\quad \int_q |q| u_{-q} u_q$ non local elasticity

$$z'_0 = z_0 + L_c \cos\theta_{eq}$$

real space $\quad E = \dfrac{\gamma}{4\pi} \displaystyle\int_{xx'} \frac{\left(u(x) - u(x')\right)^2}{(x - x')^2}$

4π or 2π?

Cracks

⇔ pulled by a spring

CL feels quadratic well

increase z'_0 ⟷ fluid level in reservoir

$(g = 0)$

$z(x, y=0) = u(x)$

$(\partial_x^2 + \nabla_y^2) z = 0$

$z = u_q e^{-|q| y + i q x}$

$$\frac{E}{\gamma} = -\frac{1}{2} \int_q^{\text{line}} u_q \partial_y u_q$$

$z'_0 = vt$

drive at fixed velocity

u fluctuates around vt

Moulinet Guthmann Rolley 2002 EPJE
xxxx

Experiment viscous liquid (overdamped dynamics)
water, glycerol/water $\gamma \sim 70\,10^{-3}$ N/m

previous He on Ce (inertial?)
superfluid
overshoots?

$L_{cap} \sim 2.5$ mm

climbs glass plate + chromium defects $\neq$ wettability contact angle
controlled artificial litho $\xi \sim 10\mu m$
+ smaller scale disorder nm? square random positions

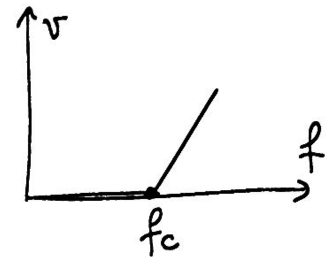
1) increase level reservoir z_0
2) observe contact line microscope $\mu m \to mm$ 2cm

$\to$ CL rough $\quad W = \overline{(u(x+L)-u(x))^2}^{\,tr} \sim L^{2\zeta}$

$\zeta \simeq 0.51 \pm 0.03$

Larkin $\int \frac{1}{q^2} \to \zeta = 1/2$

$\to$ System is near depinning
motion is quasistatic
($\neq$ creep) - not activated



$\to$ overdamped
$\to$ quasi-static

$\to$ v imposed $\quad$ v small $\quad$ CL motion discontinuous ~~[illegible]~~
pinned positions, jumps from one to next, avalanches
observe jumps, no inertial effects $\gg \xi$ $\quad$ - See no overshoots -
retardation NL

Microscopic level
$$\eta \partial_t u(x,t) = - \frac{\delta H_{el}}{\delta u(x)}\Big|_{u(x,t)} + F(x,u(x,t)) - k\,(u(x,t) - z_0'(t))$$

$\frac{\delta H_{el}}{\delta u(x)} \to -\gamma |q|\, u_q(t)$

$F \to = -\partial_u V(x,u(x))$
$\gamma'(x,u(x))$

random field
SR force

$\to$ avalanches $\quad$ fast jump + secondary jumps

define?
start - stop?
where
is 4th jump
consequence of 3 previous ones?
or new? $\quad$ criterion

will see def
$v \to 0^+$ OK

$\to$ motion deterministic $\quad (T=0)$
undergoes same sequence jumps
(almost)

avalanche $\Delta u \sim L^{\zeta}$
distribution $\quad P(L) \sim L^{-\mu}$
all scales, critical.

$\boxed{\Delta u\, L \equiv \text{SIZE} \quad (\text{magnetization jump in magnet}) \;\; M}$

- LR elasticity
~~(nonlocal)~~

$c q^2 \longrightarrow q^\mu$

$\mu < 2$

$(\mu = 1)$

+ anisotropy

$q_x, q_y, \ldots$

either genuine LR interactions

or ~~effective~~ mediated by medium (integrated on)

→ contact line of a fluid

→ LR dipolar forces for magnetic interfaces

→ ferroelectrics

→ dislocation lines

→ vortex lattices ξ, λ

- LR disorder along x
columnar defects SC

- random field spin models $\vec{h}(x) \cdot \vec{S}(x)$

$\vec{S}(x)^2 = 1$

$O(\tilde{N})$ spin waves non linearities important $\nabla\pi$ and π

Similar spirit
Calcul more involved

$(\nabla \vec{S})^2 \to (\nabla \pi)^2 + (\nabla \sqrt{1-\pi^2})^2$

only $\tilde{N} = 2 \iff$ periodic $N = 1$

$\tilde{N} = 1$ Ising
hard to control

FRG, Dina
Felman
us
Tarjus group

Clarify role of non-linéarities !

Equation of motion $(T=0)$

- 1 particle

Classical eqmo $H(p,u) = \frac{p^2}{2M} + V(u)$

$u \equiv u(t)$
$p \equiv p(t)$
$\dot{u} = \frac{du}{dt} \equiv \partial_t u$

$$\dot{p} = -\frac{\partial H}{\partial u} = -V'(u) \quad , \quad \dot{u} = \frac{\partial H}{\partial p} = \frac{p}{M}$$

$$\cancel{M\ddot{u}} + \eta\dot{u} = -V'(u) + f$$

deriv of bounded potential

applied external unbounded $H_f = -fu$

add friction
$\dot{u}$ does not grow unboundedly f
damping mechanisms

toy model
MF model

$\tau_d = \frac{M}{\eta}$

$t \gg \tau_d$
$\omega \ll \omega_d = \tau_d^{-1}$

neglect inertial terms

large f $v = \dot{u} \simeq f/\eta$
linear

$\eta \to \eta(\dot{u})$ dynamical friction
$\neq$ static one $\eta \sim$
aging contacts / contact area increases $\uparrow t$

- Elastic chain

$$\eta \dot{u}_j = K(u_{j+1} + u_{j-1} - 2u_j) + f$$

$-\sum_i f u_i$ Nf total force

$$= -\frac{\delta}{\delta u_j} \Big[\underbrace{\frac{1}{2}\sum_\ell K(u_{\ell+1} - u_\ell)^2}_{H_{el}[u]} \Big] + f$$

- Continuum

$u(x,t)$

$$\eta \partial_t u(x,t) = -\frac{\delta}{\delta u(x,t)} H[\{u(x',t)\}]$$

$F(x,u) = -\partial_u V(x,u)$

$$= +c\nabla_x^2 u(x,t) + F(x,u(x,t)) + f$$

elastic force

pinning force disorder

$H_f = -\int d^dx\, f u(x,t)$

f per unit length
fL total force

driving by quadratic well

$$H_{driv} = \int dx \frac{m^2}{2}(u(x) - w(t))^2$$

$$+f \longrightarrow + m^2(w(t) - u(x,t))$$

Eqm $T>0$

thermal noise

Langevin equation

detailed balance

glauber dynamics / Monte Carlo

Ornstein - Uhlenbeck process

relaxation in quadratic well

$V(u) = \frac{m^2}{2} u^2$

- 1 particle

$$\eta \dot{u}(t) = -m^2 u(t) + \xi(t)$$

require is Convergence to $\rightarrow P_{eq}(u) = e^{-\frac{m^2}{2T}u^2}$

choose $\xi(t)$ gaussian white noise

$$\langle \xi(t)\xi(t')\rangle = 2\eta T \; \delta(t-t')$$

$\leftarrow$ who knows ?

$t \to \eta t$ absorb η

$$u(t) = \int_0^t dt' \, e^{-\frac{m^2}{\eta}(t-t')} \xi(t') + u(0) e^{-\frac{m^2}{\eta}t} \quad \rightarrow \langle u(t) u(t')\rangle$$

assume TTI

Fourier $u(t) = \int_\omega e^{i\omega t} u_\omega$ $\qquad (\eta i \omega + m^2) u_\omega = \xi_\omega \qquad \langle \xi_\omega \xi_{\omega'}\rangle = 2\eta T \, 2\pi\delta(\omega+\omega')$

$\int_\omega \equiv \int \frac{d\omega}{2\pi}$

$$\langle u_\omega u_{\omega'}\rangle = \frac{2\pi\delta(\omega+\omega') \; 2\eta T}{\eta^2\omega^2 + m^4}$$

$$\langle u_t u_{t'}\rangle = \int \frac{d\omega}{2\pi} \frac{2\eta T}{\eta^2\omega^2+m^4} e^{i\omega(t-t')} = \frac{T}{m^2} e^{-\frac{m^2}{\eta}|t-t'|}$$

$\int \frac{dz}{2i\pi} \frac{2}{z^2+1} = 1$

residue 1

$\omega \to \omega/\eta \qquad \omega = \pm i m^2$

$$\langle u_t u_t\rangle = \frac{T}{m^2} = \langle u^2\rangle_{eq}$$

and gaussian u

more generally $\qquad \eta \dot{u} = -V'(u) + \xi(t)$

$$P(u,t) \to P_{eq}(u) = e^{-V(u)/T}$$

Fokker Planck eq. associated to Langevin eq

$$\eta \partial_t P(u,t) = -\partial_u J(u,t)$$

$$J(u,t) = -T\partial_u P(u,t) - V'(u) P(u,t)$$

$$P = P_{eq} \Leftrightarrow J_{eq} = 0$$

if $V(u)$

 $\rightarrow$ eq

if $V(u) \rightarrow V(u) - fu$

no P_{eq}

instead a stat solu $J>0$ (ring or inject current)

- interface $\qquad \eta \partial_t u(x,t) = c \nabla_x^2 u(x,t) + \xi(x,t)$

$$\langle \xi(x,t)\, \xi(x',t') \rangle = 2\eta T\, \delta(t-t')\, \delta(x-x')$$

Space-time gaussian white noise

$$\eta \partial_t u_{qt} = \underbrace{-cq^2}_{-m^2} u_{qt} + \xi_{qt}$$

$\langle \xi \xi \rangle = 2\eta T \delta(t-t')$

each mode equilibrate in its potential well

time scale $\tau_q = \frac{\eta}{cq^2}$

$$u_{q\omega} = \frac{\xi_{q\omega}}{\eta i\omega + cq^2}$$

$$\langle u_{q\omega} u_{q'\omega'} \rangle = \delta_{q',-q}\, \delta_{\omega',-\omega}\, \frac{2\eta T}{\eta^2\omega^2 + c^2 q^4}$$

$$C_{q,t-t'} = \langle u_{qt}\, u_{q't'} \rangle = \delta_{q',-q}\, e^{-\frac{cq^2}{\eta}|t-t'|}\, \frac{T}{cq^2}$$

$$R_{q,tt'} = \mathrm{TFI}\, \frac{1}{i\eta\omega + cq^2} = \frac{1}{\eta} e^{-\frac{cq^2}{\eta}|t-t'|}\, \theta(t-t')$$

Causality

$$P_{eq}[u] \sim e^{-\frac{1}{T} \underbrace{\int_q \frac{1}{2} c q^2 u_{-q} u_q}_{H_{el}[u]}}$$

more complicated dynamics

Conserved order parameter

$\int u(x)\, dx \qquad \eta \partial_t u = -\nabla^2 \frac{\delta H}{\delta u}$

imbibition / fluid invasion (one) into another..

Santucci et al..

addit. fields..

CL complicated flow liquid..

Pinning and depinning / Larkin – Ovchinnikov theory (~80)

1) periodic potential "trivial" example

$V(x,u) = g \sin u$

particle $(d=0)$

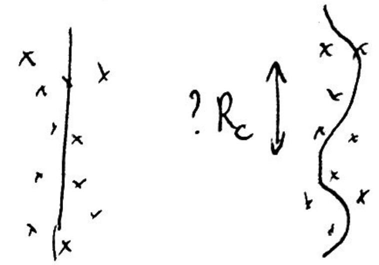
$\eta \dot{u} = -g \cos u + f \qquad f_c = g$

$= g(1-\cos u) + f - f_c$

idem for line

2) random $V(x,u)$

what is f_c for rigid line ? $(C \to \infty)$



? R_c f_c ?

$F_{\text{dis}} = \int dx\, \partial_u V(x,0) \sim \chi L^{1/2}$

$F_{\text{tot}} = -fL \qquad \Rightarrow f_c = 0$

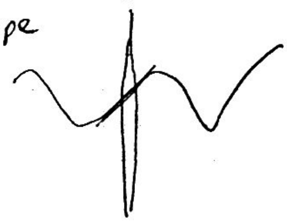
$f_c > 0$

because $u(x)$ deforms to adapt to RP.

how ?

Larkin model $V(x,u) \to -f(x)u$

idea elastic strong replace slope



Sketch

$$H = \int \frac{c}{2} (\nabla u)^2 d^d x - \int d^d x\, f(x) u(x)$$

$\overline{f(x) f(x')} = W \delta^d(x-x')$

$W = -R_0''(0) = \Delta_0(0)$

$\frac{\delta H}{\delta u(x)} = 0 \qquad u_0(x)$ min energy

$-c \nabla^2 u = f(x)$ unique solution

$H_V[u] = E^0 + \int_x \frac{c}{2} (\nabla(u-u_0))^2$

$u_q^0 = \frac{f_q}{c q^2} \qquad \overline{f_q f_{q'}} = \delta_{q',-q} W$

$\overline{\langle u_q \rangle \langle u_{q'} \rangle} = \overline{u_q^0 u_{q'}^0} = \frac{W}{c^2 q^4} \delta_{q',-q}$

disorder part

$\overline{\langle u_q u_{q'} \rangle - \langle u_q \rangle \langle u_{q'} \rangle} = \frac{T}{c q^2} \delta_{q',-q}$

Connected correl

"thermal part" $\to$ Subdominant (bounded $d > 2$)

$\to$ Identical as no disorder

TRUE FOR COMPLETE MODEL

$T=0$

$$B(x) = \overline{(u(x)-u(0))^2} = 2\frac{W}{c^2}\int \frac{d^dq}{(2\pi)^d}(1-\cos qx)$$

$$\sim \frac{W}{c^2}\tilde{A}_d |x|^{4-d} \qquad d<4$$

$$\simeq \frac{W}{c^2}\frac{1}{8\pi^2}\ln|x|\Lambda \qquad d=4$$

$$\approx \frac{W}{c^2} a_0^{4-d} \qquad d>4$$

(ordered state survive)

Larkin 70'

displacements grow $d<4$

$d<d_{uc}=\alpha$

any weak disorder

ordered state destroyed

? new FP

→ too fast growth

$u \sim x^{\zeta_L}$

$\zeta_L = \frac{4-d}{2}$

↓ ~~approx~~ approx OK

identify upper critical dim

$d>d_{uc}$ bounded ordered state

$x \leq R_c$ Larkin length

$$B(R_c) \sim r_f^2$$

$$R_c \sim \left(\frac{c^2 r_f^2}{W}\right)^{1/4-d}$$

smale W weak disorder

$R_c \gg a_0$

below R_c

- single minimum
- elastic response

above R_c

- many minima metastable states

~~$f_c R_c^d \equiv \int f(x) d^dx = \sqrt{W} R_c^{d/2}$~~

~~$f_c = R_c^{-d}\int_{R_c} d^dx\, f(x) = \sqrt{W} R_c^{-d/2}$~~

~~$\equiv R_c^{-d}\int_{R_c} d^dx \frac{c}{2}(\nabla u)^2$~~

Imry Ma

Fukuyama Lee Rice

Larkin Ov.

$$f_c \sim R_c^{-d}\int_{R_c} d^dx\, f(x)$$

$$\sim R_c^{-d/2}\sqrt{W}$$

$$\sim \frac{c r_f}{R_c^2}$$

~~$\int d^dx \frac{c}{2}(\nabla u)^2 \sim c r_f^2 R_c^d$~~

~~$\sim \int d^dx\, f(x) u \sim \sqrt{W} R_c^{d/2} r_f$~~

~~$\sim \int d^dx\, f u \sim$~~

theory of collective pinning (confirmed by more sophisticated methods)

~~$U_c \sim f_c r_f R_c^d$~~ $f u$

scale of barrier at Larkin scale

$$U_c \sim c r_f^2 R_c^{d-2} \sim W^{1/2} r_f R_c^{d/2} \sim f_c r_f R_c^d$$

elastic   disorder   driving

Leo exercise do it with replica

$$H_{rep} = \sum_{ab=1}^{n} \underbrace{\left(\frac{c}{2T} q^2 \delta_{ab} - \frac{W}{2T^2}\right)}_{\Phi(q)_{ab}} u^a_{-q} u^b_q$$

$$\langle u^a_q u^b_{q'} \rangle_{H_{rep}} = \Phi(q)^{-1}_{ab} \Big|_{n\to 0}$$

$$= \delta_{ab} \left(\overline{\langle u_q u_{q'} \rangle} - \overline{\langle u_q \rangle \langle u_{q'} \rangle} \right) + \overline{\langle u_q \rangle \langle u_{q'} \rangle}$$