Landscape of quantum phases and phase transitions

T. Senthil*

Department of Physics, Massachusetts Institute of Technology, Cambridge, MA, 02139

(Dated: July 2, 2025)

I provide a general discussion of the landscape of phases and phase transitions of equilibrium quantum matter. I discuss the physics of some illustrative examples through physical non-perturbative points of view that demonstrate the intellectual connections between these different phases. (As it stands this is a draft with incomplete referencing/acknowledgement.)

Lectures given at the Boulder Summer School 2025

*Electronic address: senthil@mit.edu

I. INTRODUCTION

In these lectures, we will study quantum phases and phase transitions of equilibrium matter. We will begin with a general survey of the landscape and will then move on to discussing aspects of particular phases/phase transitions.

By a 'quantum phase of matter', we will mean the ground state of a quantum Hamiltonian which we will usually take to be a sum of local terms[1] (such as, eg, $H = J\sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$, the Hubbard model, etc). There are many questions one could discuss in this topic. What kinds of quantum phases of matter can exist? What are their universal low energy properties? What mechanisms can stabilize them in physical realizations? What is the nature of the quantum phase transition between distinct phases? Can we develop good computational methods to explore the phase diagram in realistic microscopic settings? What experimental probes are the most revealing to identify any given quantum phase of matter?

In these notes, we will focus on a subset of these questions.

II. THE LANDCAPE

Let us orient ourselves by surveying the distinct kinds of quantum phases that can exist, and providing some rough organizing principles to characterize them.

A. Landau ordered phases of matter

These are characterized by the concepts of spontaneously broken symmetry and long range order. Examples include crystalline long range in a solid, various forms of magnetic ordering (ferromagnetism, antiferromagnetism,....), superfluids, and so on. Some of these states (such as the solid and the ferromagnet) have of course been known for milllenia. Such broken symmetry states were first discussed systematically by Landau, and are described through an order parameter that measures the extent of the broken symmetry. They are hence known as Landau ordered states.

[1] For some phenomena it will be important to include the long range Coulomb interaction. Also it is some times useful to consider toy models with all-to-all interactions that may be solvable.

B. Non-Landau order I: Topological Quantum Matter

These have been studied since the 1980s. We define these to be ground states of many body quantum systems with a gap to all excitations.

It is useful to distinguish two categories of topological quantum matter. First, we have *Symmetry Protected Topological* (SPT) phases[2]. These have no exotic excitations (such as, eg, ones with fractional quantum numbers or statistics) but yet can exist as distinct phases that cannot be smoothy connected within the space of symmetry preserving gapped Hamiltonians. Examples include topological insulators and the $d = 1$ Haldane spin-1 chain. Novel phenomena occur at the interface between two distinct SPT phases, or at a spatial boundary (regarding the vacuum as a trivial SPT). A review of SPT physics is in Ref. [1].

Second, we have matter with *'topological order"*[2]. Examples include fractional quantum Hall phases, and gapped quantum spin liquids. These have emergent *sharp* quasiparticle excitations with anyonic statistics (an infinitely long ranged 'statistical' interaction) and the possibility of fractional quantum numbers.

For both SPT and for topological ordered matter, the low energy effective theory is what is known as a Topological Quantum Field Theory.

These states are of course not captured by the concepts of broken symmetry[3] and a Landau order parameter. As a side note, just like multiple broken symmetries can coexist in the same system, we can also have co-existence of different non-Landau orders or of Landau and non-Landau order.

---

[2] A slight generalization includes gapped phases with no exotic excitations, and which are protected even in the absence of any symmetry. An example is the Kitaev Majorana chain in $d = 1$. This general class is known as "invertible topological phases". Any phase in this class has an "inverse" phase such that the combination of it and its inverse is a completely trivial phase.

[3] Actually it has become popular in some circles to say that some such phases can be captured by the Landau paradigm if one generalizes the notion of symmetry. For a review, see, eg, Ref. [3]. For instance topologically ordered states of matter are considered to spontaneously break 'higher-form' symmetries rather than ordinary (0-form) symmetries. Though interesting and useful, in applying this point of view to the systems typically of interest in condensed matter physics, we must recognize that these higher form (or other generalized) symmetries are not present in the microscopic system. Thus the full package of phenomena involves the emergence of the higher form symmetries in the first place and their spontaneous breakdown. This is different from the conventional Landau paradigm. Indeed a power of the Landau viewpoint is that we focus on *microscopic symmetries* - which are assumed to be known - and ask how we can classify different ground states according to whether these symmetries are preserved or spontaneously broken in the ground state. Incorporating topological ordered or other "non-Landau" phases into the Landau paradigm requires not only generalizing the notion of symmetry but also generalizing what is meant by the Landau paradigm itself, and I will hence not use this terminology.

C. Non-Landau order II: Beyond topological matter

Finally we can have gapless quantum phases of matter where there are gapless excitations that cannot be understood as Goldstone modes of a broken continuous symmetry. The most familiar example is the Landau Fermi liquid. This ground state has no broken symmetry/order parameter. Nevertheless, there is structure in the ground state wave function that protects gapless excitations (a large number of them!). Interesting (and simpler) variants include the Dirac/Weyl materials[4] that have been studied extensively in the last two decades. In these examples, the gapless excitation spectrum can be given a quasiparticle interpretation, famously and subtly so for the Landau fermi liquid (and more straightforwardlly for Dirac/Weyl materials).

The most difficult examples of gapless phases are those where the low energy excitation spectrum does not admit a quasiparticle description at all. Examples include models of non-fermi liquid metals, the composite Fermi liquid in the half-filled Landau level, some gapless quantum spin liquids, etc. There has been slow but steady progress in our understanding of such quantum phases of matter without quasiparticles, and there will probably be many surprises in the future.

D. Critical quantum matter

These refer to quantum matter tuned to a continuous $T = 0$ phase transition. The textbook example[5] is the phase transition between a Landau-ordered phase with a broken symmetry and a trivial gapped phase that preserves all symmetries. In this case, the phase transition is usually[4] described by a quantum version of the Landau-Ginzburg-Wilson-Fisher (LGWF) theory of a fluctuating order parameter field, In other words, in this case, the critical singularities are determined by long wavelength low frequency fluctuations of the Landau order parameter.

In all other cases, the LGWF paradigm fails. Obviously if one of the two phase itself has non-Landau order, then the phase transition involving it will not be captured through a theory of a fluctuating order parameter field. Examples of this kind include (a). Quantum Hall plateau transitions (b) Magnetically ordered phase - quantum spin liquid transitions (c) A paramagnetic Fermi liquid - magnetically ordered fermi liquid, and many others. The last example - that of the onset of broken symmetry in a Fermi liquid - must at the very least involve coupling of the order

[4] Surprisingly, even in this situation, there is an example[6] where the phase transition can happen through a route not described by LGWF theory.

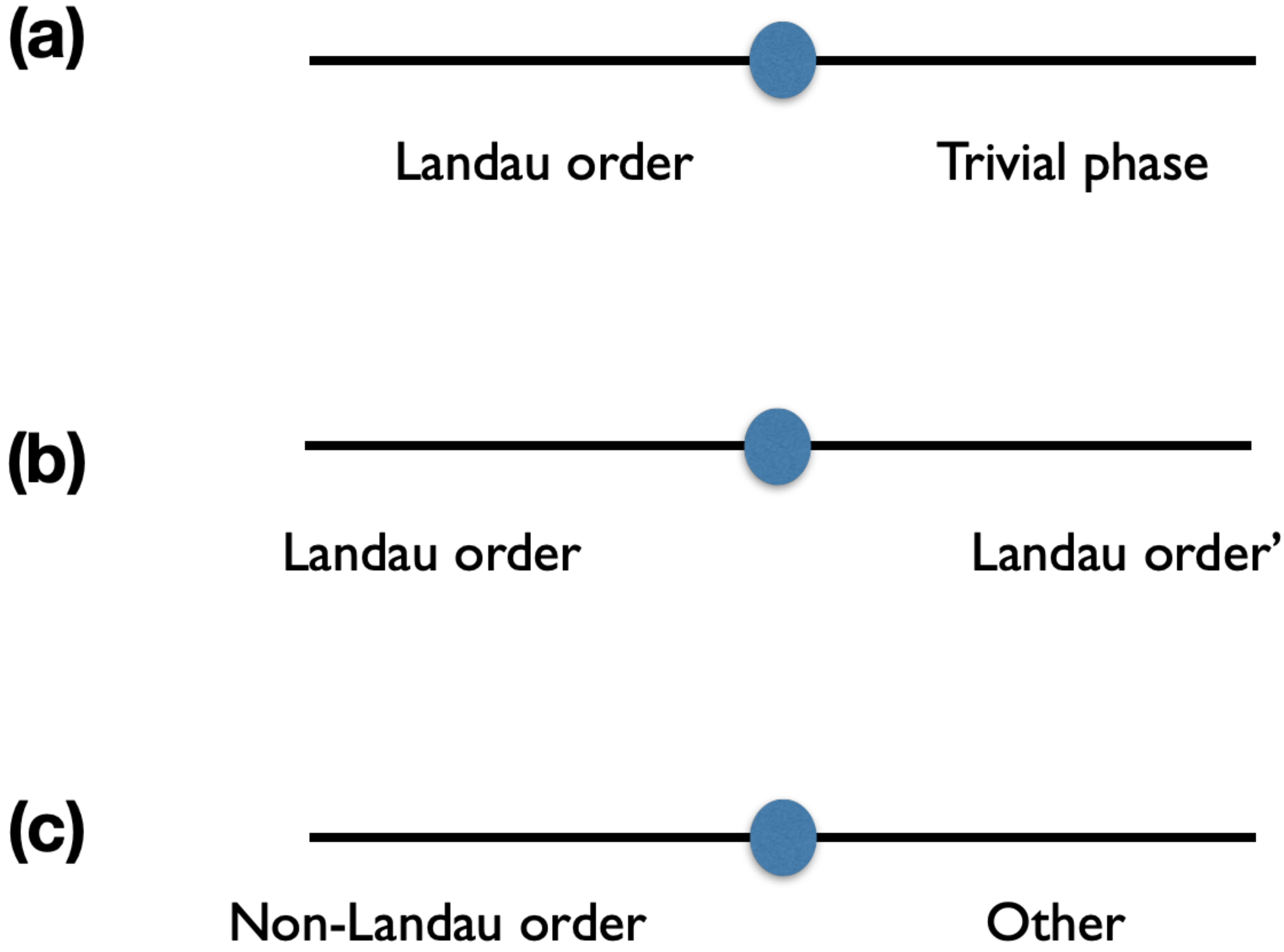

FIG. 1: Varieties of quantum critical points. (a) The only case where the traditional LGWF paradigm is usually applicable. (b) Unless the unbroken symmetry group of one phase is a subgroup of the unbroken symmetry of the other phase, the transition is Landau-forbidden (c) Non-Landau order, if present in at least one of the two phases guarantees a transition beyond the LGWF paradigm.

parameter fluctuations to the gapless excitations of the Fermi liquid (known as the Hertz theory). For a review, see Ref. [7]. This minimal modification of LGWF theory in a metallic system is not only a hard problem in itself but may also be fundamentally inadequate to capture the physics of the quantum critical point in several experimental realizations, particularly in heavy fermion systems.

A different kind of example is provided by Landau-forbidden continuous phase transitions between phases that themselves are Landau allowed[8, 9]. In these examples, the critical theory can be expressed in terms of emergent gauge fields coupled to gapless matter fields with fractional quantum numbers though these are not excitations of either phase. These are known as Deconfined Quantum Critical Points, and are best established for the Neel to Valence Bond Solid phase transition in $2d$ square lattice $SU(N)$ magnets for large enough N. See Ref. [10] for a review.

A crucial point that has emerged in the last few decades is that there are intimate intellectual connections between the theories that describe these diverse phases and phase transitions. Thus the study of topological quantum matter provides important insights into the study of gapless quantum matter and vice versa. So it makes sense to have a good understanding of all of these distinct ground states. In the next sections, we will provide some vignettes of this landscape through a few illustrative examples. We will revisit familiar non-Landau phases of matter from possibly unfamiliar (to some, if not all, readers) points of view so as to deepen our understanding of them.

III. ILLUSTRATIVE EXAMPLES

A. Topological insulators revisited

Consider a time reversal invariant system of charged particles in $3d$. We assume charge conservation so that there are global $U(1)$ and time reversal $\mathcal{T}$ symmetries. Further, as usual, we assume that the electric charge is even under time reversal. We are interested in gapped insulating phases of such a system such that the only excitations are electrons or their composites (thus there are no excitations with fractional charge/braiding statistics).

Let us start with some simple but powerful observations. All excited states can be labeled by an integer charge ne ($n \in \mathcal{Z}$ where e is the electron charge. Excitations with odd n will be fermions while those with even n will be bosons. Consider a 'gedanken' experiment where we probe the system by placing a magnetic monopole M (of basic strength $q_m = h/e \equiv 2\pi/e$) inside the medium. Note that the monopole is an external probe and *not* an excitation of the system. We know that electric fields are even under $\mathcal{T}$ while magnetic fields are odd. Thus, while the electric charge is even under $\mathcal{T}$, the magnetic charge q_m is odd.

Now suppose that the probe monopole M nucleates some electric charge q_e. Its time reversed partner TM will have the opposite magnetic charge $-q_m$ but must have the same electric charge q_e. Imagine bringing together M and TM to get rid of all the magnetic charge but have a total electric charge $2q_e$. But once the magnetic charge is gone, the result must be an excitation of the insulator we are probing. Thus we must have

$$2q_e = ne \tag{1}$$

where n is an integer. This equation has two distinct classes solutions. We may have $q_e = 0, e, 2e, \cdots$ or we may have $q_e = \pm e/2, \pm 3e/2, \cdots$. In either class of solution, we can go between different options by adding or removing electrons. However it is impossible to go from one class to the other by binding electrons.

Thus we conclude that time-reversal invariant $3d$ insulators with no exotic excitations come in (at least) two distinct varieties: those in which a probe monopole has integer electric charge and those in which it has a charge $1/2$ (+integer) of e. We take a conventional insulator to be one in which the probe monopole has charge (in units of e) $0 \mod 1$. We see that it is possible to have a distinct $\mathcal{T}$-protected topological insulator where the probe monopole has charge $1/2 \mod 1$.

Note the following important points. (i) If $\mathcal{T}$ is broken, then M and TM need not have the same electric charge and we cannot conclude anything about the charge of M. Thus $\mathcal{T}$ protects the quantization of the electric charge of a probe monopole. (ii) The monopole is not an excitation, and hence can have a fractional electric charge, even though (by decree) dynamical excitations do not. (iii) Quantization of the electric charge of the monopole guarantees stability of the putative topological insulator to all perturbations (interactions, disorder etc) that preserve the charge conservation and time reversal symmetries.

So what kind of insulator supports fractional charge on probe monopoles? To understand this, let us consider in greater detail, the bound state of M and TM when both have fractional charge $q_e = e/2$. The classical angular momentum of the associated electromagnetic field is readily computed and is seen to be $\mathbf{L} = \frac{\hbar}{2}\mathbf{R}$ (where $\mathbf{R}$ is the relative vector separating M and TM). We can also solve the two-particle quantum mechanics problem of M and TM. The solution shows that the bound state has an orbital angular momentum $\frac{\hbar}{2}$ suggesting that it is has a "spin" 1/2. This may be understood by considering the Berry phase seen when M moves through a closed loop C. This phase is $\frac{1}{2}\frac{1}{2}(2)\Omega$ where Ω is the solid angle subtended by the loop C. One factor of $1/2$ comes from the usual Berry phase of a charge e moving around a monopole, another factor $1/2$ takes into account the charge $e/2$ of M, and there is a factor of 2 ue to the fact that the magnetic charge of M also sees the electric charge of TM. This Berry phase is the same as that of a unit electric charge moving around a neutral 2π magnetic monopole where the bound state is well known to have internal angular momentum $1/2$ in the ground state.

Under $\mathcal{T}$, $M \leftrightarrow TM$, and thus the relative vector $\mathbf{R} \to -\mathbf{R}$. In the ground states, when we project to the spin-1/2 doublet subspace, $P\mathbf{R}P = \mathbf{J}$ (the spin-1/2 operator). Thus under $\mathcal{T}$, $\mathbf{J} \to -\mathbf{J}$ which implies that the bound states is a Kramer's doublet[11, 12] under $\mathcal{T}$. Finally the half-integer spin suggests that this bound state is a fermion, and this can be shown explicitly[13].

Thus we conclude that the possibility that a probe 2π monopole have quantized electric charge $e/2$ in a $\mathcal{T}$- symmetric $3d$ insulator requires that the basic charged particles are charge-e fermions which are Kramers doublets (and not for insulators of, say, bosons or Kramers singlet fermions).

Can these fermions have a conserved $SU(2)$ spin-$S = 1/2$? The answer is no. We know that the fermion is a bound state of M and TM both of which must have the same S. Therefore their bound state cannot have $SU(2)$ spin-1/2.

It follows that we need an electronic insulator where the electrons are spin-orbit coupled (so that electron spin is not conserved).

There is a lot more that one can understand from this point of view. For instance we could discuss the surface of the topological insulator (or equivalently an interface with the trivial insulator). In the vacuum, a probe monopole will have $q_e = 0$. Thus when a monopole passes through the interface into the bulk of the topological insulator, it must pick up an electric charge $\pm e/2$. Thus the surface cannot be completely trivial. Rather there must be a surface state such that a 2π flux passing through it sucks in a charge-$\pm e/2$.

A formal characterization of the surface is that it realizes the $U(1)$ and $\mathcal{T}$ symmetries with an "'t Hooft anomaly". If we couple in a probe $U(1)$ gauge field, the surface theory alone will not be $\mathcal{T}$-invariant, but the combined theory of the surface and the bulk will be $\mathcal{T}$-invariant.

It is a good exercise for the reader to check that the familiar free fermion topological insulator fits the considerations above.

B. Fractional quantum Hall revisited

A $2d$ system of charged particles showing a Fractional Quantum Hall (FQH) effect has a conductivity tensor (at $T = 0$)

$$\sigma_{ij} = \frac{\sigma_H}{2\pi} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ij} . \tag{2}$$

with $\sigma_H = p/q$ where p, q are co-prime integers with $q > 1$. (We are working in units with $e = \hbar = 1$). The FQH is usually seen in $2d$ electron gases in a uniform magnetic field (in platforms like $GaAs$

or graphene). Very recently it has been seen in systems (twisted $MoTe_2$, multilayer rhombohedral graphene) which are microscopically time-reversal invariant. $\mathcal{T}$ is then spontaneously broken thereby enabling an FQH state. In this context these are known as Fractional Quantum Anomalous Hall (FQAH) states. These are a special case of a larger class of FQH states known as Fractional Chern Insulators (FCI) which refer to any lattice $2d$ system (with or without a B-field) that shows an FQH. Examples at non-zero B include bilayer graphene aligned with a hexagonal Boron-Nitride substrate and twisted bilayer graphene.

While there is a well-developed microscopic theory for the FQH state in Landau levels, it is far less developed in lattice systems. Ths it is interesting to ask what measuring a fractional σ_H implies about the low energy theory. We will describe a small portion of recent results, described in Ref. [14], on this question. Ref. [14] also has extensive references to the original literature.

We begin with an argument, due originally to Laughlin, that shows that fractional σ_H implies the existence of fractionally charged excitations. We put the FQH system on an annulus and thread 2π flux adiabatically at a rate $d\Phi/dt$. This induces a circulating emf at a radius r given by $E_\theta(2\pi r) = -d\Phi/dt$. Correspondingly there is a radial Hall current density $j_r = \sigma_{xy} E_\theta = \frac{\sigma_H}{2\pi r}\frac{d\Phi}{dt}$. The total charge transferred as Φ ramps up from 0 to 2π is

$$Q = 2\pi r \int dt j_r \ = \sigma_H \tag{3}$$

Thus fractional σ_H implies that we can pump a fractional charge from the outer to the inner edge, and thus that a fractionally charged excitation (the 'vison' v) is nucleated at the inner edge.

A later refinement showed that the vison must have fractional statistics $\theta = \pi\sigma_H$. To see this, note that though the 2π flux cannot be detected by electrons, it will lead to a non-trivial braiding phase on fractionally charged anyons. Consider a single bulk anyon a localized far away from the edge, and drag it around the inner hole. With 2π flux, there is an Aharanov-Bohm phase $2\pi Q_a$. Hence a and v have a braiding phase

$$B(a, v) = B(v, a) = Q_a \mod 1 \tag{4}$$

Thus v detects the fractional charge of any other anyon. In particular $B(v.v) = Q_v = p/q \mod 1$. This implies that $2\theta_v = 2\pi p/q \mod 2\pi$. We can actually show a stronger result that $\theta_v = \pi p/q \mod 2\pi$. Finally it can be shown that the vison is an abelian anyon (see Ref. [14] for a discussion of these statements). Thus we conclude that fractional σ_H implies the existence of a vison v with (a) fractional charge $Q(v) = \sigma_H$ (b) fractional self-statistics $\theta_v = \pi p/q$, and (c) braiding statistics with any other anyon given by a braiding phase $B(a, v) = Q_a \mod 1$.

So the IR theory has fractionally charged anyons and hence is topologically ordered. What is the "minimal" theory consistent with the above? For concreteness, we define minimal as the theory with the smallest number of anyons.

Let us start with some simple observations. From v we can get v^2 by fusing v with itself. This has charge $Q(v^2) = 2p/q$ and statistics $\theta(v^2) = 4\pi p/q$. Clearly v^2 is also abelian. In general we can consider v^n with $Q(v^n) = np/q$ and $\theta(v^n) = \pi n^2 p/q$. Note that in an electronic system we can identify $(Q(a), \theta(a)) \sim (Q(a) + 1, \theta(a) + \pi)$ by binding an electron to the anyon.

Let us restrict to odd q to begin with. Clearly we have q distinct anyons $(1, v, v^2, ..., v^{q-1})$. Consider v^q. It has $(Q(v^q), \theta(v^q)) = (p, \pi pq)$ which can be identified with $(0, 0)$ by binding powers of electrons (irrespective of whether p is odd or even. Thus we get a minimal theory by identifying v^q with a local particle. The minimal theory - which we denote $\mathcal{V}^{q,p}$ - is an abelian topological order with q distinct anyons.

Note that most of the observed quantum Hall states (on the order of about a 100 states) have odd q. Interestingly essentially all these states, save a few exceptions (< 5), are described by these minimal states!

For even q, the analysis is a bit more involved. Now v^q is a boson with odd integer electric charge. It thus cannot be a local excitation in an electronic system even though it braids trivially with all other v^n. There must therefore be anyons not contained in $(1, v, v^2, ...)$ that v^q braids non-trivially with. (HW: Show that this implies there must be anyons with charge $\frac{1}{2q}$ in the theory.) Thus we must have v^{lq} be a local excitation with l an even integer bigger than 1. It is possible to show that the miminal choice is $l = 2$, and that there are four distinct minimal theories which are all non-abelian, and each have $3q$ distinct anyons.

C. Fermi liquids revisited

We will focus on some general properties of Landau Fermi liquids that follow from their symmetries and their realization in the low energy theory. Our discussion will follow Ref. [15].

Let us start with some general observations on global symmetry in quantum many body physics. Suppose the microscopic (the "UV theory") system may have a global symmetry group G_{UV}. We will be interested in situations where the G_{UV} is not spontaneously broken in the ground state. The low energy theory will have a (possibly different) global symmetry group G_{IR}. There will be an image of G_{UV} that embeds into G_{IR}. Note that G_{IR} may be bigger than G_{UV}, *i.e* the IR theory may have an emergent symmetry. In addition, G_{IR} may have an 't Hooft anomaly. Such anomalies are 'topological' properties of how symmetries are realized; they are robust to deformations within the same phase of matter. We already encountered them before in the context of surfaces of SPT matter. A concrete example is the chiral anomaly of massless Dirac fermions in $d = 1$ or $d = 3$ dimensions which have been of interest in condensed matter physics in recent years. The characterization of anomalies and their consequences is an important source of connection between topological and other kinds of quantum matter.

In this subsection, we are concerned with a microscopic system of electrons (which, for simplicity, we take to be spinless) on a lattice at some filling ν. For concreteness we restrict to $d = 2$. The microscopic global symmetries are charge conservation and discrete lattice translations (so that $G_{UV} = U(1) \times Z^2$).

We assume that the ground state is a Landau Fermi liquid. The IR theory then has sharp quasiparticles near a well-defined Fermi surface. What is the symmetry of the low energy theory? Colloquially, we have separate conservation of quasiparticles at each point of the Fermi surface, and hence we might expect a very large amount of symmetry. Let us make this precise. For each point θ (a periodic coordinate that parametrizes the Fermi surface, which we assume is just a single closed oriented curve), there is a conserved n_θ such that $n_\theta d\theta$ is the number of quasiparticles between θ and $\theta + d\theta$. These are the conserved generators of the IR symmetry. A general symmetry element[5] can be taken to be

$$e^{i \int d\theta f(\theta) n_\theta}, \tag{5}$$

for smooth functions $f(\theta)$.

Let us now ask about how the microscopic symmetries (G_{UV}) embed into this low energy symmetry group. The total charge $n \sim \int d\theta n_\theta$. Unit lattice translations along direction $\alpha = x, y$ embed as

$$T_\alpha \sim e^{-i a_\alpha \int d\theta K_{F\alpha}(\theta) n_\theta} \tag{6}$$

where a_α is the lattice spacing in the α direction. We can take this action fo translations as the *definition* of the Fermi momentum $K_{F\alpha}(\theta)$. Thus both the $U(1)$ and translation symmetries of the microscopic theory map to elements of the low energy $LU(1)$ symmetry.

The IR symmetry has an anomaly. One definition of the anomaly is that when we couple the

[5] These define smooth maps from the circle described by θ to $U(1)$, and form a group known as the loop group of $U(1)$, and is denoted $LU(1)$.

system to gauge fields, the symmetry and hence the associated conservation laws are lost. In the present context, we can see that there must be an anomaly by asking about the effect of an external electromagnetic field on the Fermi surface. Then we lose separate conservation of the n_θ (though of course the total charge is conserved). This is clear with either an electric or a magnetic field. In the latter case, semiclassicaly, the quasiparticle moves around in one direction on the Fermi surface due to the Lorentz force, thereby destroying separate conservation of n_θ.

A useful physical picture of the anomaly is as follows: the Fermi surface is the boundary in k-space of the rigidly occupied Fermi sea. In a B-field, the interior of the Fermi surface stays rigid but there is a 'chiral' edge state at the k-space boundary which is the Fermi surface.

It can actually be shown that in a B-field, the Fermi sea can be thought of as showing an integer quantum Hall effect in momentum space, and the one-way quasiparticle motion at the Fermi surface is the corresponding chiral edge state.

A more formal manifestation of the same physics is obtained by turning on a 2π magnetic flux of the external vector potentials $A_{x,y}$. For the chiral k-space edge state at the Fermi surface, the n_θ now satisfy the chiral commutation algebra

$$[n_\theta, n_{\theta'}] = -\frac{i}{2\pi}\frac{d}{d\theta}\delta(\theta - \theta') \tag{7}$$

familiar from usual IQH edge states.

The emergent symmetry and anomaly of the Fermi liquid by themselves determine many (but not all) universal properties. These 'kinematic' universal physics should be distinguished from other universal properties that depend on the dynamics of the theory (as captured by the low energy Hamiltonian). As an application, let us show how the celebrated Luttinger's theorem follows from the emergent symmetry/anomaly. This theorem relates the area of the Fermi surface to the lattice filling. It was shown within perturbation theory by Luttinger in the 1960s. A non-perturbative topological argument was given by Oshikawa[16] much later in 2000. This argument involves adiabaticaly threading 2π flux through a cycle of a torus the system is placed in, and calculating the (crystal) momentum transferred in both the UV and IR theories. Here we will obtain it as a universal kinematic property as an explicit consequence of the more fundamental statements on the emergent symmetry and its anomaly.

Consider the UV theory in the presence of a spatially uniform 2π flux. Then $T_{x,y}$ do not commute. Rather we have

$$T_x T_y T_x^{-1} T_y^{-1} = e^{2\pi i \nu} \tag{8}$$

In the IR theory, we should use Eqn. 6, and the commutation algebra in Eqn. 7. Then we find (HW: show this using the Baker-Campbell-Hausdorff formula)

$$T_x T_y T_x^{-1} T_y^{-1} = e^{i\frac{V_F a_x a_y}{2\pi}} \tag{9}$$

where V_F is the volume of the Fermi surface. Matching Eqn. 9 to Eqn. 8, we obtain Luttinger's theorem.

We emphasize that the emergent symmetry and its anomaly are the fundamental kinematic properties. They imply a number of other universal phenomena (response to electric fields, quantum oscillations, etc) in addition to Luttinger's theorem.

We will stop here though there are a number of other questions one could discuss. Can the IR theory with its anomaly be regarded as the boundary of a bulk theory with the same symmetry? The answer is yes but the bulk theory lives in $4 + 1$ dimensions though we are discussing Fermi liquids in $2 + 1$ dimensions. A useful physical point of view is that the 'bulk' theory should really be thought of as a theory in phase space. The bulk direction can then be interpreted as going into the interior of the occupied Fermi sea.

These considerations on ordinary Fermi liquids might give us a possible angle through which to make progress on the much more difficult problem of compressible non-Fermi liquid metals. Perhaps they have universal kinematic properties that are easier to understand than their full dynamics.

[1] T. Senthil, Annu. Rev. Condens. Matter Phys. **6**, 299 (2015).

[2] X.-G. Wen, *Quantum Field Theory of Many-body Systems : From the Origin of Sound to an Origin of Light and Electrons.*, Oxford Graduate Texts (OUP Premium, 2004), ISBN 9780198530947, URL `http://libproxy.mit.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=nlebk&AN=186592&site=ehost-live`.

[3] J. McGreevy, Annual Review of Condensed Matter Physics **14**, 57 (2023).

[4] N. Armitage, E. Mele, and A. Vishwanath, Reviews of Modern Physics **90**, 015001 (2018).

[5] S. Sachdev, *Quantum phase transitions* (Cambridge university press, 2011).

[6] Z. Bi, E. Lake, and T. Senthil, Physical Review Research **2**, 023031 (2020).

[7] H. v. Löhneysen, A. Rosch, M. Vojta, and P. Wölfle, Reviews of Modern Physics **79**, 1015 (2007).

[8] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004).

[9] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Phys. Rev. B **70**, 144407 (2004), URL `http://link.aps.org/doi/10.1103/PhysRevB.70.144407`.

[10] T. Senthil, 50 Years of the Renormalization Group: Dedicated to the Memory of Michael E Fisher pp. 169–195 (2024).

[11] C. Wang, A. C. Potter, and T. Senthil, Science **343**, 629 (2014).

[12] M. A. Metlitski, C. Kane, and M. P. Fisher, Physical Review B **92**, 125111 (2015).

[13] A. S. Goldhaber, Physical Review Letters **49**, 905 (1982).

[14] M. Cheng, S. Musser, A. Raz, N. Seiberg, and T. Senthil, arXiv preprint arXiv:2505.14767 (2025).

[15] D. V. Else, R. Thorngren, and T. Senthil, Physical Review X **11**, 021005 (2021).

[16] M. Oshikawa, Physical Review Letters **84**, 3370 (2000).