Computability in Physics

We have seen that we can learn interesting things about physical systems from complexity theory. (E.g. QMA-hardness of Local Hamiltonian → glassy behaviour. And there are many other examples, not just in many-body quantum physics!)

What about computability theory?

There are fewer results, but there are some nice ones. We're going to see two examples, one classical, one quantum.

Note: Any computational problem on a finite number of instances is decidable.

Exercise: Prove this!

→ In any undecidability result, there must be an ∞ somewhere.

Undecidability results necessarily concern some form of idealised, ∞ limit of a physical system.

But deep down, all physics results are about idealised models of the real world!

Undecidability in some idealised ∞ limit typically implies some "echo" of undecidable behaviour manifests for non-idealised, finite cases.

Keep a lookout for where the ∞ comes into our examples!

Example 1

Undecidability of Halting!

A TM is something you can construct in the real world.

Halting is a natural question about its dynamics.

Only idealisation is ∞ tape.
(Where ∞ comes in!)

Can keep adding more sections of tape as needed to approach ∞.

Let's see some examples of undecidability in (somewhat) more natural physical systems...

Classical Example

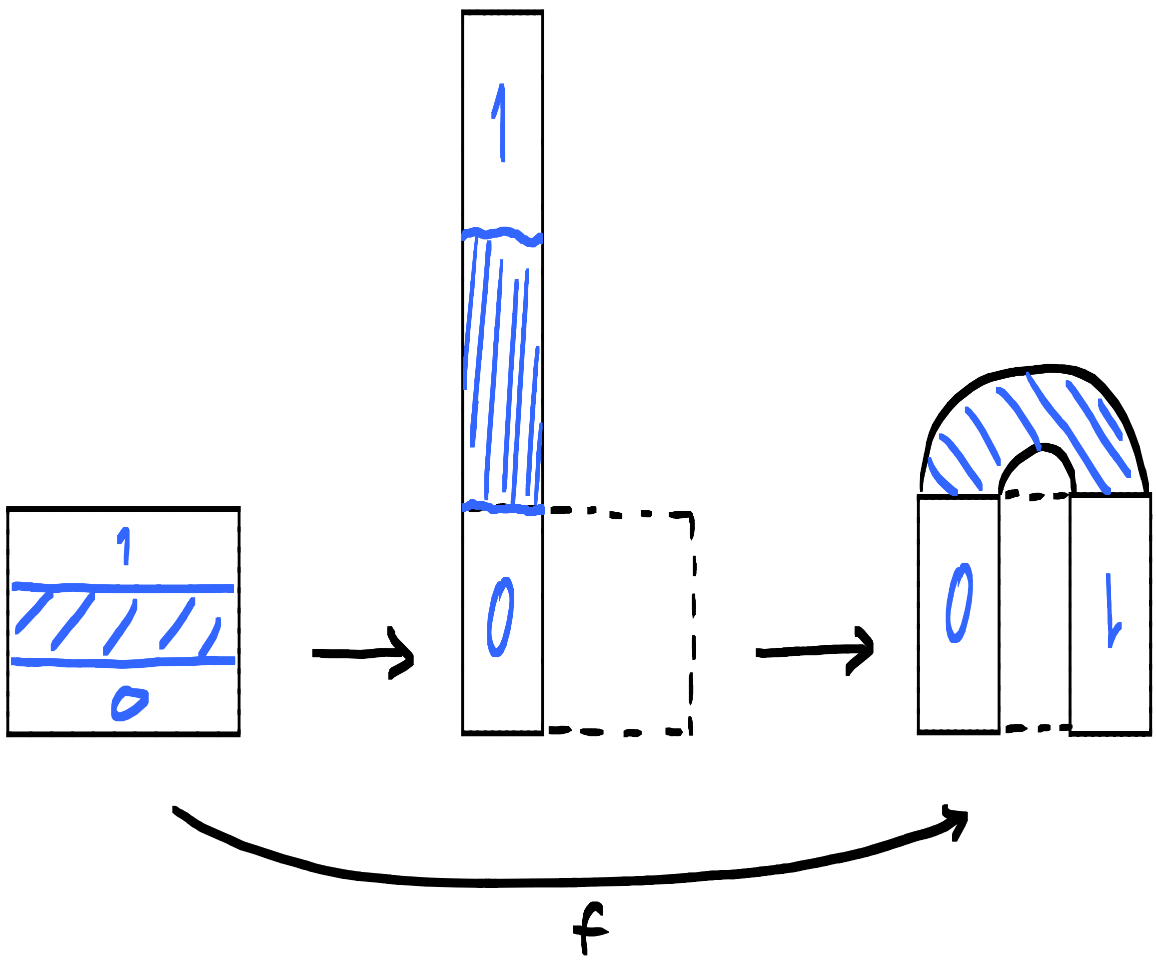
Particle dynamics in 3D

[Chris Moore, '90]

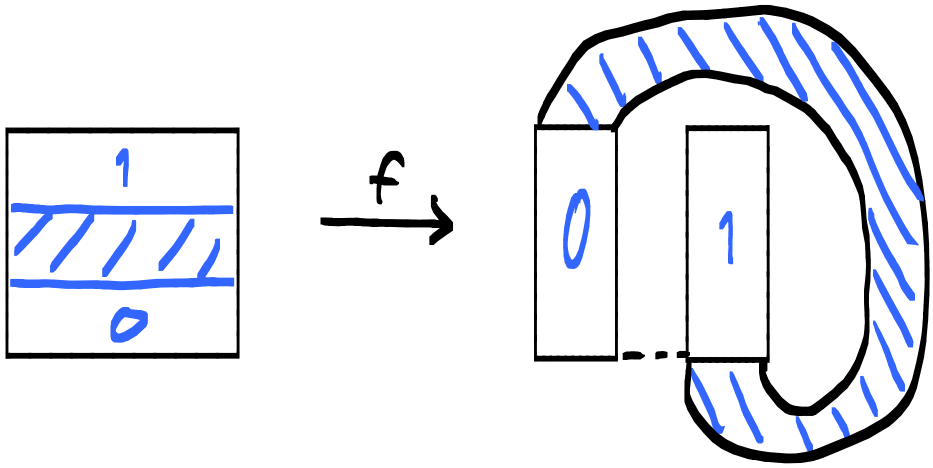
Smale horseshoe map

Classic example of chaotic dynamics.

"Stretch and fold" square onto itself:

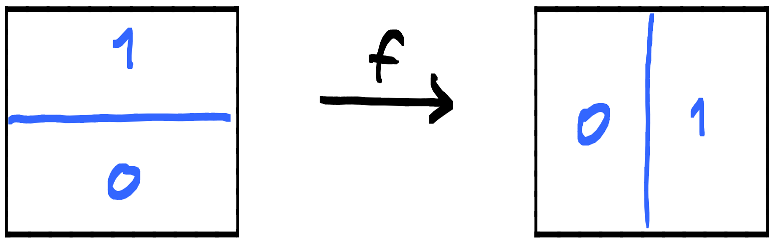


For convenience, consider variant called the <u>bakers map</u>:

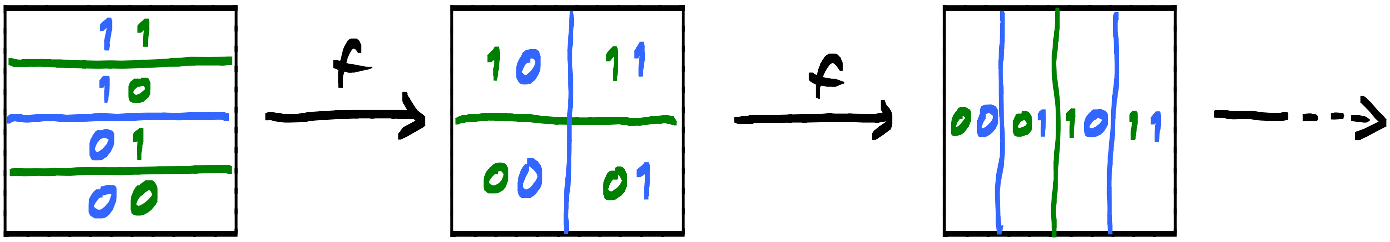


Only difference is left side isn't "flipped".

For further convenience, don't draw sections outside square & close up gaps inside square in diagrams:



What happens when we iterate f ?



Each iteration of map shifts one digit of vertical coordinate to horizontal coordinate.

Write horizontal/vertical coordinates of point within square (in binary) before/after decimal point (horiz. coordinate in "reverse", most significant bit last:

$$(x, y) = (0.x_1 x_2 x_3 \ldots, \; 0.y_1 y_2 y_3 \ldots)$$
$$\downarrow$$
$$n = \ldots x_3 x_2 x_1 . y_1 y_2 y_3 \ldots$$

Then each iteration of map shifts decimal point one step to the right:

$$f(x,y) \equiv \sigma(n)$$

$$\sigma : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$$

$$n_i \longrightarrow n_{i+1}$$

Formally, have homomorphism φ between baker's map & binary shift map:

$$\begin{tikzcd}
(x,y) \arrow[r, "f"] \arrow[d, "\varphi"'] & f(x,y) \arrow[d, "\varphi"] \\
n \arrow[r] & \sigma(n)
\end{tikzcd}$$

$f^{-1} \cong \sigma^{-1}$ shifts in other direction.

Note: if we put "gaps" between regions back in, see that we're considering action of baker's map on the 2D Cantor set.

Generalised shift map

Consider more general map on bi-infinite binary strings, of form:

$$\Phi : \{0,1\}^{\mathbb{Z}} \longmapsto \{0,1\}^{\mathbb{Z}}$$

$$n \longrightarrow \sigma^{F(n)}(G(n))$$

σ = shift map, as before

$F : \{0,1\}^k \longmapsto \mathbb{Z}$, $k < \infty$

Extend to $\{0,1\}^{\mathbb{Z}}$ by acting only on the k digits after decimal point, leaving rest unchanged

Tells us how much to shift & in which direction, depending on finite # digits after decimal point.

$G : \{0,1\}^l \longmapsto \{0,1\}^l$, $l < \infty$

Extend to $\{0,1\}^{\mathbb{Z}}$ similarly to G.

Replaces l digits after decimal point with new ones.

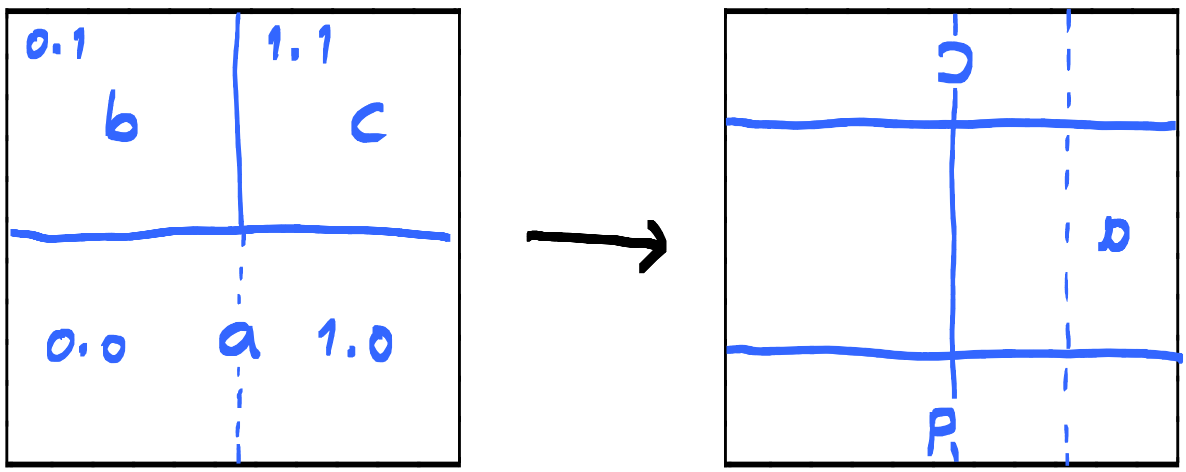
Do generalised shift maps correspond to some mapping of unit square?

Yes!

$\sigma^{F(n)}$ shift corresponds to stretching & folding, as before. But now different regions get stretched & folded different amounts (& in either horiz / vert direction depending on sign of F).

$G(n)$ corresponds to swapping regions of the square.

E.g. $F(0) = +1$, $F(1) = -1$;
$G(0) = 1$, $G(1) = 0$



Lemma

Any generalised shift map is homomorphic to a piece-wise map (with a finite # components) of the 2D Cantor set onto itself.

Conversely, any C^2 area-preserving diffeomorphism of $\mathbb{R}^2$ preserving the 2D Cantor set is homomorphic to a generalised shift map.

Proof: Exercise.

(Or see [Moore '91])

Generalised shift maps are just Turing Machines in disguise!

Lemma (Church-Turing thesis for G.S.)
GS maps are Turing-equivalent.

Proof

TM (Q, Σ, δ).

Represent each element of $Q \cup \Sigma$ by a distinct binary string.
Represent blank symbol $\perp \in \Sigma$ by 0.

Represent tape + head pos. + state as concatenation of these strings.

Put string for $q \in Q$ immediately after decimal point, string for tape symbol $\sigma_0 \in \Sigma$ head is currently over immediately after this.

Tape to left of head goes before decimal point; tape to right of head goes to right of decimal point, after q & σ.

$$n = x.y = \cdots \sigma_{-3}\sigma_{-2}\sigma_{-1}.\,q\,\sigma_0\,\sigma_1\,\sigma_2 \cdots$$

$$G(y_1 \ldots y_k) = (q, \sigma') \text{ part of } \delta(q, \sigma_0)$$

$$F(y_1 \ldots y_k) = D \text{ part of } \delta(q, \sigma_0)$$
where $L \to -1$, $R \to +1$.

$\square$

Let Φ_u be G.S. map corresponding to Universal TM.

Def. G.S. Reachability Problem

Input: point (x,y) in unit square, finite subregion $R \subset$ unit sq.

Output:

YES if (x,y) ever lands within R when iterating Φ_u.

NO otherwise

Thm.

G.S. Reachability is undecidable.

Proof

Take $(x,y) \equiv x \cdot y = 0.q_0 \sigma_1 \sigma_2 \cdots \sigma_n$

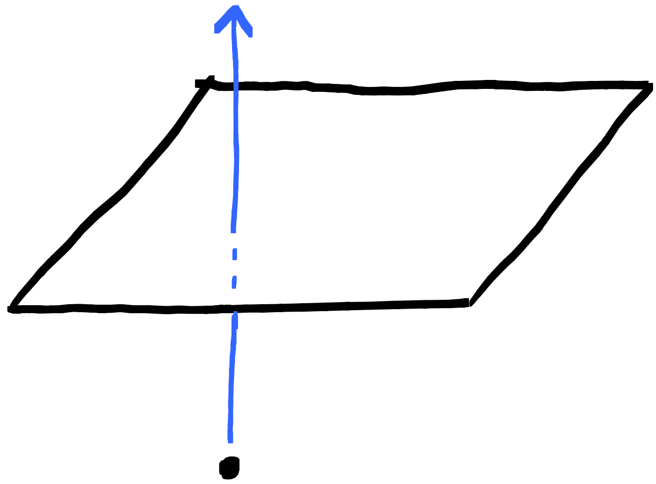
R = region containing points $0.q_f \sigma_1 \sigma_2 \cdots$

G.S. Reachability is equivalent to Halting problem for UTM on input $\sigma_1 \sigma_2 \cdots \sigma_n$.

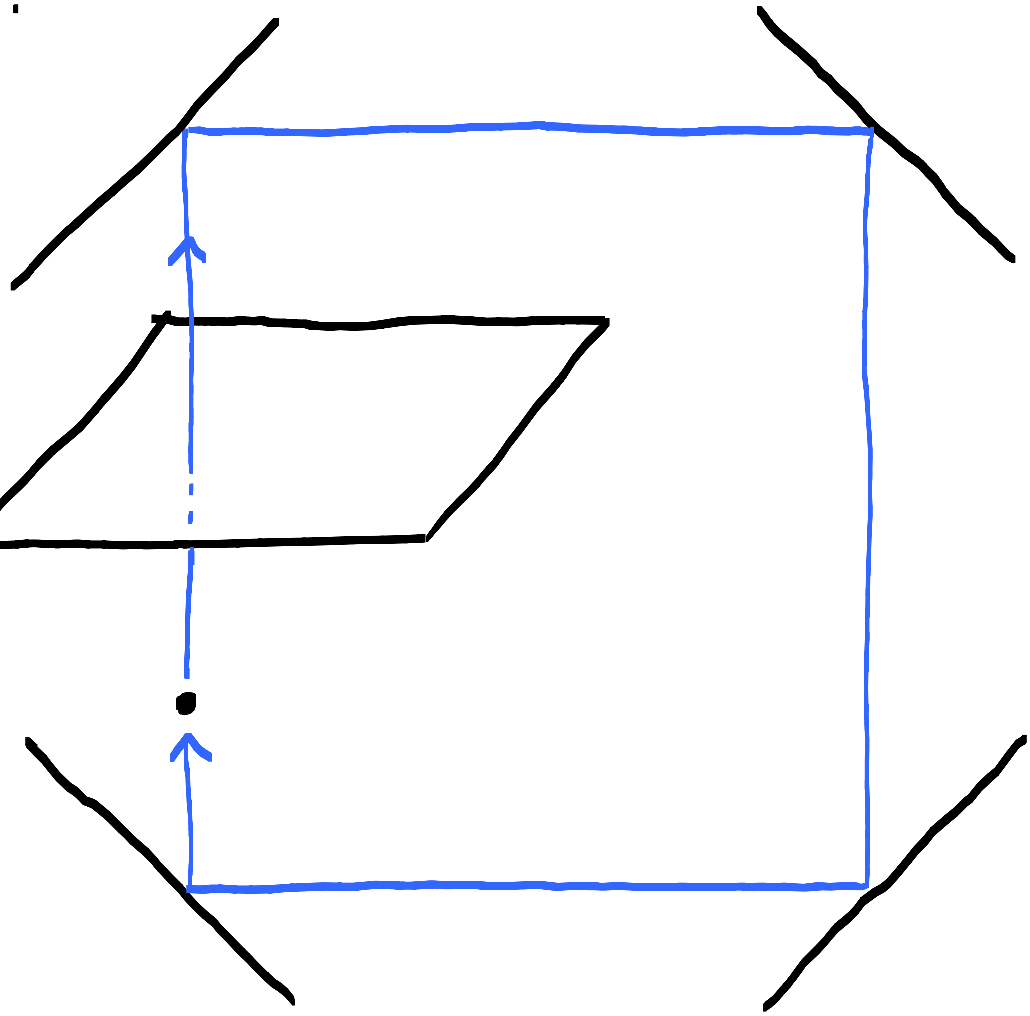
$\square$

Particle in a 3D box

Send particle upwards through unit square:

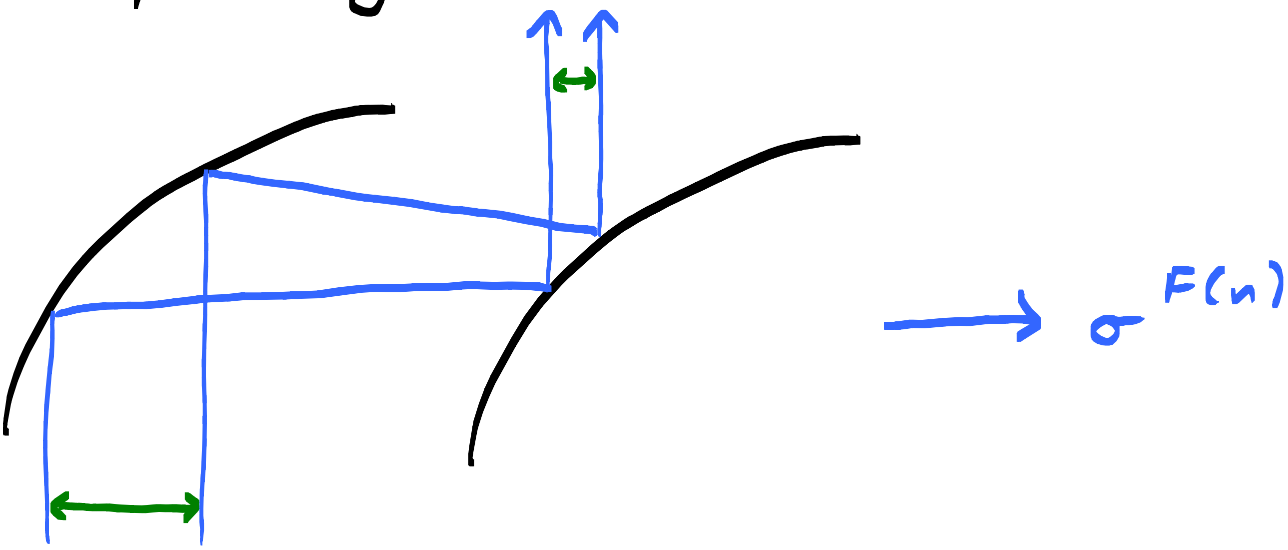


Arrange planar "mirrors" around square so particle repeatedly goes back through square:

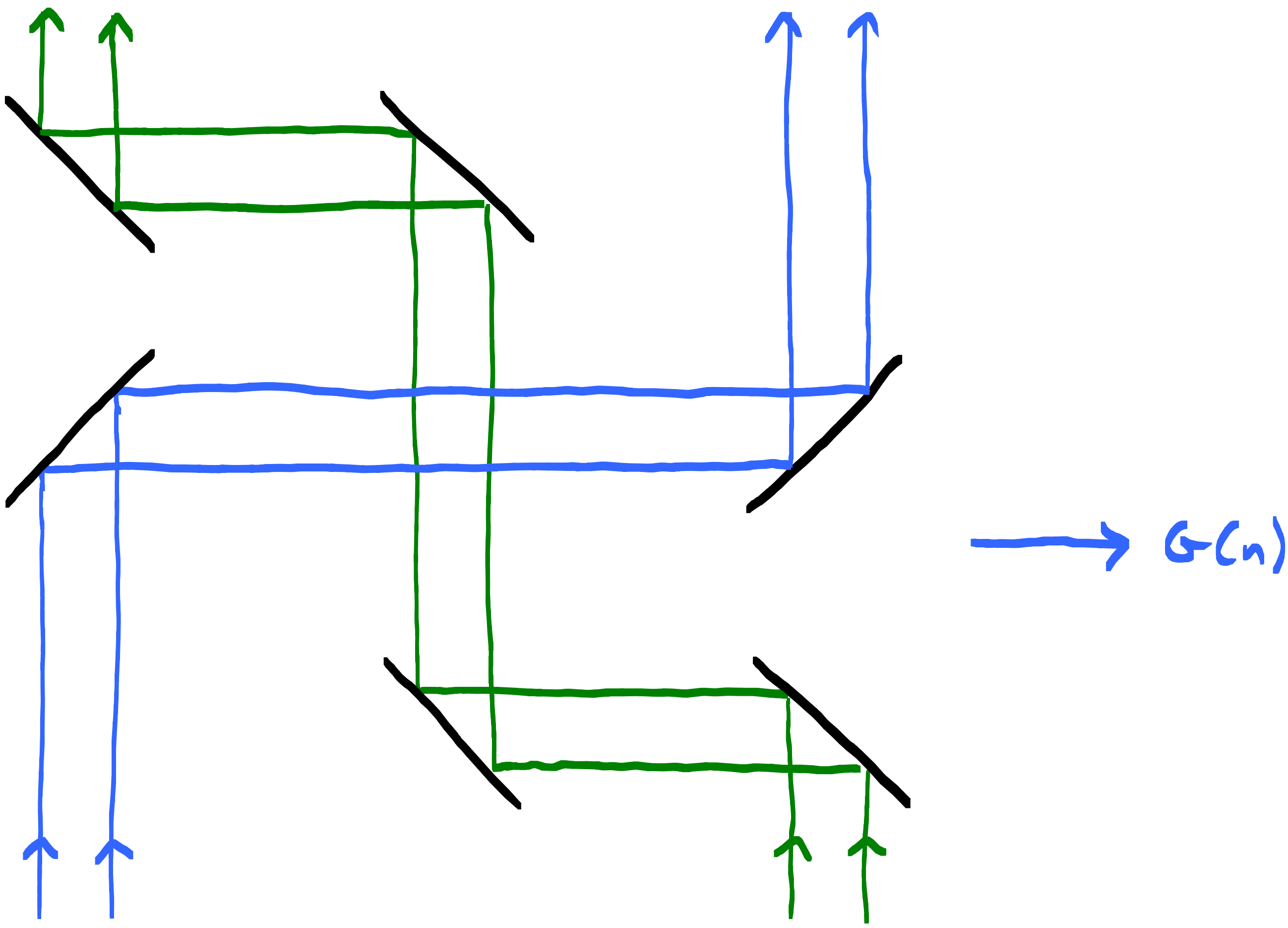


Squeezing & stretching ($\equiv \sigma(n)$) of a subregion ($\equiv \sigma^{F(n)}$) can be implemented by parabolic reflectors.

E.g. squeezing:



Exchanging subregions can be implemented by linear reflectors:



Combining a <u>finite</u> # of reflectors, can implement $\Phi_u(n) = \sigma^{F(n)}(G(n))$ for universal G.S. map Φ_u.

Put the whole thing inside a box, with a hole in the bottom under region corresponding to $q_0 \in Q$, & hole in the top above region corresponding to $q_f \in Q$.

<u>Def.</u> Does-it-come-out-the-box problem

Input: position (x, y) within bottom hole

Output:

YES if particle entering box at (x, y) moving upwards eventually exits box

NO otherwise

<u>Thm</u>

Does-it-come-out-the-box problem is undecidable.

<u>Proof</u>

Equivalent to G.S. Reachability Problem for Φ_u (which we've seen is equivalent to Halting).

<u>Exercise</u>: Where's the ∞ hiding here?

What physics do we learn from this?

Shift map is classic example of chaotic dynamics.

If n, n' differ only after k'th decimal place, $|n-n'| \leq 2^{-k}$.

$$|\sigma^t(n) - \sigma^t(n')| = 2^t \, |n-n'|$$

$\longrightarrow$ exponentially growing divergence from arbitrarily small differences in initial state (Butterfly effect)

If only know initial state to finite precision ε, cannot predict dynamics beyond time $\sim \log(1/\varepsilon)$.

On the other hand, if we know initial state exactly, can predict dynamics perfectly to time $= \infty$.

G.S. map typically diverges <u>less</u> fast, as sometimes shifts right, sometimes left.

Might expect initial states to diverge as $\sim 2^{\sqrt{t}}$ rather than 2^t.

However, G.S. dynamics is <u>harder</u> to analyse than chaotic dynamics.

Even if we know initial state <u>exactly</u>, undecidability implies we cannot predict the future dynamics.

In fact, almost all properties of the dynamics are uncomputable: Lyapunov exponents, fixed points / cycles, basins of attraction, etc.
(by Rice's Thm.)

$\longrightarrow$ Qualitatively different type of complex dynamics than chaotic.

Particle Dynamics in Smooth Potential

Put gaps back between the regions in the unit square & filling in gaps with C^∞ functions

$\longrightarrow$ smooth (C^∞) 2D potential
except for regions transformed
by -ve $F(n) + G(n)$ substitutions.

Embed into 3D manifold
$\longrightarrow$ smooth (C^∞) 3D potential

(See [Moore '91] for details.)

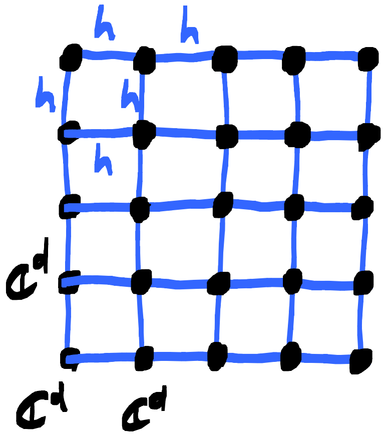
Quantum Example

Undecidability of the Spectral Gap

Consider translationally invariant Hamiltonians with nearest-neighbour interactions on a 2D square lattice of size L:

$$H(L) = \sum_{\substack{\langle i,j\rangle \\ i,j=1\ldots L}} h_{ij}$$

$$h \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$$



Phase transitions occur when spectral gap between ground state & excited states vanishes.

Finite dimensional matrix $\Rightarrow$

$\Rightarrow$ discrete spectrum

$\Rightarrow$ spectral gap > 0 for all finite L

$\longrightarrow$ Interested in thermodynamic limit $L \to \infty$.

Huge amount of q. cond-mat concerns spectral gaps. Many important, long-standing open problems, e.g.

- Haldane gap conjecture (1D spin-1 Heisenberg)
- Existence of topological spin-liquid phase
- Yang-Mills mass-gap problem (\$$10^6$)

Def. Spectral gap

$$\Delta(H(L)) := \lambda_1(H(L)) - \lambda_0(H(L))$$

second-smalles eigval. min. eigval.

Def. Gapped

Will say H is gapped if
$\exists L_0, \delta > 0$ s.t. $\forall L > L_0$

- $H(L)$ has unique ground state
- $\Delta(H(L)) > \delta$.

Spectral gap of H is then

$$\Delta(H) := \lim_{L\to\infty} \Delta(H(L)) \geq \delta$$

if lim exists.

Def. Gapless

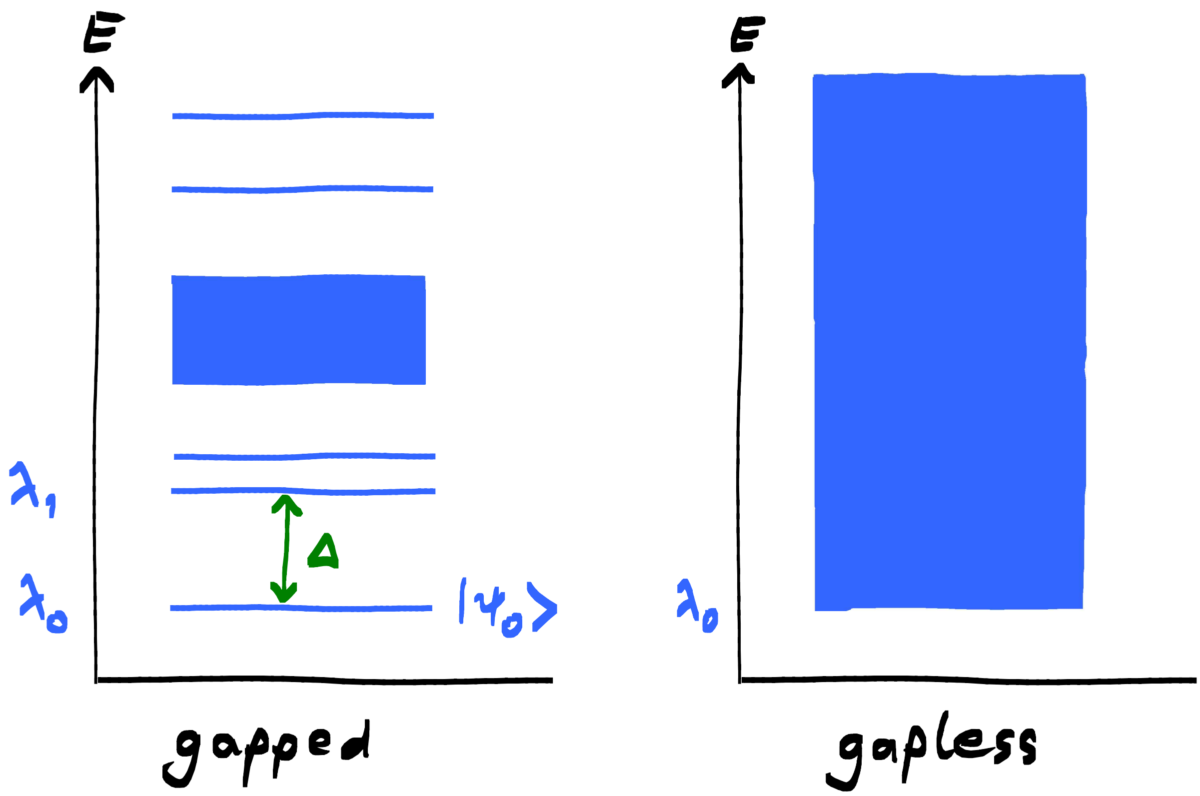
Will say H is gapless if
$\forall\, \varepsilon, c > 0 \quad \exists L_0 > 0$ s.t. $\forall L > L_0$
any point in interval $[\lambda_0, \lambda_0 + c]$
(where $\lambda_0 = \lambda_0(H(L))$) is within ε
of some eigval. of $H(L)$.

I.e. H is gapless if it has continuous spectrum above the g.s. in thermodynamic limit.

Note: "gapless" is <u>not</u> defined as the negation of "gapped" here.

There are Hamiltonians that are neither gapped <u>nor</u> gapless according to these Defs. E.g. degenerate g.s.s with gap above.

However, disallowing such ambiguous cases ($\rightarrow$ Promise) only makes our undecidability result <u>stronger</u>.



Def Spectral Gap Problem

Input: $h \in \mathcal{M}_{d^2}(\mathbb{A})$ ← $d^2 \times d^2$ matrices with algebraic matrix elems.

Promise: $H = \sum_{\langle i,j \rangle} h_{ij}$ is either gapped or gapless (by above Defs)

Output: YES if gapped

NO otherwise

Critical to restrict matrix elems. of h to at least be computable, or problem is trivially undecidable as you can never even compute what the Hamiltonian is, let alone it's spectral gap!

For similar reasons, critical H is described by finite amount of data. OK here as H is translationally invariant.

Thm [Cubitt, Perez-Garcia, Wolf 2015]

The Spectral Gap Problem is undecidable for all $d \geq$ a fixed (huge!) value d_0.

Proof (impressionist sketch!)

Idea:

Use history state encoding of evolution of circuit that's equiv. to universal Turing Machine

+ energy penalty on halting state

$\longrightarrow H = H_{prop} + H_{halt}$ $|q_f\rangle\langle q_f|$

cf. output penalty in Kitaev construction

$$|\Psi_0\rangle = \frac{1}{\sqrt{T}} \sum_t |\psi_t\rangle |t\rangle$$

$$|\psi_T\rangle = |q_f\rangle |\sim\rangle |T\rangle$$

Problems:

- Need to encode ∞ uniform circuit family & ∞ time-evolution (otherwise trivially decidable!)
- H changes non-trivially as circuit size changes.
- H definitely not translationally-inv. nearest-neighbour on 2D lattice.
- $\Delta(H(n)) \sim \frac{1}{\mathrm{poly}(n)}$

  H always gapless $\Rightarrow$ trivially decidable.

But have another, nicer, uniform model of computation:
(quantum) Turing Machines.

Idea:
Encode evolution of QTM in history state, instead of circuit.

This idea + ~30 pages of hard work

→ QMA_{EXP}-hardness of Local H. problem on 1D translationally-inv. nearest-neighbour spin chain.
[Gottesman & Irani 2009]

Get 1D + trans. inv. + nearest-neighbour for "free" from fact that QTM tape is 1D, transition rules don't depend on where head is located, & only involve head + tape cell under it.

Hard part is encoding QTM in a history state H in first place!

→ ~30 pages to redo G-I construction because we need different properties for computability rather than complexity results

Problems

1. How do you initialise input to QTM?
   (G-I use chain length in a clever way. But for spectral gap problem we're in thermodynamic limit: chain length $\longrightarrow \infty$.)

2. H always gapless $\Rightarrow$ trivially decidable.

1. Need to somehow encode arbitrarily large # input bits into <u>finite</u> # matrix elems. of h.

   Then somehow "extract" this & "feed" it as input to universal TM encoded in history state.

Idea

Encode input as binary digits of a phase parameter φ in h.

Use q. phase-estimation algorithm (encoded in QTM history state construction) to extract φ.

Feed output of phase-estimation QTM as input to universal TM (all encoded in history state construction).

Need phase estimation QTM with peculiar properties to make this idea work: need exact (i.e. 0-error) phase estimation $\forall \varphi \in \mathbb{Q}$ that runs with 0 space overhead

(Comes close to contradicting some old no-go Thms!)

$\longrightarrow$ ~30 pages to prove (by explicit construction!) that phase-estimation QTM with these properties exists.

Problem:

2. H always gapless $\Rightarrow$ trivially decidable

Idea:

(Inspired by classical undecidability of Tiling proofs from '60s & '70s)

Go to 2D & use aperiodic tiling to "place" non-zero density of phase-estimation QTM + UTM history state Hamiltonians of all finite chain lengths, all "running" on same phase φ.

(But do this fully trans. invariantly!)

Short chain lengths where not enough run-time to finish UTM computation contribute ≈ 0 energy.

Chain lengths where have enough run-time to halt contribute energy <u>density</u> $E_p > 0$
(though E_p uncomputably small!)

$\longrightarrow$ g.s. energy $\sim L^2$
(i.e. $\nearrow \infty$ instead of $\searrow 0$ as before!)

$\longrightarrow$ ~ 10 pages to prove required properties of aperiodic tiling due to [Robinson '71]

<u>Problem</u>:

Gives undecidability of deciding if $E_p = 0$ or > 0
(But <u>no</u> promise bounding E_p away from 0 by finite const. in > 0 case.)

Idea

"Glue" Tiling + QTM + UTM H together with simple gapped & gapless Hamiltonians, in such a way that continuous spectrum is "pushed" up by energy $\nearrow \infty$ in Halting case, to reveal gapped spectrum.

$\longrightarrow$ ~ 30 pages to glue all parts of construction together & prove required spectral properties.

$\square$

(See Supplementary Material to Nature paper for extended, high-level overview of construction & proof.
Available as ancillary file at arXiv: 1502.04135.

See arXiv: 1502.04573 for full (127-page!) proof.)

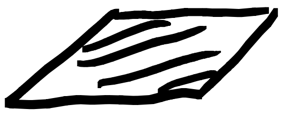
What physics do we learn from this?

Cannot have ∞ lattice ($\equiv$ infinitely large sample of material!) in lab.

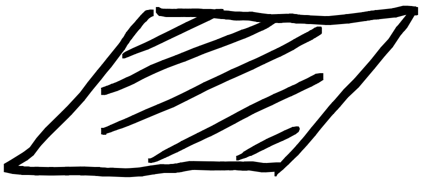
$\longrightarrow$ Cannot see undecidability per se.

But see "echos" of undecidability in physics that in principle <u>is</u> experimentally accessible:

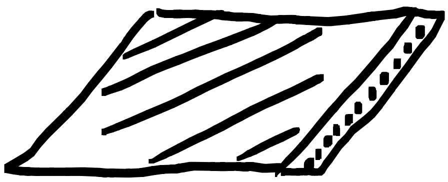
- Sudden change in spectrum & ground state at arbitrarily large finite lattice size.
  (Size-driven "phase transition")



create sample, measure $\rightarrow$ gapless



double sample size, measure $\rightarrow$ gapless

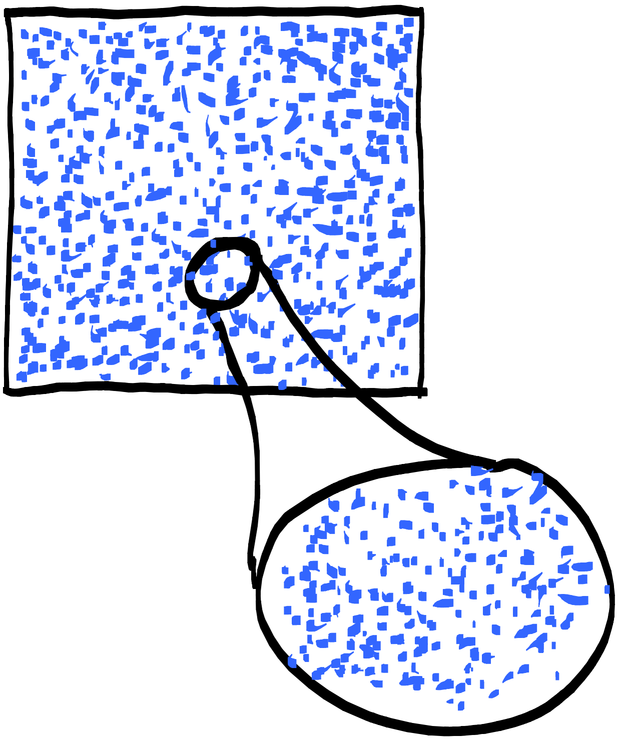


add one more row of atoms
measure $\rightarrow$ gapped!

Undecidability $\Rightarrow$ <u>impossible</u> to predict if and when transition occurs.
Threshold lattice size uncomputably large.

Never know if just one more atom would completely change physics!

• Uncomputably fractal phase diagrams.



Find gapped points arbitrarily close to gapless points in phase diagram.

And gapless points arbitrarily close to gapped points.

Moving arbitrarily small distance in phase diagram drives system through ∞ many phase transitions.

Fractal phase diagrams well-known in cond-mat (e.g. Hofstadter butterfly).

This is a qualitatively different (uncomputable) type of complex phase diagram.

(Cf. generalised shift dynamics vs. chaotic dynamics.)

- Open up intriguing possibility (but comes nowhere close to proving!) some of big open problems in theoretical physics may be undecidable (Haldane conjecture, Yang-Mills...)

However, these are <u>single</u> instances, not ∞ families of problems
$\Rightarrow$ cannot be (algorithmically) undecidable.

Would need to strengthen techniques to prove axiomatic independence of specific, physically relevant instances

$\longrightarrow$ Needs new techniques.

(Cf. Gödel's 2nd incompleteness Thm. vs. 1st incompleteness Thm.)

- Undecidability of spectral gap construction
$\Rightarrow$ undecidability of <u>any</u> property that distinguishes gapped systems with unique, product g.s. from gapless systems (≈ Rice's Thm. for cond-mat):
  - spectral gap
  - decay of g.s. correlation functions
  - ...

Open Problems

- Undecidability of spectral gap in 1D?
  (Aperiodic tilings do not exist in 1D)
- Minimum local Hilbert space dimension for undecidability?
  (Is spectral gap problem decidable for nearest-neighbour qubit models?)
- ...

Exercise: Prove one of these!