Universality in (slightly) out-of-equilibrium dynamics in correlated matter: Hydrodynamic EFTs and transport bounds

Lectures for BSS 2025

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July 15, 2025

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1 Lecture 1 – Thermal Dynamics

The goal of these lectures will be to develop nonperturbative tools to study the (slightly) out-of-equilibrium of generic quantum many-body systems. For example, we will be interested in the response of our system, initially in an equilibrium state, e.g.

$$\rho \propto e^{-\beta H}\,, \qquad \rho \propto e^{-\beta(H-\mu Q)}\,, \tag{1.1}$$

to external probes: $\vec{E}$, $\vec{B}$, δT, $\delta\mu$ (doping/in plane field), etc. These probes will take the system out of equilibrium

$$H_0 \to H(t) = H_0 + \int d^d x\, \phi(t,\vec{x})\mathcal{O}(t,\vec{x})\,, \tag{1.2}$$

but in the limit where these probes are weak, we can expand in them and end up computing dynamical correlation functions in the thermal state:

$$\mathrm{Tr}(\rho\mathcal{O}_1(t_1,\vec{x}_1)\mathcal{O}_2(t_2,\vec{x}_2)\cdots)\,. \tag{1.3}$$

These can either be viewed as measuring fluctuations and correlations in the thermal state, or near equilibrium dynamics as a response to external probes. A typical example of this is the conductivity, which can either be viewed as an equilibrium two-point function of the current, or its expectation value in the presence of a small electric field

$$j_i(\omega) = \sigma_{ij}(\omega)E_j(\omega)\,, \qquad \sigma_{ij}(\omega) \sim \frac{1}{\omega}\langle j_i(\omega)j_j\rangle\,. \tag{1.4}$$

This correspondence goes beyond linear response, with higher-point correlation functions corresponding encoding the nonlinear response to external probes. However, for this expansion to work we will always be near thermal equilibrium – hence "slightly" out-of-equilibrium.

These objects are very difficult to compute in interacting quantum many-body systems. At nonzero temperature $T > 0$, perturbation theoy essentially always breaks down at late enough times. Even in weakly coupled theories, resummation is extremely difficult. These lectures will present an alternative approach: an effective field theory describing fluctuating hydrodynamics, which is expected to emerge in any interacting local many-body system.

1.1 Thermal equilibrium

As a warm-up, we will first consider a simpler set of observables: equilibrium observables, i.e. the response of the system to time-independent (static) probes. In terms of correlation functions, this corresponds to integrating every operator in (1.3) over time.

First note that if we analytically continue time, thermal correlators can be represented as a path integral on a cylinder $\mathbb{R}^d \times S^1_\beta$:

$$\mathrm{Tr}\left(\rho T_E \left\{\mathcal{O}_1(\tau_1, \vec{x}_1)\mathcal{O}_2(\tau_2, \vec{x}_2)\cdots\right\}\right) = \int D\psi\, \mathcal{O}_1(\tau_1, \vec{x}_1)\cdots e^{-S_E[\psi]}\,, \tag{1.5}$$

where T_E denotes Euclidean time-ordering, and $\mathcal{O}(\tau) = e^{\tau H}\mathcal{O}e^{-\tau H}$ [you can derive this the usual way you derive a path integral – the cylinder of length β arises from $\rho \propto e^{-\beta H}$ and the trace which imposes (anti-)periodic boundary conditions for bosons (fermions)]. This establishes a correspondence between imaginary time observables of quantum many-body systems in $d+1$ spacetime dimensions, and observables of statistical mechanics on a $d+1$ dimensional spatial cylinder.

To generate these correlation functions, we couple our system to spatially-dependent probes that source the operators of interest. One could source any operator, let us start with a charge density:

$$Z[\beta, \mu(\vec{x})] = \mathrm{Tr}\, T_E e^{-\int_0^\beta d\tau (\hat{H}(\tau) - \int_x \mu(\vec{x})\hat{n}(\vec{x},\tau))} \equiv e^{-W[\beta,\mu]} \tag{1.6}$$

Note that if we set $\mu(\vec{x}) = \mathrm{const}$, we recover $Z = \mathrm{Tr}\, e^{-\beta(H-\mu Q)}$. The generating functional (or thermal effective action, or free energy) W generates connected correlators of time integrated densities. For example,

$$\left.\frac{\delta^2 W}{\delta\mu(\vec{x})\delta\mu(0)}\right|_{\mu=\mathrm{const}} = \beta \int_0^\beta d\tau\, \mathrm{Tr}\left[\rho\, \hat{n}(\tau, \vec{x})\hat{n}\right]_c = \beta G^E_{nn}(\omega_n = 0, \vec{x}) \tag{1.7}$$

where we defined the (connected) Euclidean Green's function $G^E_{nn}(\tau, \vec{x}) \equiv \mathrm{Tr}\,(\rho n(\tau, \vec{x})n) - (\mathrm{Tr}\,\rho n)^2$ and its Fourier transform.

You may worry that W should be a horribly complicated function of the function $\mu(x)$ ("functional"). In most situations, this functional has a simple long-wavelength expansion thanks to the fact that system has a finite **thermal correlation length** $\xi < \infty$, such that equilibrium correlators decay exponentially at large distances:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \leq e^{-|\vec{x}|/\xi} \qquad \text{as } |\vec{x}| \to \infty\,. \tag{1.8}$$

When the thermal correlation length is finite, one can integrate everything out to obtain a local generating functional W, in a derivative expansion suppressed by the correlation length $\xi\nabla$. The first few terms are

$$W = \int d^d x \left[f_0(\beta, \mu(x)) + f_1(\beta, \mu(\vec{x}))(\nabla\mu)^2 + O(\nabla^4)\right] \tag{1.9}$$

The coefficients $f_0, f_1, \ldots$ are unknown "Wilsonian coefficients" – we will see that this expression has predictive power despite these unknowns. We have assumed isotropy for simplicity, but this assumption can be straightforwardly lifted. One can show that the first coefficient, which is the only one entering at zeroth order in derivatives, is related to the pressure as

$$f_0(\beta, \mu) = -\beta P(\beta, \mu) \,. \tag{1.10}$$

To see this, consider the simpler case of a homogenous source $\mu = \text{const}$. In this case, the functional reduces to

$$Z[\beta, \mu] = \operatorname{Tr} e^{-\beta(H-\mu Q)} = e^{V f_0(\beta,\mu)} \tag{1.11}$$

It is then straightforward to verify that the function $P(\beta, \mu) \equiv -f_0(\beta, \mu)/\beta$ satisfies the usual identities of pressure:

$$dP = sdT + nd\mu \,, \qquad \varepsilon + P = sT + \mu n \,, \tag{1.12}$$

where $n = \langle \hat{n} \rangle$ and $\varepsilon = \langle \hat{h} \rangle$ are the equilibrium charge and energy density.

So the thermal effective action encodes the equation of state of the system. It also encodes static response: for example the charge susceptibility can be obtained from (1.7)

$$\chi(q) \equiv G^E_{nn}(\omega_n = 0, q) = \frac{-1}{\beta} \int d^d x \, e^{-ikx} \left. \frac{\delta^2 W}{\delta\mu(\vec{x})\delta\mu(0)} \right|_{\mu=\text{const}} = \frac{d^2 P}{d\mu^2} - \frac{2}{\beta} f_1 q^2 + O(q^4) \tag{1.13}$$

At $q = 0$, this static susceptibility measures the charge compressibility $dn/d\mu$. The thermal effective action tells us that this object has an analytic expansion in q^2. In Exercise 1, you will generalize our construction to capture static correlators of charge and energy density.

Our results (in particular, analyticity of $\chi(q)$) relied on the assumption of a finite thermal correlation length $\xi < \infty$. There are two notable situations where this assumption fails:

- Ordered phases of continuous symmetries ($\Rightarrow$ Nambu-Goldstone modes)
- Thermal phase transitions

In the first, one can simply keep the long range fields (the Goldstones) in the effective action, we will see an example of this. Thermodynamics and out-of-equilibrium dynamics is richer near thermal phase transitions (see in particular [1]) – the methods described in these lectures can be generalized to these situations, but we will not do so here.

Exercise 1: *Practice with thermal effective actions*

(i) Check Eq. (1.12), and Eq. (1.13).

(ii) Let us generalize the thermal effective action so that it can also generate static correlators of energy density:

$$Z[\beta(\vec{x}),\mu(\vec{x})] = \mathrm{Tr}\, T_E e^{-\int_0^{\beta_0} d\tau \int d^dx \frac{\beta(x)}{\beta_0}(\hat{h}(x,\tau)-\mu(\vec{x})\hat{n}(\vec{x},\tau))} \equiv e^{-W[\beta,\mu]}\,. \tag{1.14}$$

For $\beta(\vec{x}) = \beta_0 = \text{const}$, it reduces to our previous expression. Still using the assumption of a finite correlation length, generalize Eq. (1.9) to obtain a functional of $\beta(\vec{x})$ and $\mu(\vec{x})$. Then, generalize Eqs. (1.7) and (1.13) to obtain the matrix of static susceptibilities $\chi_{AB}(q)$ for heat and charge $A.B = \hat{h},\hat{n}$.

1.2 Inevitability of hydrodynamics (1): existence of long-lived excitations

We now turn to the actual (real time) dynamics. We saw that the response to static probes was captured by a local generating functions (or thermal effective action) of the probe fields. What is the organizing principle for dynamical response? The answer, which we will slowly build towards, is hydrodynamics.

We start with a simple argument that shows the existence of long-lived excitations in thermal states. Consider the retarded Green's function

$$G^R(t,x) = i\theta(t)\,\mathrm{Tr}\,(\rho[n(t,x),n]) \tag{1.15}$$

for the charge density satisfying a continuity relation $0 = \dot{n} + \nabla\cdot j$. Then, the $\omega\to 0$ and $q\to 0$ limits of its Fourier transform do not commute:

$$\lim_{q\to 0}\lim_{\omega\to 0} G^R(\omega,q) = \chi \neq 0 = \lim_{\omega\to 0}\lim_{q\to 0} G^R(\omega,q)\,. \tag{1.16}$$

The limit on the RHS simply follows from the fact that $n(\vec{q}=0)$ is a conserved charge (which commutes with itself). The LHS follows from the thermal effective action: indeed, G^R is the analytic continuation of G^E:

$$G^R(i\omega_n,q) = G^E(\omega_n,q). \tag{1.17}$$

You will show this and further explore thermal Green's functions in Exercise 2. We know how to compute $G^E(\omega_n=0,k)$ from the thermal effective action: it picks up a static susceptibility (charge compressibility, specific heat, magnetization susceptibility, etc.).

This non-commutativity does not quite prove hydrodynamics (and in fact applies to free theories as well). But it does require long-lived, long range, excitations to produce an IR singular behavior capable of making the two limits not commute. In the context of diffusion, this non-commutativity is realized by the presence of a collective diffusive pole in the Green's function

$$G^R(\omega, q) = \frac{\chi D q^2}{-i\omega + Dq^2}\,. \tag{1.18}$$

This becomes analytic again in q when $\omega \to 0$, as expected from our local thermal effective action. In the opposite limit, it vanishes as required by charge conservation.

Exercise 2: *Fun with thermal Greens functions*

(i) The retarded and Euclidean Green's functions are defined as[a]

$$\text{Retarded:} \qquad G^R_{AB}(t) = i\theta(t)\langle [A(t), B]\rangle \tag{1.19}$$

$$\text{Euclidean:} \qquad G^E_{AB}(\tau) = \langle A(\tau) B\rangle\,, \tag{1.20}$$

where thermal expectation values are denoted by $\langle \cdot \rangle = \mathrm{Tr}(\rho\ \cdot)$ with $\rho = e^{-\beta H}/\,\mathrm{Tr}\, e^{-\beta H}$. We are assuming the QFT is in a finite (but very large) volume so that the spectrum is discrete. Recall the convention for Heisenberg operators $A(t) = e^{iHt}Ae^{-iHt}$, and $A(\tau) = e^{H\tau}Ae^{-H\tau}$. Using spectral representations (i.e., inserting a complete basis of energy eigenstates), show that the Fourier transforms of these functions are the analytic continuation of each other:

$$G^R(i\omega_n) = G^E(\omega_n)\,. \tag{1.21}$$

(ii) Many other Green's functions are useful in different contexts, for example:

$$\text{Wightman:} \quad G^+_{AB}(t) = \langle A(t) B\rangle \tag{1.22a}$$

$$\text{Symmetric:} \quad G^S_{AB}(t) = \frac{1}{2}\langle\{A(t), B\}\rangle \quad = \quad \frac{1}{2}\left(G^+_{AB}(t) + G^+_{BA}(-t)\right) \tag{1.22b}$$

$$\text{Feynman:} \quad G^F_{AB}(t) = \langle \mathcal{T} A(t) B\rangle \quad = \quad \theta(t) G^+_{AB}(t) + \theta(-t) G^+_{BA}(-t) \tag{1.22c}$$

$$\text{Retarded:} \quad G^R_{AB}(t) = i\theta(t)\langle [A(t), B]\rangle \quad = \quad i\theta(t)\left(G^+_{AB}(t) - G^+_{BA}(-t)\right) \tag{1.22d}$$

$$\text{Two-sided:} \quad G^2_{AB}(t) = \mathrm{Tr}\left(\rho^{1/2} A(t) \rho^{1/2} B\right) \quad = \quad G^+_{AB}(t - \tfrac{i\beta}{2}) \tag{1.22e}$$

$$\text{Euclidean:} \quad G^E_{AB}(\tau) = \langle A(\tau) B\rangle \tag{1.22f}$$

In a thermal state, these are all related. Show, using either the spectral decomposition method of **(i)** or the $i\epsilon$ prescription of **(ii)**, some or all of the following relations:

$$G^\pm(\omega) = e^{\pm\beta\omega/2} G^2(\omega)\,, \quad G^S(\omega) = \cosh\frac{\beta\omega}{2} G^2(\omega)\,, \quad \mathrm{Im}\, G^R(\omega) = \sinh\frac{\beta\omega}{2} G^2(\omega)\,. \tag{1.23}$$

These relations are sometimes called fluctuation dissipation relations, or KMS identities – they rely on the fact that the thermal density matrix corresponds to evolution in imaginary time. Thermal higher-point functions are instead *not* all related to each other, and in particular cannot all be obtained by analytically continuing Euclidean correlators.

[a] Note that while our convention for G^R is indubitably the best one, it is sadly only adopted in a minority of textbooks, including Chaikin&Lubensky. It differs from the definition used in, e.g., Altland&Simons, Kapusta, Kamenev, and Wen, by a minus sign. With our convention, $\omega \operatorname{Im} G^R(\omega)$ is *positive* and G^R is the analytic continuation of $+G^E$.

1.3 Inevitability of hydrodynamics (2): long-lived excitations are collective

The argument in the previous section shows that there are long-lived excitations in thermal states. However, quasiparticles typically acquire a finite lifetime at finite temperature, even if they are stable. For example, in a Fermi liquid interactions lead to an imaginary part of the self-energy

$$G^R_{\psi^\dagger\psi}(\omega,k) = \frac{1}{\omega - \epsilon_k - \Sigma(\omega,k)}\,, \qquad \operatorname{Im}\Sigma(\omega,k_F) = \frac{\#}{E_F}(\pi^2 T^2 + \omega^2)\,. \tag{1.24}$$

See [?] or [2] (You will derive a similar thermal broadening of fermionic quasiparticles due to phonons in Exercise 3). Due to the particle-hole continuum, the quasiparticle in a Fermi liquid is broadened ($\operatorname{Im}\Sigma \neq 0$) even at $T=0$, but the excitation is still sharp at low energies $\omega \to 0$. However, when $T>0$ even low energy quasiparticles decay: $\lim_{\omega\to 0}\operatorname{Im}\Sigma \propto T^2/E_F$. In time domain, this leads to exponential decay of correlation functions $e^{-t/\tau}$, with a time scale $\tau \sim E_F/T^2$. This finite relaxation rate of quasiparticles at finite temperature is very general, it occurs even if the particle is completely stable $\operatorname{Im}\Sigma(\epsilon_k,k)=0$ at $T=0$ (for example, a quasiparticle in a gapped system).[1]

We therefore see that the long-lived excitations responsible for the non-commutativity of limits in (1.16) must be *collective*. The effective theory of these collective excitations is hydrodynamics.

Exercise 3: *Thermal broadening in a metal*

To illustrate the inevitable thermal broadening of interacting quasiparticles at finite temperature, we will consider a simple model of a Fermi gas coupled to an Einstein (non-dispersive)

[1] The one exception to this are Nambu-Goldstone bosons. This is because, in a sense, they already are (one of) the appropriate degrees of freedom that should be kept in the hydrodynamic description.

phonon:

$$S = \int dt d^d x \, \psi^\dagger (i\partial_t - \epsilon_k)\psi - \frac{1}{2}\phi(\partial_t^2 + \omega_{\rm ph}^2)\phi + g\phi\psi^\dagger\psi \,. \tag{1.25}$$

While thermal broadening is a real time phenomenon, it is simplest to derive in the imaginary time formalism, followed by analytic continuation. . The Euclidean (imaginary frequency) correlators are

$$G^E_{\psi\psi^\dagger}(\omega_m, k) = \frac{1}{i\omega_m + \epsilon_k} \equiv G(i\omega_m, k)\,, \qquad G^E_{\phi\phi}(\Omega_n, q) = \frac{1}{\Omega_n^2 + \omega_{\rm ph}^2} \equiv D(i\Omega_n, q)\,, \tag{1.26}$$

where $\Omega_n = 2\pi n T$ and $\omega_m = (2m+1)\pi T$ with $n, m \in \mathbb{Z}$ are the bosonic and fermionic Matsubara frequencies (fermion fields must obey antisymmetric boundary conditions around the thermal cylinder because of anticommutation relations).

(i) Show that the fermion self-energy at one loop is given by

$$\Sigma(i\omega_m, k) = g^2 T \sum_{\Omega_n} \int \frac{d^d q}{(2\pi)^d} D(i\Omega_n, q) G(i\omega_m + i\Omega_n, k+q)\,. \tag{1.27}$$

The frequency sum can be evaluated by the following useful trick: the function $f_{\rm BE}(\omega) = \frac{1}{e^{\beta\omega}-1}$ (the Bose-Einstein distribution) has simple poles at the bosonic Matsubara frequencies $\omega = i\Omega_n$, with residue T. Use Cauchy's theorem to write the self-energy as

$$\Sigma(i\omega_m, k) = \frac{g^2}{2\omega_{\rm ph}} \sum_{\pm} \int_q \pm \frac{f_{\rm FD}(\epsilon_q) + f_{\rm BE}(\pm\omega_{\rm ph})}{\epsilon_q - i\omega_n \pm \omega_{\rm ph}}\,. \tag{1.28}$$

(Note that Σ is independent of k here).

(ii) We now analytically continue $i\omega_m \to \omega + i0^+$ to obtain the self-energy appearing in the retarded Green's function. Show that its imaginary part is given by

$$\operatorname{Im}\Sigma(\omega, k) = \frac{\pi}{2}\frac{g^2}{\omega_{\rm ph}}\nu(0) \sum_{\pm} \pm \left(f_{\rm BE}(\pm\omega_{\rm ph}) + f_{\rm FD}(\omega \pm \omega_{\rm ph})\right) \tag{1.29}$$

where $\nu(0)$ is the density of single-particle states at the Fermi surface (for a spherical Fermi surface, $\nu(0) = \frac{S_{d-1}}{(2\pi)^d}\frac{k_F^{d-1}}{v_F}$). Assuming $\omega_{\rm ph} \ll T \ll E_F$, show that this leads to a linear-in-T decay rate of fermionic quasiparticles:

$$\Gamma \sim \operatorname{Im}\Sigma(0, k_F) = \pi \frac{g^2}{\omega_{\rm ph}^2}\nu(0) T\,. \tag{1.30}$$

1.4 Schwinger-Keldysh contour for real time dynamics

You showed in Exercise 2 that the retarded Green's function is the analytic continuation of the Euclidean Green's function to real frequencies. One could therefore imagine always working in Euclidean (imaginary) time, and analytically continuing at the end. However, this is usually impractical. The most important reason is that in essentially any case of interest, we will not be able to analytically solve correlators. One therefore typically only has access to them in an asymptotic high or low frequency expansion, which cannot be analytically continued. Another reason is that for higher point functions, Euclidean correlators analytically continue to "fully retarded" Green's functions (nested commutators), which do not form a complete basis of real time observables: the other time orderings are not related to these by fluctuation dissipation relations like the ones you showed for the two-point function.

We are interested in correlators of the form

$$\mathrm{Tr}\left(\rho\mathcal{O}(t_1)\mathcal{O}(t_2)\cdots\right) . \tag{1.31}$$

Unlike in the ground state, where we can time evolve $e^{-iHt}|0\rangle$, now we'd like to time evolve a mixed state:

$$\rho(t) = e^{-iHt}\rho e^{iHt} . \tag{1.32}$$

Let us construct a generating functional to produce correlators like (1.31). To do so, we couple any operator of interest to background sources J: $S \to S + \int \mathcal{O}J$. The time evolution unitary is now:

$$U(t_f, t_i)[J] = Te^{-i\int_{t_i}^{t_f} H(t)+J(t)\mathcal{O}(t)} . \tag{1.33}$$

We can turn on different sources on both legs, to produce the following generating functional:[2]

$$Z[J_1, J_2] \equiv \mathrm{Tr}\Big(U(\infty,-\infty)[J_1]\rho U^\dagger(\infty,-\infty)[J_2]\Big) \tag{1.34}$$

which has the following pictorial representation shown in Fig. 1.

There are several general properties that this generating functional satisfies:

1. $Z[J,J] = 1$ "collapse rule" (from trace cyclicity. More generally, latest time cond.)

2. $Z[J_1,J_2]^* = Z[J_2,J_1]$ unitarity (note that Z is not a pure phase)

[2] This will actually only produce a subset of all possible time orderings, see Eq. (1.37). To obtain out-of-time ordered correlators, more "switchbacks" would be needed.

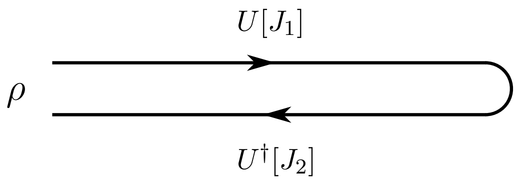


Figure 1: Schwinger-Keldysh contour with sources. Time increases from left to right.

These apply for any density matrix. For the case of the thermal state $\rho = e^{-\beta H}/\operatorname{Tr} e^{-\beta H}$, Z satisfies an additional KMS condition. This is entirely parallel to the relations between various Green's functions that you derived in your Problem set. The condition reads:

3. $Z[J_1, J_2] = Z[J_1(-t + i\beta), J_2(-t)]$ KMS + TR

The proof is simple:

$$\begin{aligned}
Z[A_1, A_2] &= \operatorname{Tr}\left(\left[Te^{i\int J_1 \mathcal{O} dt}\right] \rho \left[\bar{T}e^{i\int J_2 \mathcal{O} dt}\right]\right) \\
&= \operatorname{Tr}\left(\rho \left[Te^{i\int J_1(t+i\beta) \mathcal{O} dt}\right] \left[\bar{T}e^{i\int J_2 \mathcal{O} dt}\right]\right) \\
&= \operatorname{Tr}\left(\rho \left[Te^{i\int J_1(t+i\beta) \mathcal{O} dt}\right] \left[\bar{T}e^{i\int J_2 \mathcal{O} dt}\right] \mathcal{T}^{-1}\mathcal{T}\right)^* \\
&= \operatorname{Tr}\left(\rho \left[\bar{T}e^{-i\int J_1(-t-i\beta) \mathcal{O}_1 dt}\right] \left[Te^{-i\int J_2(-t) \mathcal{O}_1 dt}\right]\right)^* \\
&= \operatorname{Tr}\left(\left[\bar{T}e^{i\int J_2(-t) \mathcal{O}_1 dt}\right] \left[Te^{i\int J_1(-t+i\beta) \mathcal{O}_1 dt}\right] \rho\right) \\
&= Z[J_1(-t+i\beta), J_2(-t)]
\end{aligned} \tag{1.35}$$

In the second line we time translated $e^{\beta}\left[Te^{i\int J_1 \mathcal{O}_1 dt}\right] e^{-\beta} = \left[Te^{i\int J_1 \mathcal{O}_1(t-i\beta) dt}\right]$ and then changed variable in the integral. In the third line, we acted with the antiunitary operator $\mathcal{T}$ which we then commuted across the trace. We have assumed time-reversal for simplicity, it is in fact not necessary (as you might expect given your proofs of fluctuation-dissipation relations) [3]. We are free to use time translation invariance to impose KMS as

$$Z[J_1, J_2] = Z[J_1(-t + \tfrac{1}{2}i\beta), J_1(-t - \tfrac{1}{2}i\beta)]\,. \tag{1.36}$$

One can then generate various correlators as

$$\langle \mathcal{O}_1(t)\mathcal{O}_1(t')\cdots \mathcal{O}_2(\tilde{t})\mathcal{O}_2(\tilde{t}')\cdots\rangle \equiv \frac{\delta^n \log Z}{\delta J_1(t)\cdots} = \operatorname{Tr}\left(T\left[\mathcal{O}(t)\mathcal{O}(t')\cdots\right]\rho\bar{T}\left[\mathcal{O}(\tilde{t})\mathcal{O}(\tilde{t}')\cdots\right]\right)\,. \tag{1.37}$$

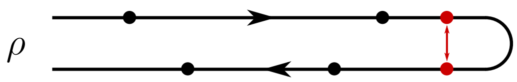
It is useful to introduce the Keldysh basis

$$\mathcal{O}_r \equiv \frac{1}{2}\left(\mathcal{O}_1 + \mathcal{O}_2\right)\,, \qquad \mathcal{O}_a \equiv \mathcal{O}_1 - \mathcal{O}_2\,. \tag{1.38}$$

Notice that any correlator involving $\mathcal{O}_a$ at the latest time vanishes, from trace cyclicity (this generalizes the "collapse" rule)

$$\langle \mathcal{O}_{a/r}(t_1)\cdots\mathcal{O}_{a/r}(t_{n-1})\mathcal{O}_a(t_n)\rangle = 0\,, \qquad t_1 < t_2 < \cdots < t_n\,. \tag{1.39}$$

as illustrated below:



In particular, $\langle\mathcal{O}_a\mathcal{O}_a\rangle$ vanishes, and

$$\begin{aligned}
2\langle\mathcal{O}_r(t)\mathcal{O}_a\rangle &= \langle(\mathcal{O}_1(t)+\mathcal{O}_2(t))(\mathcal{O}_1-\mathcal{O}_2)\rangle \\
&= \langle\mathcal{O}_1(t)\mathcal{O}_1\rangle + \langle\mathcal{O}_2(t)\mathcal{O}_1\rangle - \langle\mathcal{O}_1(t)\mathcal{O}_2\rangle - \langle\mathcal{O}_2(t)\mathcal{O}_2\rangle \\
&= \langle T\mathcal{O}(t)\mathcal{O}\rangle + \langle\mathcal{O}(t)\mathcal{O}\rangle - \langle\mathcal{O}\mathcal{O}(t)\rangle - \langle\bar{T}\mathcal{O}(t)\mathcal{O}\rangle \\
&= 2\theta(t)\langle[\mathcal{O}(t),\mathcal{O}]\rangle\,,
\end{aligned} \tag{1.40}$$

so $\langle\mathcal{O}_r(t)\mathcal{O}_a\rangle = -iG^R(t)$. Similarly,

$$\begin{aligned}
4\langle\mathcal{O}_r(t)\mathcal{O}_r\rangle &= \langle T\mathcal{O}(t)\mathcal{O}\rangle + \langle\mathcal{O}(t)\mathcal{O}\rangle + \langle\mathcal{O}\mathcal{O}(t)\rangle + \langle\bar{T}\mathcal{O}(t)\mathcal{O}\rangle \\
&= 2\langle\{\mathcal{O}(t),\mathcal{O}\}\rangle\,,
\end{aligned} \tag{1.41}$$

so that $\langle O_r(t)\mathcal{O}_r\rangle = G^S(t)$.

In Exercise 2, you showed the following fluctuation dissipation relation:

$$G^S(\omega) = \coth\frac{\beta\omega}{2}\,\mathrm{Im}\,G^R(\omega)\,. \tag{1.42}$$

You can show it again by using the KMS property of $Z[J_1,J_2]$ above.

Example: Einstein phonon. In Exercise 3 you studied the damping of fermions coupled to an Einstein phonon

$$S = \int dt\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega_{\mathrm{ph}}^2\phi^2 \tag{1.43}$$

Its Euclidean two point function on the thermal cylinder is

$$G^E(\omega_n) = \frac{1}{\omega_n^2+\omega_{\mathrm{ph}}^2}\,, \qquad \omega_n = 2\pi nT\,, \tag{1.44}$$

which implies that the retarded Green's function is

$$G^R(\omega) = \frac{1}{-(\omega+i0^+)^2+\omega_{\mathrm{ph}}^2}\,. \tag{1.45}$$

Note that G^R is state independent for free fields, since $[\phi(t),\phi]$ is a c-number! Other correlators will depend on β. For example, we know that the symmetric Green's function satisfies

$$G^S(\omega) = \coth\frac{\beta\omega}{2}\,\mathrm{Im}\,G^R(\omega) = \coth\frac{\beta\omega}{2}\pi\delta(\omega^2-\omega_{\mathrm{ph}}^2)\,. \tag{1.46}$$

We therefore already know the relevant Schwinger-Keldysh propagators:

$$\langle \phi_r(\omega)\phi_a \rangle = -iG^R(\omega) \tag{1.47}$$

$$\langle \phi_r(\omega)\phi_r \rangle = G^S(\omega) \tag{1.48}$$

You will use these and the corresponding fermionic real time propagators in the following Exercise.

Exercise 4: *Thermal broadening in real time formalism*

Let us revisit, directly in the real time formalism, the broadening of quasiparticles in Fermi liquid due to phonons studied in Ex. 3. The path integral on the Schwinger Keldysh contour takes the form

$$Z = \int D\psi_{1,2} D\phi_{1,2}\, e^{i(S_1 - S_2)} \tag{1.49}$$

with $S_i \equiv S[\phi_i, \psi_i]$ given by Eq. (1.25). The interaction takes the form

$$S_1^{\text{int}} - S_2^{\text{int}} = g\left(\phi_1 \psi_1^\dagger \psi_2 - \phi_2 \psi_2^\dagger \psi_2\right) . \tag{1.50}$$

(i) Write the interaction in Keldysh basis, and show that the electron self-energy at 1-loop is

$$\Sigma(\omega, k) = g^2 \int \frac{d\Omega d^d q}{(2\pi)^{d+1}} \left[D^S(\Omega, q) G^R(\omega + \Omega, k + q) + D^R(\Omega, q) G^S(\omega + \Omega, k + q)\right] , \tag{1.51}$$

where G and D are the fermion and phonon Green's functions.

(ii) Perform the Ω integral by residue, and show that one recovers our previous expression (1.28). In the real time approach, the Fermi-Dirac and Bose-Einstein distributions do not arise from Matsubara sums, but from the expressions for the symmetric Green's functions.

2 Lecture 2 – Modern EFTs for Fluctuating Hydrodynamics

The previous lecture showed that interacting quasiparticles have a finite lifetime at nonzero temperature. Conserved quantities, instead, cannot decay, which hints at an effective description containing only these collective excitations. This is the theory of fluctuating hydrodynamics: the dynamics of conserved densities. In the case of liquids like water, these conserved quantities are mass, energy, and momentum. For correlated materials the conserved quantities may be charge, energy, spin, etc. depending on the situation under consideration.

Fluctuating hydrodynamics is an Effective Field Theory (EFT), sharing many similarities with zero temperature EFTs for ordered phases such as magnets or superfluids. Instead of providing a microscopic formulation, it directly describes the emergent dynamics. As such, EFTs have a few unfixed parameters (Wilsonian coefficients), whose relation to the microscopic pararmeters are not always known. Nevertheless, we will see that EFTs are not just "phenomenological" but have substantial predictive power, relating many otherwise independent observables.

Hydrodynamic EFTs apply very generally to local many-body systems, on the lattice or in the continuum, disordered or not, strongly or weakly interacting, quantum or classical, including even systems with discrete time evolution such as unitary circuits or Floquet systems. In a sense, hydrodynamics is the most universal EFT. For this reason, it also has a very long history, and some of the contents of the EFT description presented here have been long known:

- Their equations of motion (turning off noise fields) reproduce the diffusion equation, Navier Stokes equation, etc. {19th century physics}

- To leading order, they predict simple poles (diffusive, or sound modes) in response functions of many-body systems {early 20th century physics: Landau-Lifschitz, Kadanoff-Martin [4]}

- More generally, they describe *fluctuating hydrodynamics*, noise etc. {late 20th century physics: Martin-Siggia Rose, KPZ, etc. [5, 6, 7, 8]}

- Beyond: adaptable to purely quantum observables, which can also have hydrodynamic signatures – spectral form factor [9], OTOCs, entanglement dynamics, open quantum systems [10]. {21st century physics}

- Further modern developments of these EFTs include: addressing systems with exotic symmetries (higher-form, or dipole/fracton like) [11, 12, 13]; ...

EFTs for hydrodynamics will be presented below in an ahistorical way, following the very recent realization that these EFTs can be constructed entirely based on a symmetry breaking pattern unique to mixed states [14]. This approach of course rests on pioneering work constructing EFTs on Schwinger-Keldysh contours [3, 15, 16, 17], as well as early theories of fluctuating hydrodynamics [6].

2.1 Symmetries of mixed state time evolution

We have seen that even at weak coupling, real time thermal dynamics is difficult to study from microscopics. We are in need of an EFT. The cleanest EFTs in theoretical physics come from spontaneous symmetry breaking (SSB), where Goldstone's theorem guarantees the existence of gapless excitations. These are carried by fields that nonlinearly realize the broken symmetry. This nonlinear realization highly constrains the EFTs, which therefore have remarkable predictive power [18, 19]. Recently, a SSB pattern applied to mixed states has been proposed to protect the long-lived nature of hydrodynamic excitations [14, 20, 21, 22]. The resulting EFTs are essentially identical to those used for many decades; however this approach is appealing as it unifies hydrodynamics with EFTs for SSB.[3]

To understand the SSB pattern, we first notice that there is a natural action of a doubled symmetry on density matrices. Consider for simplicity a $U(1)$ symmetry with Noether charge Q. One can consider the action

$$\rho \to e^{-i\alpha_1 Q}\rho e^{i\alpha_2 Q}\,. \tag{2.1}$$

Clearly, since $[H,Q]=0$ this action commutes with time evolution. In that sense, this $U(1)\times U(1)$ is is a symmetry. This doubling of symmetries is somewhat awkward and seems unnecessary; it was first discussed in the context of open systems, where it is useful and important [23] (this reference also explains more precisely what is meant by the doubled symmetry (2.1)). In that context, the density matrix evolves according to the Lindblad equation[4]

$$\partial_t\rho = \mathcal{L}\rho \equiv -i[H,\rho] + \sum_i \left(2L_i\rho L_i^\dagger - L_i^\dagger L_i\rho - \rho L_i^\dagger L_i\right)\,. \tag{2.2}$$

This open system Lindblad dynamics opens the door to breaking only *one* of the two symmetries. Specifically, if charge is exchanged with the bath, then the Lindblad operators can be charge (e.g., $L=\Phi$). This will break the symmetry (2.1) down to the diagonal $\alpha_1=\alpha_2$.

Why should we care about this in closed systems, where both symmetries are preserved and would usually be thought of as a single symmetry? The proposal of [20, 14] is that

[3]

[4]The evolution is not unitary, because we have integrated out (traced out) the environment. The Lindblad equation ignores non-localities in time that this could yield (Markovian assumption); it is the most general trace preserving $\mathrm{Tr}[\rho(t)]=1$, completely positive, local in time (and time-independent) evolution. "Completely positive" is a stronger condition than "positive" ($\rho(t)>0\,\forall t$); Ref. [24] provides a gentle introduction to the Lindblad equation that explains why this is desired.

thermal states generically *spontaneously* break $U(1) \times U(1)$ down to the diagonal. Loosely, while a pure state transforms just by a phase

$$|E,Q\rangle\langle E,Q| \to e^{i(\alpha_1-\alpha_2)Q}|E,Q\rangle\langle E,Q|\,, \tag{2.3}$$

the Gibbs state is only invariant if $\alpha_1 = \alpha_2$ (one cannot pull the phase out)

$$\rho = \sum_{E,Q} e^{-\beta E}|E,Q\rangle\langle E,Q| \to \sum_{E,Q} e^{i(\alpha_1-\alpha_2)Q}e^{-\beta E}|E,Q\rangle\langle E,Q|\,. \tag{2.4}$$

This generalizes to any symmetry (including spacetime and higher-form symmetries): the proposal is that all continuous symmetries have SWSSB in the thermal states, and corresponding modes.[5]

We will not further justify this assumption here, but we will study its consequences.

2.2 Warm-up: EFT for ordered phase

Let us start by considering a system with a $U(1)$ symmetry that is spontaneously broken in the conventional sense, say the 3d XY model in the ordered phase. In the Schwinger-Keldysh language, both $U(1)_{1,2}$ are broken and we therefore have two Goldstones

$$\phi_a = \phi_1 - \phi_2\,, \qquad \phi_r = \frac{1}{2}\left(\phi_1 + \phi_2\right)\,. \tag{2.5}$$

We therefore expect to have a local representation of the generating functional in terms of these Goldstones:

$$Z[A_1, A_2] = \int D\phi_1 D\phi_2 e^{iS_{\rm eff}[A^1_\mu+\partial_\mu\phi^1, A^2_\mu+\partial_\mu\phi^2]} \tag{2.6}$$

(note that this generating functional is automatically gauge invariant). Let us build the quadratic action at leading order in derivatives:

$$\mathcal{L} = c_1\dot\phi_a\dot\phi_r + c_2\partial_i\phi_a\partial_i\phi_r + ic_3(\partial_i\phi_a)^2 + ic_4\dot\phi_a^2 + \cdots\,, \tag{2.7}$$

with $c_i \in \mathbb{R}$ to satisfy the unitarity constraint $Z[A_a, A_r]^* = Z[-A_a, A_r]$. Furthermore, the latest time condition forbids ϕ_r^2 terms in the quadratic action.[6] The final condition we still need to impose is KMS. KMS is a nonlocal (in time) condition on the effective action, but

[5]SWSSB of discrete symmetries have also been subject of interest in the context of mixed state topological phases [25].

[6]Note that we are perturbatively imposing the conditions on the generating functional below (1.34). See Ref. [15] for a discussion.

the nonlocality is at a scale β that is earlier than the expected hydrodynamic cutoff.[7] We will therefore impose it perturbatively in $\beta\partial_t$. It then acts on the fields as

$$\begin{cases} \phi_1 \to -\phi_1(-t+\frac{1}{2}i\beta) \simeq -\phi_1(-t) - \frac{1}{2}i\beta\dot\phi_1(-t) \\ \phi_2 \to -\phi_2(-t-\frac{1}{2}i\beta) \simeq -\phi_1(-t) + \frac{1}{2}i\beta\dot\phi_1(-t) \end{cases} \quad \text{or} \quad \begin{cases} \phi_a \to -(\phi_a + i\beta\dot\phi_r) \\ \phi_r \to -(\phi_r + \frac{1}{4}i\beta\dot\phi_a) \end{cases} \tag{2.8}$$

Note that $\phi_a\phi_r \to \phi_a\phi_r$ + T.D., so both c_1 and c_2 are unconstrained. Indeed, we had already obtained them from a microscopic model, $S_1 - S_2$. However the ϕ_a^2 terms do not have this structure, they couple the two legs! (we saw how such terms could be generated from interactions, cf. $\Gamma \sim \lambda^2$). While

$$\phi_a^2 \to \phi_a(-t)^2 + i2\beta\phi_a(-t)\dot\phi_r(-t) \tag{2.9}$$

is not invariant, the following combination is:

$$\phi_a(\phi_a + i\beta\dot\phi_r) \to (\phi_a + i\beta\dot\phi_r)\phi_a\,. \tag{2.10}$$

So we find the following action:

$$S = \int c_1\dot\phi_a\dot\phi_r + c_2\partial_i\phi_a\partial_i\phi_r + ic_3\dot\phi_a\left(\dot\phi_a + i\beta\ddot\phi_r\right) + ic_4\partial_i\phi_a\left(\partial_i\phi_a + i\beta\partial_i\dot\phi_r\right) \tag{2.11}$$

is KMS invariant, to leading order. The retarded Green's function features a pair of poles at frequencies[8]

$$\omega = \pm c_s k - iDk^2 + \cdots \tag{2.12}$$

Instead of having a finite decay rate $-i\Gamma$, the superfluid Goldstones are protected even at finite T and simply have a sound attenuation rate $\Gamma \to Dk^2$.

2.3 Strong to Weak SSB of $U(1)$: theory of fluctuating diffusion

Our main interest is the "normal" phase, where there isn't spontaneous symmetry breaking in the conventional sense. We will focus on a system with a global $G = U(1)$ symmetry first, but generalizing this construction to $G = \mathbb{R}^{d+1}$ will produce a fluctuating theory of fluid mechanics (Navier-Stokes equations and beyond). We have argued that the normal phase is characterized by strong to weak SSB:

$$U(1)_a \times U(1)_r \to U(1)_r\,. \tag{2.13}$$

[7] This is the conjectured "Planckian bound" [26, 27, 28, 29].

[8] Inverting this Gaussian action also produces gapped or relaxed poles $\lim_{k\to 0}\omega(k) \neq 0$ that are however beyond the regime of validity of the EFT.

We will therefore only have a single Goldstone, ϕ_a. However, to realize KMS symmetry (and collapse rules) in the simplest possible way, it will need a Schwinger-Keldysh partner, which we will call μ_r. This is a "matter field" which linearly realizes the symmetry, but forms a KMS multiplet with ϕ_a as follows:

$$\begin{cases} \phi_a \to -(\phi_a + i\beta\mu_r) \\ \mu_r \to \mu_r + \frac{1}{4} i\beta\ddot{\phi}_a \end{cases} \tag{2.14}$$

I.e., μ_r transforms like $\dot{\phi}_r$ from the previous section. We are now ready to build the EFT. The c_1, c_3 and c_4 terms from above are allowed:

$$S = \int c_1\dot{\phi}_a\mu_r + iTc_3\partial_i\phi_a\left(\partial_i\phi_a + i\beta\partial_i\mu_r\right) + iTc_4\dot{\phi}_a\left(\dot{\phi}_a + i\beta\dot{\mu}_r\right) \tag{2.15}$$

Interestingly, the absence of the c_2 term changes the leading scaling behavior: the $\phi_a\mu_r$ part of the action shows that we have diffusive behavior $\omega \sim k^2$. This implies that to leading order, we can drop c_4 w.r.to c_3. We are then left with the following leading order action

$$S = \chi\int \dot{\phi}_a\mu_r + iTD\partial_i\phi_a\left(\partial_i\phi_a + i\beta\partial_i\mu_r\right) + \cdots \tag{2.16}$$

We have given names to the remaining coefficients c_1, c_3: suceptibility χ and diffusivity D. The first name will be justified shortly, whereas D was chosen because the μ_r satisfies a (noisy) diffusion equation

$$\frac{\delta S}{\delta\phi_a} = 0 \quad \Rightarrow \qquad \partial_t\mu_r - D\partial_i^2\mu_r = -iTD\partial_i^2\phi_a\,. \tag{2.17}$$

To identify the charge, we can couple the system to background fields. This is simple for ϕ_a: $\partial_\mu\phi_a \to \nabla_\mu\phi_a \equiv \partial_\mu\phi_a - A_{\mu,a}$. Now μ_r is already gauge invariant, but KMS requires A_μ^r to enter through F_{0i}^r as

$$Z[A_a, A_r] = \int D\mu_r D\phi_a e^{iS}\,, \quad S = \chi\int \nabla_0\phi_a\mu_r + iTD\nabla_i\phi_a\left(\nabla_i\phi_a + i\beta\partial_i\mu_r + i\beta F_{0i}^r\right) + \cdots \tag{2.18}$$

You will show this in the problem set. One can also add gauge invariant contact terms such as $F_{\mu\nu}^a F_{\mu\nu}^r$, but this enters at one higher order in derivatives.

We are now ready to compute observables. The retarded Green's function of the charge density is

$$G^R(p) = i\langle j_r^0(p) j_a^0\rangle = i\frac{\delta^2}{\delta A_a(-p)\delta A_r}\log Z = i\chi^2 Dk^2\langle\mu_r(p)\phi_a\rangle\,. \tag{2.19}$$

We will need the propagators. Inverting the action without sources:

$$S = \frac{1}{2}\chi\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\begin{pmatrix}\phi_a & \mu_r\end{pmatrix}_{-p}\begin{pmatrix}2iTDk^2 & i\omega + Dk^2 \\ -i\omega + Dk^2 & 0\end{pmatrix}\begin{pmatrix}\phi_a \\ \mu_r\end{pmatrix}_p \tag{2.20}$$

gives

$$\langle \mu_r(p)\phi_a\rangle = \frac{-i/\chi}{-i\omega + Dk^2}\,, \qquad \langle \mu_r(p)\mu_r\rangle = \frac{2TDk^2/\chi}{\omega^2+(Dk^2)^2}\,. \tag{2.21}$$

So we have

$$G^R(\omega,k) = \frac{\chi Dk^2}{-i\omega+Dk^2} \tag{2.22}$$

Simple result, but remarkable: a pole in the lower half plane. In finite volume, all discontinuities are along the real axis! They coalesce in the thermodynamic limit. Another important feature is that we have a *single* pole: half the mode counting as in $T=0$ Goldstones: we have "half" the symmetry breaking.

Part of this Green's function is accessible by equilibrium physics:

$$\lim_{\omega\to 0} G^R(\omega,k) = \chi = G^E(\omega_n=0,k)\,. \tag{2.23}$$

This is a parameter of the TEA, and measures the susceptibility $\chi = dn/d\mu$. One other important parameter in thermal matter is the conductivity $j=\sigma E$. It can be measured by the following Kubo formula:

$$\sigma = \lim_{\omega\to 0}\lim_{k\to 0}\frac{-i}{\omega}G^R_{jj}(\omega,k) = \lim_{\omega\to 0}\lim_{k\to 0}\frac{-i\omega}{k^2}G^R_{nn}(\omega,k) = \chi D \tag{2.24}$$

I used the WI to relate the GFs, but you could also derive σ directly. This is the "Einstein relation" – the EFT ties several independent observables (χ, D, σ).

In the limit $\beta\omega \ll 1$, the Wightman function $\mathrm{Tr}(\rho j^0(\omega,k)j^0) = \frac{2}{e^{\beta\omega}+1}G^S(\omega) \simeq G^S(\omega)$ are equal and given by

$$\langle j^0(\omega,k)j^0\rangle \simeq \frac{2}{\beta\omega}\,\mathrm{Im}\,G^R(\omega,k) = \frac{2T\chi Dk^2}{\omega^2+(Dk^2)^2}\,. \tag{2.25}$$

Its Fourier transforms are

$$\langle j^0(t,k)j^0\rangle = \int\frac{d\omega}{2\pi}e^{-i\omega t}\frac{2T\chi Dk^2}{\omega^2+(Dk^2)^2} = \chi T e^{-Dk^2|t|}\,, \tag{2.26}$$

and

$$\langle j^0(t,x)j^0\rangle = \chi T\int\frac{d^dk}{(2\pi)^d}e^{ikx}e^{-Dk^2t} = \frac{\chi T}{(4\pi D|t|)^{d/2}}e^{-x^2/(4D|t|)}\,. \tag{2.27}$$

Of course these results will receive corrections from irrelevant operators. We will revisit them soon.

Exercise 5: *KMS invariance of diffusion EFT*

We showed during the lecture that the combination

$$\partial_i\phi_a\left(\partial_i\phi_a + i\beta\partial_i\mu_r\right) \tag{2.28}$$

is KMS invariant (to leading order in derivatives). Show that the correct way to gauge this term while preserving KMS is:

$$\nabla_i\phi_a\left(\nabla_i\phi_a + i\beta\partial_i\mu_r + i\beta F^r_{0i}\right)\,, \tag{2.29}$$

with $\nabla_\mu\phi_a \equiv \partial_\mu\phi_a - A_{\mu,a}$.

2.4 EFT of momentum conserving fluids

In certain clean correlated systems, electron fluids or electron-phonon soups may conserve momentum well enough to exhibit a hydrodynamic regime which includes momentum density (in addition to charge or energy density) as a long-lived collective degree of freedom [30, 31] (see Levitov lectures).

The additional degree of freedom arises from strong-to-weak SSB of the translation symmetry:

$$\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\,. \tag{2.30}$$

Let us consider a system with charge and momentum conservation, ignoring energy conservation for simplicity. We will construct the EFT following App. A of [29]. The EFT will then contain $d+1$ Goldstones that shift under the $U(1)\times\mathbb{R}^d$ symmetry:

$$\phi^a \to \phi^a + c\,, \qquad X^a_i \to X^a_i + c_i\,. \tag{2.31}$$

Let us denote the KMS partners of these fields as μ and v_i. The KMS symmetry then is realized as (2.14) and

$$\begin{cases} X^i_a \to X^i_a + i\beta v^i + \cdots \\ v^i \to -\left(v^i + \frac{1}{4}i\beta\ddot{X}^i_a + \cdots\right)\end{cases} \tag{2.32}$$

Note the overall minus sign compared to (2.32) coming from the fact that momentum flips under time-reversal symmetry.

Let us construct an EFT for these fields that is invariant under (2.31) and (2.32). We can first consider each sector separately, as follows:

$$S_{U(1)}[\mu,\phi_a] = \int \chi\mu\dot{\phi}_a + iT\sigma_{ij}\partial_i\phi_a\left(\partial_j\phi_a + i\beta\partial_j\mu\right) + \cdots \tag{2.33a}$$

$$S_{\mathbb{R}^d}[v,X] = \int \chi_{PP}v_i\dot{X}^i_a + iT\eta_{ijkl}\partial^iX^j_a\left(\partial^kX^l_a + i\beta\partial^kv^l\right) + \cdots \tag{2.33b}$$

The additional indices carried by the translation Goldstones allows one to write potentially many "momentum conductivities", parametrized by the viscosity tensor η_{ijkl}. Let us for simplicity consider isotropic systems with reflection symmetry: this requires to express the tensors σ_{ij} and η_{ijkl} in terms of the tensor δ_{ij} (you will generalize this to systems without reflection or time-reversal symmetry in Ex. 6, in which case one can also use $\epsilon_{i_1\cdots i_d}$). This leads to

$$\sigma_{ij} = \sigma\delta_{ij}\,, \qquad \eta_{ijkl} = \eta_1\delta_{ij}\delta_{kl} + \eta_2\delta_{ik}\delta_{jl} + \eta_3\delta_{il}\delta_{jk}\,. \tag{2.34}$$

For the viscosity tensor, notice that after integration by parts it is contracted with a tensor symmetric in $i \leftrightarrow k$ – one can therefore wlog either η_1 or η_3 to zero, and are left with two parameters. These are called the bulk and shear viscosities, and are arranged as

$$\eta_{ijkl} = \zeta\delta_{ij}\delta_{kl} + \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right)\,. \tag{2.35}$$

If this were all, both sectors would simply diffuse. However we can also couple the two – this will lead to a hybridization of these diffusive modes into a sound mode. To leading order in derivatives, there are two terms that one can

$$S_{\text{mix}} = \int \alpha_1\mu\partial_i X^i_a + \alpha_2 v^i\partial_i\phi_a\,. \tag{2.36}$$

This action is only invariant under KMS if $\alpha_1 = \alpha_2$. Moreover, consistency with static equilibrium requires $\alpha_1 = n$. The full action for linearized fluctuating hydrodynamics is then

$$S_{U(1)} + S_{\mathbb{R}^d} + S_{\text{mix}} \tag{2.37}$$

Its equations of motion are the linearized, noisy Navier-Stokes equations, which arise in the context of electron hydrodynamics. Solving for the modes as we did for the diffusive EFT, one finds a sound mode carried by charge and longitudinal momentum

$$\omega = \pm c_s k - i\gamma_s k^2\,, \qquad c_s^2 = \frac{n^2}{\chi\,\chi_{PP}}\,, \quad \gamma_s = \frac{1}{\chi_{PP}}\left(\frac{2}{d}(d-1)\eta + \zeta\right)\,, \tag{2.38}$$

as well as a diffusive mode carried by transverse (shear) momentum excitations

$$\omega = -iDk^2\,, \qquad D = \frac{\eta}{\chi_{PP}}\,. \tag{2.39}$$

Exercise 6: *Parity odd transport in the hydro EFT*

Try to extend the EFTs of diffusion (Sec. 2.3) and the EFT of fluids (Sec. 2.4) to systems without parity or time-reversal symmetry.

Hint: for diffusion, try to introduce a term corresponding to the Hall conductivity σ_H. You may need to keep background fields activated to see this term. For parity-violating fluids, you should be introduce a Hall viscosity η_H, as well as a few other parameters (see, e.g., [32]).

Exercise 7: *EFT for nonlinear fluctuating hydrodynamics*

...

2.5 Systematics of the EFT (if time permits)

The point of setting up an EFT is to be systematic. This will take us beyond "20th century" hydrodynamics, and there will be surprises. In particular, studying corrections will reveal exactly in which regime the EFT is valid. We will find that all new operators in the EFT are irrelevant. One would therefore expect corrections to Eqs. (2.25-2.27) to be small at small ω, k or large t, x. While this is correct for Eqs. (2.25) and (2.27), it is not for Eq. (2.26), i.e. $\langle j^0(t,k)j^0\rangle$. This "dangerous irrelevance" in the simple theory of diffusion shows the subtleties of EFTs in non-Lorentz-invariant contexts.

Let us classify operators of the EFT by their scaling dimension. Scaling is a little richer in this context. By scaling the leading Gaussian action as $S \sim 1$ one finds

$$\omega \sim k^2\,, \qquad \phi_a \sim \mu_r \sim k^{d/2}\,. \tag{2.40}$$

Note that the density 2pt function (2.27) exemplifies this scaling ($n \sim \mu_r \sim k^{d/2}$).

What corrections can one have to the EFT? We can have higher derivative corrections:

$$\partial_t \sim \partial_i^2 \sim k^2\,, \tag{2.41}$$

which will give $1/t$ corrections at late times (and are clearly irrelevant). We can also have nonlinearities: because ϕ_a needs to enter with a derivative, the leading non-linearities will come from extra factors of

$$\mu_r \sim k^{d/2}\,. \tag{2.42}$$

These will produce cubic vertices in the EFT, for example:

$$\mathcal{L} \supset (D + D'\mu_r + \cdots)\partial_i\phi_a(\partial_i\phi_a + i\beta\partial_i\mu_r)\,. \tag{2.43}$$

The D term is the one from the Gaussian action (2.16), the D' term produces a cubic vertex.[9] Because we need two cubic vertices to give a correction to the 2pt function, the overall suppression of these corrections is $k^d \sim 1/t^{d/2}$, they are also irrelevant.

[9] Notice that we have not written the new cubic term with an arbitrary coefficient, but rather with $D' \equiv dD(\mu)/d\mu$, the derivative of the diffusion with respect to chemical potential – if we know the density (or μ)-dependence of the diffusion constant, we know about inevitable nonlinear terms of the EFT [33].

We therefore expect the following structure for corrections to observables

$$\begin{aligned}\langle j^0(x,t)j^0\rangle = \frac{\chi T}{(4\pi Dt)^{d/2}}\Big[\quad & F_{0,0}(y)+\frac{1}{t}F_{0,1}(y)+\frac{1}{t^2}F_{0,2}(y)+\cdots \\ & +\frac{1}{t^{d/2}}\left(F_{1,0}(y)+\frac{1}{t}F_{1,1}(y)+\frac{1}{t^2}F_{1,2}(y)+\cdots\right) \\ & +\frac{1}{t^{d}}\left(F_{2,0}(y)+\frac{1}{t}F_{2,1}(y)+\frac{1}{t^2}F_{2,2}(y)+\cdots\right)+\cdots\ \Big],\end{aligned} \tag{2.44}$$

where the $F_{\ell,n}$ are universal scaling functions (up to an overall prefactor or two) of the dimensionless scaling variable $y\equiv x/\sqrt{Dt}$, arising at ℓ-loops and nth order in the derivative expansion – they are predictions of the EFT (predictive power despite Wilsonian coefficients).

One could write similar expressions for (2.25) or (2.26) in terms of scaling functions of $\omega/(Dk^2)$ or tDk^2.

2.6 Loop correction to diffusion and dangerous irrelevance (time won't permit)

The fact that hydrodynamics should receive loop corrections, as anticipated in the previous section, was in fact discovered numerically in the 70s [34], and is what lead to the first action formulations for hydrodynamics [6, 7]. These loops are traditionally called "hydrodynamic long-time tails". See Refs. [35, 36] for a discussion of their relevance for the QGP. They are suppressed in holographic theories, where they arise from graviton loop corrections in the bulk [37] (indeed, the hydrodynamic EFT action is proportional to the free energy, through χ, and hence a large free energy $\chi\sim N^2$ leads to suppressed loop corrections).

See Ref. [38] for a recent treatment of loop corrections within the diffusive EFT. We quote the result:

$$G^R_{j^0j^0}(\omega,k)=\frac{\chi Dk^2}{(D+\delta D(\omega,k^2))k^2-i\omega}\,,\qquad \delta D(\omega,k^2)=\frac{\chi D'^2}{D^2}(-i\omega)\frac{\left(\frac{2i\omega}{D}-k^2\right)^{\frac{d}{2}-1}}{(16\pi)^{d/2}\Gamma(\frac{d}{2})}\,. \tag{2.45}$$

This result can almost be guessed without any calculation, except for the overall numerical factor: (i) D'^2 comes from two insertions of the vertex (2.43); (ii) the scaling of the correction is $O(k^d)$, as anticipated below (2.43); (iii) the branch cut at $\omega=-\frac{i}{2}Dk^2$ can be found by cutting rules, see below; (iv) the overall ω guarantees that the static Green's function is analytic, as required from the thermal effective action (Sec. 1.1) [39]; (v) the remaining factors of D,χ are fixed by dimensional analysis.

The location of the branch point can be found similarly with a similar argument that leads to the location of the branch point at $s=-(2m)^2$ for two-particle threshold in QFT.

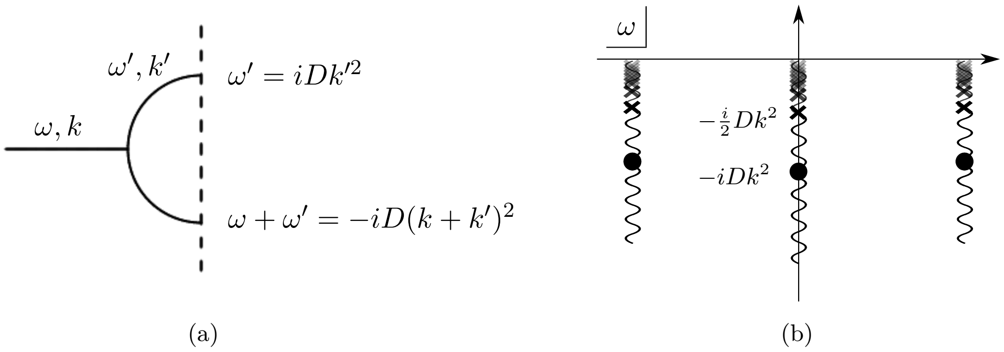


Figure 2: (a) On-shell condition for the two internal legs, leading to a branch point at $\omega = -\frac{i}{2}Dk^2$. Further loops would lead to additional branch points at $-\frac{i}{n}Dk^2$. The branch points and poles associated with a sound mode $\omega = \pm c_s k - ik^2$ are also shown on the left and right. (Figure adapted from [40, 41]).

Imagine cutting the one-loop diagram as in Fig. 2, and placing both internal legs on shell. The smallest (imaginary) value of external frequency that allows for this is $\omega = -\frac{i}{2}Dk^2$, and arises when $k' = -k/2$.

You may worry about the fact that this 1-loop branch point is closer to the origin of the complex ω plane, and is therefore more singular in the IR than the leading non-analyticity, the diffusive pole. The fact that the non-analyticity across the cut is k^d suppressed implies that it is still a small correction to $G^R(\omega, k)$. However, your concerns are warranted: Fourier transforming to time $\omega \to t$, the non-analyticity is picked up and leads to a correction:

$$\langle j^0 j^0 \rangle(t,k) = \chi T \left[e^{-Dk^2} + \frac{\#}{t^{d/2}} e^{-\frac{1}{2}Dk^2 t} + \cdots \right] . \tag{2.46}$$

Despite the $1/t^{d/2}$ suppression, the 1-loop correction starts to dominate at late times $t \gtrsim 1/(Dk^2)$. Eventually, higher and higher loops dominate, with n-loop contributions scaling as $\frac{1}{t^{nd/2}} e^{-\frac{1}{n}Dk^2 t}$. This series can be approximately resummed, yielding an entirely different, stretched exponential decay of the correlator [41][10]

$$\langle j^0 j^0 \rangle(t,k) \approx e^{-\sqrt{Dk^2 t}} . \tag{2.47}$$

This phenomenon highlights the subtleties of non-relativistic EFTs: while correlators can be expanded in terms of scaling functions as in Eq. (2.44), the behavior of the scaling functions can be singular in certain parameter regions, leading to a breakdown of the EFT in those regions.

[10] This "diffuson cascade" and breakdown of the EFT was also observed numerically [42], albeit in a stochastic system where the resummed behavior appears to be different.

3 Lecture 3 – Bounds on Transport

EFTs for fluctuating hydrodynamics capture dynamical (and equilibrium) observables in thermalizing systems in terms of a handful of parameters ("Wilsonian coefficients"), such as diffusivities D or susceptibilities χ. They have predictive power, in that many experimentally independent observables are expressed in terms of these parameters.[11]

However, the EFTs do not predict the value of parameters like χ or D. These must be typically obtained from a microscopic calculation. The fact that microscopic calculations for most strongly correlated systems of experimental relevance are not under theoretical control motivates finding universal constraints that these parameters must satisfy, irrespective of the microscopics. This lecture will cover such constraints: bounds on transport.

The search for universal bounds on transport shares the philosophy of other UV/IR constraints on quantum many-body dynamics (e.g., Lieb-Schultz-Mattis, Lieb-Robinson, sum rules, etc.), where one attempts to constrain emergence based on general principles (unitarity, symmetries and their anomalies, causality) rather than perturbative calculations.

One experimental motivation for bounds on transport is the linear-in-T resistivity seen in the normal ("strange metal") phase of a host of high-Tc superconductors: while establishing a universal mechanism for strange metal phenomenology has proven challenging, the existence of a universal bound may explain why many strongly correlated systems with no small parameter push against this limit.

3.1 Mott-Ioffe-Regel limit

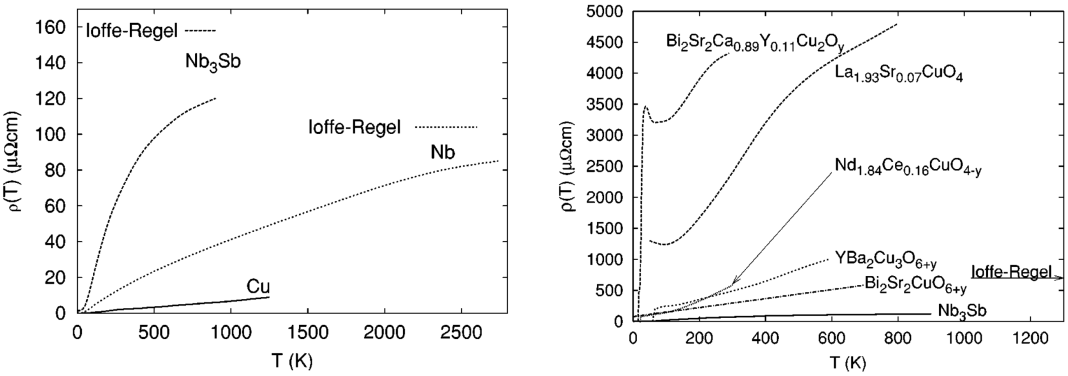
Trasport can often be studied in a semiclassical approach using Boltzman kinetic theory. Consider a Fermi liquid, where the collision integral produces a mean free path ℓ for the low energy quasiparticles with momentum close to p_F. For this semiclassical picture to be self-consistent at all, this mean-free-path should be larger than the spatial resolution of the particles, which according to the uncertainty principle must satisfy

$$p_F \ell \gtrsim \hbar \, . \tag{3.1}$$

We set $\hbar = 1$ in the following. Inserting this in the Drude formula for the dc conductivity, which involves the mean free time $\tau = \ell / v_F$, gives a limit conductivity that can be achieved by kinetic theory:

$$\sigma_{\rm dc} = \frac{ne^2}{m_*}\tau \sim e^2 k_F^{d-1} \ell \gtrsim e^2 k_F^{d-2} \, . \tag{3.2}$$

[11] For example, nonlinear response depends on the same coefficients as linear response [33].



(a) (b)

Figure 3: (a) The resistivity of very good metals such as copper grows with temperature but stays well below the MIR limit for all T below the melting temperature. Less good metals such as Nb show resistivity saturation near the MIR limit. (b) Instead, bad metals are defined as metals that do not show such resistivity saturation, signalling transport beyond the quasiparticle kinetic theory paradigm. (Figure from [46]).

This Mott-Ioffe-Regel (MIR) limit is not a bound per se: when it is reached, the formalism of kinetic theory breaks down. Interestingly, many metals show *resistivity saturation* as temperature is increased and their resistivity approach this limit (see [43] for a review). This phenomenon, even in conventional metals, is not entirely understood.[12] [13]

Part of the interest for the MIR limit is that it is violated in many strongly correlated materials, called "bad metals" [47]. That this is possible is no surprise, given that the argument above is not a bound. However that argument suggests that bad metals cannot be described in terms of quasiparticles obeying kinetic theory.

3.2 Strong coupling bound on non-quasiparticle transport

We now lift the assumption of quasiparticle transport, and explore possible universal bounds on transport. First note that transport is singular in the limit of weak coupling at weak coupling $\lambda \ll 1$ (at least in clean systems), because free systems do not thermalize. Transport parameters such as diffusivities, conductivities, viscosities, and the local equilibration time

[12] Diffusivity saturation at the Mott-Ioffe-Regel limit also seems to occur in the context of spin diffusion in certain cold atom systems. The diffusion constant of a system of quasiparticles is related to the mean free time or length as $D \sim v^2\tau \sim v\ell = \frac{p}{m}\ell = \frac{\hbar k\ell}{m} \gtrsim \frac{\hbar}{m}$, where the last step is the MIR limit for the diffusivity.

[13] See [44, 45] for controlled large N models of a Fermi liquid coupled to acoustic phonons, where a feature occurs in the resistivity at the MIR limit

diverge in this limit

$$D,\, \sigma,\, \eta,\, \tau_{\text{eq}} \sim \frac{1}{\lambda^2}\,. \tag{3.3}$$

We therefore do not expect an absolute upper bound on these quantities. As coupling is increased, these parameters decrease; however this expression only holds at weak coupling. Certain solvable strongly coupled models (mostly: holographic QFTs and SYK-like models) exhibiting $O(1)$ transport parameters have suggested that these do not continue to decrease indefinitely as coupling becomes strong. This has lead to a number of conjectured quantum lower bounds on transport parameters. There are several versions of this conjecture, for various different observables, discussed below.

Planckian bound: The idea that there is a quantum limit to dissipation is perhaps most concisely formulated as the Plankian bound on the local equilibration time

$$\tau_{\text{eq}} \gtrsim \frac{\hbar}{T}\,, \tag{3.4}$$

which states that there is a quantum limit to how fast a many-body system can thermalize [26, 27, 28]. In quasiparticle systems, transport parameters are often expressed in terms of τ_{eq} (or a mean-free time), so in a sense the Planckian bound can be thought of as a seed for other transport bounds. However, one challenge to establish (3.4) has been to provide a precise definition of τ_{eq}. We will discuss this below.

Diffusion bound: The calculation of transport parameters in holographic QFTs gave a key analytic insight on dissipation in strongly coupled quantum many-body systems [48]. Holographic QFTs have conserved momentum, whose transport is characterized by viscosities. The shear viscosity at strong coupling was found to not decrease indefinitely, but settle at an order one value (when divided by the entropy density). A similar behavior arises in the quark-gluon plasma:

$$\text{QCD high } T\text{:} \quad \frac{\eta}{s} \sim \frac{1}{\lambda^2} \qquad\qquad \text{QCD intermediate } T\text{:} \quad \frac{\eta}{s} \sim 1 \tag{3.5}$$

This has lead to the conjecture that this ratio is bounded by an order one number (times $\hbar$). While this bound is not directly useful for transport in materials, it can be formulated in a way that is amenable to generalization, as a bound on the diffusion constant of shear momentum modes [49]

$$D_\eta = \frac{\eta}{\chi_{PP}} = c^2 \frac{\eta}{sT} \gtrsim \frac{c^2}{T} \tag{3.6}$$

The second equality holds in relativistic QFTs where the momentum susceptibility is fixed by Lorentz invariance. However this suggests a generalization [49]

$$D \gtrsim v^2 \frac{\hbar}{T} \tag{3.7}$$

where v is the velocity relevant to dynamics in the system under consideration – perhaps $v = v_F$ for a metal. One appeal of this bound is that metals saturating it exhibit linear-in-T resistivity:

$$\rho = \frac{1}{\sigma} = \frac{1}{\chi}\frac{1}{D} \sim \frac{v_F}{k_F^{d-1}}\frac{1}{v_F^2} T \sim \frac{m_*}{n} T \tag{3.8}$$

If one were to extracting a time scale from this resistivity using the Drude formula (3.2) (not necessarily a reasonable thing to do), this would yield a Planckian time scale $\tau \sim 1/T$.

There has been no proof of these conjectures (even the η/s one which has stronger assumptions and were the choice of the velocity is clear). However, progress on related bounds but in an opposite direction have been made using causality, we will discuss this below.

Bound on Lyapunov exponent: this formulation of the bound relies on the strongest assumptions, and applies to an observable that is difficult to measure experimentally; the flipside is that a sharp bound can actually be established, using analyticity of thermal correlation functions [50]. It is based on the observation that certain semiclassical or large N systems feature an exponential growth of out-of-time-ordered correlators

$$\langle \mathcal{O}(t)\mathcal{O}\mathcal{O}(t)\mathcal{O}\rangle \sim 1 - \frac{1}{N}e^{\lambda_L t}\,, \tag{3.9}$$

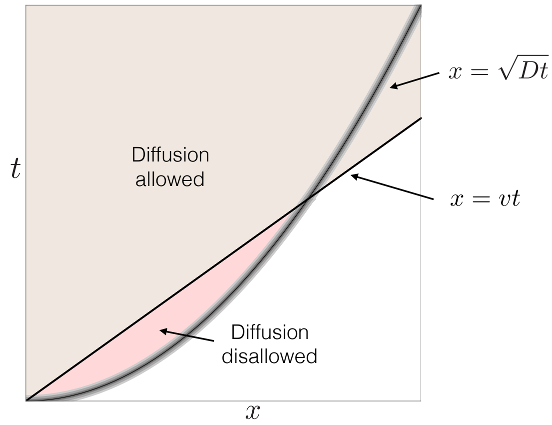
a semiclassical analog of the "butterfly effect" of classical chaos. If one assumes that correlators indeed have this behavior, then the corresponding timescale must satisfy a Planckian bound:[14]

$$\frac{1}{\lambda_L} \geq \frac{1}{2\pi}\frac{1}{T}\,. \tag{3.10}$$

3.3 Causality bound on diffusion

Some progress in establishing universal bounds on transport has been made by harnessing the tension between hydrodynamic dispersions and causality [51]. Consider a system with a lightcone, meaning that the commutator of local operators $[\mathcal{O}(t,x), \mathcal{O}(0,0)]$ vanishes

[14] Technically, the correlator that is studied is a regulated version of the OTOC $\mathrm{Tr}\,\sigma O(t)\sigma O\sigma O(t)\sigma O$ with $\sigma = \rho^{1/4}$, which can be shown to be analytic in a strip $\mathrm{Im}\,t < \beta/2$. In a nutshell: bounded functions that are analytic in a region cannot vary too fast.



(a) (b)

Figure 4: (a) Hydrodynamics cannot emerge too soon or it would predict spectral densities outside of lightcones (from [51])

for $x > ct$. This applies to relativistic QFTs, where c is the speed of light, as well as quantum circuits (cellular automata). The Lieb-Robinson bound implies that it also applies approximately to local Hamiltonians on a lattice, where in this case c is replaced by the Lieb-Robinson velocity $v_{\rm LR}$. . Instead, notice that diffusive spreading $x \sim \sqrt{Dt}$ is faster than any lightcone at sufficiently early times, see Fig. 4a (this also applies to diffusively broadened sound modes, but we focus on diffusive modes for simplicity). Of course this is not a contradiction: like any EFT, hydrodynamics is an emergent description that holds only after some timescale, which in the context of hydrodynamics is the local equilibration time $\tau_{\rm eq}$. Loosely, causality requires the diffusive front to be subluminal $\sqrt{Dt} < v_{\rm LR} t$, leading to an upper bound on the diffusivity

$$D \lesssim v_{\rm LR}^2 \tau_{\rm eq} \,. \tag{3.11}$$

This is not a sharp bound since it involves evaluating the Green's function near the cutoff, at which point corrections become large.

There are several improvements one can make to this argument. First, the Lieb-Robinson bound is very conservative (it holds in any state!) and is typically far from being saturated in the thermal state. A slower state-dependent velocity that bounds dynamics is the butterfly velocity $v_{\rm B} \leq v_{\rm LR}$ (usually defined by a spatially-resolved version of the OTOC (3.9)), so that under certain conditions the bound can be improved to $v_{\rm LR} \to v_{\rm B}$ [52, 53]. Second, the bound (3.11) can be made sharp in certain special situations, including open systems [54] and large N QFTs [55].

3.4 Planckian bound on thermalization

Let us return to the Planckian conjecture:

$$\tau_{\rm eq} \gtrsim \frac{\hbar}{T} \,. \tag{3.12}$$

Is there a sharp notion or definition for the local equilibration for which this conjecture is true (and interesting)? Consider first a system of quasiparticles. A natural notion of equilibration time is the collision time (mean free time between collisions). However, even here this notion is ambiguous, as there are several collision time scales. Indeed, free fermions in a disordered landscape will collide frequently with defects, but this is not even truly a many-body system! (Eigenstates are still tensor products of single-particle states, even though these are more complicated than Bloch states). What distinguishes this type of scattering from thermalizing scattering off dynamical degrees of freedom (including the fermions off themselves) is that scattering off static disorder is elastic: ther energy of the fermion is conserved as it bumps off defects. This is illustrated in Fig. 5.

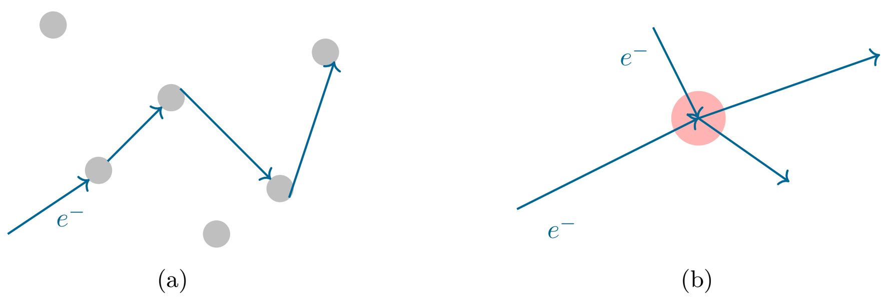


Figure 5: (a) Elastic scattering from static disorder. (b) Inelastic electron-electron collisions (or electron-phonon, etc.) redistribute energy and drive many-body thermalization.

Note that the line between elastic and inelastic scattering can be a little fuzzy. What if the disorder can move, but only very slowly? A similar situation arises when electrons scatter off phonons at temperatures above the Debye temperature, such that the typical energy of electrons is $\gg$ the available energies of phonons. See Ref. [28] for a detailed discussion.

We would like a definition that does not rely on a quasiparticle picture [29]. One definition that is tempting is to define it as the decay rate of local operators at late times

$$\langle \mathcal{O}(t)\mathcal{O}\rangle \sim e^{-\Gamma t} \,. \tag{3.13}$$

However, hydrodynamics implies that bosonic operators generically decay polynomially, because of overlap with hydrodynamic modes. Such a definition could apply in sectors that

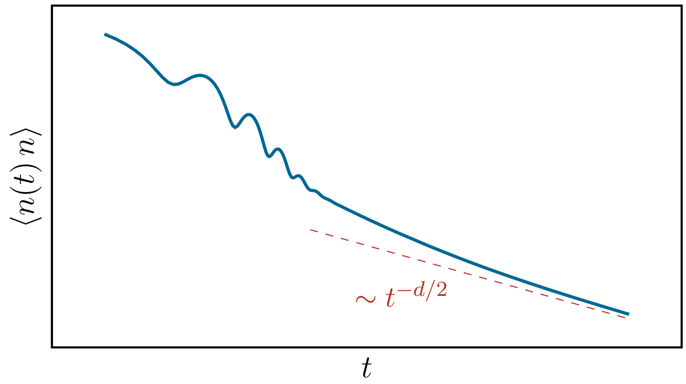


Figure 6: The time scale of emergence of hydrodynamics provides a mechanism-independent definition of the local equilibration time τ_{eq}.

decouple from hydrodynamics, but in many situations of interest there is no such sector. A definition that can apply to any model or system is to identify the local equilibration time as the time scale of emergence of hydrodynamics (said differently, it is the UV cutoff of hydrodynamics). This is illustrated in Fig. 6. There are several ways to make this definition sharp, see [29].

Now hydrodynamic EFTs predict much beyond the asymptotic form of the 2pt function. They also capture intermediate time corrections (see Eq. (2.44)). These corrections must die off for standard diffusive behavior to emerge – this leads to a Planckian bound on the equilibration time [29, 56].

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