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Big Ideas in the Understanding of Fractions: A Learning Progression

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Abstract

Learning progressions describe how students typically develop understanding of key concepts and are promising tools that teachers can use to create and to interpret assessments. This study proposes a learning progression for fractions, which are a foundational concept in mathematics. We suggest that there are five increasingly sophisticated ways to interpret fractions: as a part-whole relationship, as a quotient, as a measurement, as a ratio/rate, or as an operator. We argue that these five interpretations form a coherent learning progression that has great potential to facilitate formative assessment of student learning. We provide both theoretical and empirical support for this learning progression through the examination of mathematics content standards and curricular lesson orderings. We leverage the common vertical scale of a large-scale standardized mathematics assessment to demonstrate that items associated with more sophisticated interpretations of fractions are more difficult than are those at the earlier stages of our progression.

Keywords: formative assessment; mathematics education; learning progression; construct map; vertical scale

For some time now, *learning progressions* (LPs), which are descriptions of the developmental path that students are likely to take when learning the concepts in a given domain (Clements & Sarama, 2004), have been viewed as a promising means of coordinating three elements that are critical to student learning: curriculum, instruction and assessment (Clements & Sarama, 2004; Lobato & Walters, 2017). By emphasizing the “big picture” ideas within a subject, these trajectories provide an organizing structure that makes explicit connections between what might otherwise be viewed as long lists of discrete knowledge and skills. Learning trajectories, as the name implies, shift emphasis away from a model of teaching and learning in which a unit of content is taught, and students either demonstrate mastery of the content through their performance on assessment tasks, in which case they move on, or they do not, in which case the next steps are seldom well articulated. Instead, focus is placed on the different ways that students understand an important idea, and how this understanding, with the right support, is expected to become more sophisticated over time.

Ideally, a learning progression can help teachers and students alike to notice the aspects of reasoning that a student uses to solve mathematical problems and to determine whether that reasoning might be productive in one context, but not in another. Engaging in this type of formative assessment of students’ understanding is a crucial part of teachers’ efforts to connect the curricular standards and the instructional activities related to an LP. Formative assessment can take many forms, from listening to students’ conversations while they work on instructional tasks to periodically administering more formal assessments. Teachers typically engage in a range of these practices, but more and more school districts are using large-scale, predominately multiple-choice tests for this purpose[1]. The *i-Ready Diagnostic*, for example, provides a list of "can-do's and next steps" based on each student's assessment scores that are meant to help teachers tailor instruction to meet students' individual needs. While all instances of formative assessment, including these large-scale assessments, present opportunities for both providing feedback to students and adjusting instruction, the more immediate the adjustment, the stronger the impact on student learning is likely to be. There may, therefore, be a disconnect between the kinds of local, teacher- and researcher-developed assessments that are most frequently used in the development and validation of LPs (e.g., Arieli-Attali & Cayton-Hodges, 2014; Wilkins & Norton, 2018; Wright, 2014; Yulia et al., 2019), and the large-scale interim and summative assessments that are commonly used by teachers in K-12 public school settings in the United States. The former tend to allow for more immediate and iterative feedback during the learning process, while the latter are typically only administered a few times a year after learning has occurred.

In this paper we address this disconnect by using the results from a widely used commercial assessment, Curriculum Associates' *i-Ready Diagnostic*, to support an LP for fractions. We present compelling empirical evidence to support the validity of this LP by taking advantage of *i-Ready*'s vertical scale, which allows us to compare the difficulties of items designed for students in different grade levels. By using the vertical scale to investigate the relative difficulties of items that are more and less mathematically sophisticated, according to the LP, we provide an example where an LP can provide a productive structure for making sense of results of an externally mandated assessment. We note up front that although the research

[1] Three of the commercial assessments commonly purchased by U.S. school districts for this purpose are the MAP Growth Assessments (NWEA), *i*-Ready Diagnostic assessments (Curriculum Associates) and Star Assessments (Renaissance Learning).

presented in this study did occur within the context of a collaboration with Curriculum Associates that was intended to contribute to their ongoing efforts to improve upon their assessment and curricular products, our research team was given full autonomy in the development and validation efforts related to an LP for fractions.

In what follows, we begin by situating our study within the curriculum and assessment context of the *i-Ready Learning* and *i-Ready Diagnostic* products that Curriculum Associates provides to school districts in the United States. We motivate our specific focus on an LP for the understanding of fractions, and then provide a summary of the results from our review of pre-existing fraction LPs. We then introduce an LP that takes a slightly larger scope and grain-size than do the LPs found in the pre-existing mathematics education literature. Our intention in using this grain-size is to provide teachers with a handful of large categories that they can hold in their minds more effectively than a series of hyper-specific standards. Furthermore, it is easy to get bogged down in specifics and to focus on individual skills rather than on the concepts that underly those skills. Our intension in this work is to encourage teachers to focus on the big ideas that make up students' understandings of fractions rather than thinking of fractions as a list of facts and skills to be memorized and reproduced. Since our large LP levels span several grades, and multiple grades have standards associated with each level, we expect that this large-grained LP could also allow teachers to have cross-grade-level discussions using common language (e.g., Suh & Seshaiyer, 2015). We characterize the qualitative distinctions between the levels of the LP and illustrate each level with an exemplar item. We then turn to evidence of empirical support for the LP using data from the *i-Ready Diagnostic* assessment. We show that the levels of our LP are associated with both the difficulty of items across the *i-Ready Diagnostic* assessment's vertical scale, and with the curricular ordering of lessons that have been coded according to each

LP level. We conclude with a discussion about how the LP can be used to support formative inferences about student learning in classroom contexts that feature a combination of locally developed and externally mandated assessments.

The Curriculum and Assessment Context

The *i-Ready Diagnostic* assessment comprises two grade-specific standardized tests in reading and mathematics that are intended to be administered during the fall, the winter and the spring of each academic school year. Students take each test on a digital interface, and tests are designed to be adaptive such that each new multiple-choice item to which a student is exposed depends upon whether they answered prior items correctly. The mathematics test for students in grades K-12 consists of up to 66 items, and the content of these items is organized into four strands: Algebra, Geometry, Measurement, and Number and Operations. The test was developed to serve the following four purposes (Curriculum Associates, 2018, p. 8):

1. Establish a metric that will allow for an accurate assessment of student knowledge that can be monitored over a period of time to gauge student improvement.
2. Accurately assess student knowledge for different content strands within each subject.
3. Provide information on what skills students are likely to have mastered and likely need to work on next.
4. Link the assessment results to instructional advice and student placement decisions about Curriculum Associates' *i-Ready Instruction* curricula and print products.

A distinguishing feature of Curriculum Associates' approach to working with school districts is that it seeks to bundle its *i-Ready Diagnostic* assessment with curricular resources that teachers can use as part of their efforts to facilitate student learning. These resources come in two forms:

i-Ready Learning, a library of online instructional modules, and Ready Learning, a series of grade-specific printed books containing instructional lessons.

Curricular priorities: The importance of fractions

The reasoning skills that students typically develop with regard to fraction operations and equality in grades three through six help to build a sense of mathematical structure that facilitates the learning of more formal algebraic concepts later on (Common Core State Standards Initiative, 2010; Empson et al., 2011). In particular, students need to understand conceptually what fractions are and how they interact with one another (Byrnes & Wasik, 1991), which largely involves seeing fractions as real number values that can be placed at a unique point on a number line (Hansen et al., 2015; Siegler et al., 2011) and developing proportional reasoning and visualization skills (Hansen et al., 2015). There is also empirical evidence that students with a solid understanding of fractions are more likely to be successful in future mathematics coursework in middle school and beyond (Bailey et al., 2012; Booth & Newton, 2012; Siegler et al., 2012; Torbeyns et al., 2015).

Given the foundational nature of fractions understanding, we developed a learning progression for fractions that is meant to help students, teachers, and parents track student growth in this domain. During the development process, we reviewed existing LPs and the broader literature around students' understandings of fractions as well as the Common Core State Standards for Mathematics (CCSS-M) and the curricular focus and ordering of the fractions-related content in the *i-Ready* curriculum.

A review of pre-existing learning trajectories for fractions

It is evident from the order of clusters of standards in the CCSS-M that students should be exposed to increasingly more sophisticated uses of fractions as they move through grades three through six (CCSSI, 2010). The first cluster of fractions-related standards (3.NF.1-3) appears in third grade and has to do with understanding fractions as numbers, including equipartitioning to form a unit fraction, representing fractions on a number line, and understanding equivalent fractions. Next, in fourth grade, students are expected to extend their understanding of equivalence (4.NF.1-2), apply their existing understandings of operations to fractions with common denominators (4.NF.3-4), and learn the connections between fraction and decimal notation (4.NF.5-6). Fifth grade introduces additive operations on fractions with different denominators (5.NF.1-2) and multiplicative operations with fractions (5.NF.3-7). The bulk of fraction learning articulated in the CCSS-M ends in sixth grade, in which students refine their understandings of multiplication and division with fractions (6.NS.1) and learn about ratios and rates (6.RP.1-3). While the standards serve as a good curricular reference, and were designed with learning theories in mind, they have not been adopted by all states, and countries other than the United States may certainly have different ideas about curricular and developmental ordering.

Over the past decade, there have been four notable developments of LPs for fractions that complement the implicit LP implied by the CCSS-M: Arieli-Attali & Cayton-Hodges (2014), Wilkins & Norton (2018), Wright (2014), and Yulia et al. (2019). Other important related research can be found in Confrey's extensive work on equipartioning (e.g., Confrey, Maloney, & Corley, 2014) and proportions (Confrey et al., 2019) and Nizar's work on fraction division (Nizar et al., 2017). We do not focus on these as a motivation for the LP we introduce in the following section because they tend to focus on student ideas as they relate to fractions at a much finer grain size, whereas our goal was to propose an LP that takes a high-level view of fraction learning that spans multiple grades during elementary and early middle school.

Arieli-Attali and Cayton-Hodges (2014) developed an LP for Grades 3-5 mathematics as part of the Cognitively Based Assessment *of*, *for*, and *as* Learning (CBAL®) research initiative run by Educational Testing Service (ETS) in an attempt to expand upon the Grades 6-8 work they had already developed in response to the release of the CCSS-M (Bennett, 2010). They began by reviewing the available literature on cognitive science and mathematics education and consulting with internal and external panels of experts in mathematics education and cognitive psychology. They then collected evidence in support of their hypothesized LP using semistructured cognitive interviews in which 14 students in grades three through five and two teachers, all from the United States, were asked to complete a series of tasks that were designed to provide evidence of reasoning about rational numbers. Their LP begins with (1) understanding fractional units followed by (2) fractions as numbers. They then shift to (3) additive structure and (4) multiplicative structure.

Around the same time, Wright (2014) developed a learning progression based upon movement through four of Kieren's (1980, 1976) five ways to conceptualize fractions: as a rate or a ratio, as a quotient, as an operator, or as a measure. Wright used case studies of six 12- and 13-year-old students in New Zealand, analyzing their progress through each conceptualization using a combination of cognitive interviews and quantitative analysis in which Wright claimed to have identified patterns in students' responses to relevant test items (though he did not specify how this was accomplished). Wright used the collected evidence to develop a matrix in which students move through levels of sophistication within each of the four selected ways to conceptualize fractions. In Wright's LP, students move through different levels of each conceptualization starting with (1) unit forming, followed by (2) unit coordinating, (3) equivalence, and (4) comparison. As we will describe in the following section, we take the four fraction conceptualizations to be part of our LP, and we argue that the four levels Wright presents actually map onto those conceptualizations.

Wilkins and Norton (2018) published another fractions LP in which they proposed a hierarchy among the fraction schemes that Steffe and Olive (2010) put forward in their foundational book on children's development of fraction understandings. Wilkins and Norton developed a test with items designed to measure each of the schemes of interest (four items per scheme) and administered the test to 300 students in grades five through eight from the United States and China. Their first level indicates that students should be able to (1) produce fractions by disembedding parts of a divided whole. Next, students should be able to (2) iterate a unit fraction in order to recreate the whole and use this process to determine the size of that unit fraction. Their next level involves students (3) using equipartitioning and iteration to recreate a whole from a proper fraction. Finally, students should (4) recreate a whole from an improper fraction using the same method described in level (3). These authors did not cite either of the earlier LPs as a motivation for their work. As support for this LP, the authors cited earlier work presenting quantitative analyses of responses to their scheme-based fractions test. For example, Wilkins and Norton (2011) provided several analyses supporting an earlier version of the LP. The authors used the gamma statistic (Goodman & Kruskal, 1954) and visual inspection of 2x2 contingency tables to investigate the pairwise ordering of their schemes into LP levels[2].

[2] For each adjacent scheme (corresponding approximately to a level of their 2018 LP), the authors examined correlations among scheme-specific sum scores (over four dichotomous items). In replications such as Norton et al. (2018), the authors again computed gamma statistics to establish ordering among LP levels that closely resemble those in Wilkins and Norton (2018).

The final LP that we identified was produced by Yulia and colleagues (2019). They reviewed the extant literature to develop a hypothetical LP and associated tasks, which they checked with teachers participating in their study, then revised as needed. Validation of this LP consisted of classroom observations and task-focused cognitive interviews with 25 students in Indonesia. Their LP order is (1) fractions as a part-whole relationship, (2) determining fractional equivalence, (3) comparing fractional values, and (4) operating with fractions.

The four LPs described above each provide valuable conjectures regarding the paths that students tend to take when learning about fractions. However, each LP features either a level or developmental conceptualization of fractions that is unique to that study. For example, the concepts of ratios and rates are only included in Wright's (2014) work, and even then, they are treated as their own distinct construct through which students move rather than as an ordered level within a larger fractions construct. Furthermore, two of the LPs used qualitative methods with fairly small samples in local contexts (Arieli-Attali & Cayton-Hodges, 2014; Yulia et al., 2019). It is also an open question whether the order implied by these pre-existing LPs has been sufficiently validated. For example, the LP by Wright (2014) used a small sample of only six students in one classroom and was unclear in describing how the patterns in test scores were identified. The LP by Wilkins & Norton (2018) had a larger and more geographically-diverse sample, but the assessment they used only contained four items for each of the four schemes that they investigated.

A New Learning Progression for Understanding Fractions

The preexisting LPs described above tended to have fairly fine-grained levels and/or did not span the full range of concepts that students would be exposed to in curricula aligned to CCSS-M (CCSSI, 2010) or that are similar in nature to what can be found in the Ready curriculum. Similarly, the feedback that is presently provided about assessment results through the *i-Ready* system is based upon fine-grained claims about student's proficiency with individual standards. As our intention was to design a big-picture LP that could serve as a guiding framework for teachers across multiple grades when making instructional decisions, we used the larger fractions literature to help us synthesize and consolidate the levels of existing LPs and the *i-Ready* claims about what students should know and be able to do to form an LP with five broad levels that represent different ways that students can conceptualize fractions.

Kieren (1976, 1980) identified five main ways to conceptualize fractions: as representing a part-whole relationship, as a quotient, as a measurement/number, as an operator, and as a ratio or a rate. Each of these conceptualizations has been acknowledged as playing an important role in building students' understanding of how fractions can be used in various problem settings and have been cited in the development of other fractions LPs. To our knowledge, however, they have not been used as the central elements in forming such a progression without some amount of alteration. The five conceptualizations are often referred to as "sub-constructs" (e.g., Kieren, 1980; Norton & Boyce, 2013; Thompson & Saldanha, 2003; Wright, 2014), which implies that they are interrelated, but distinct, parts of some overall construct of fractions knowledge (Charalambous & Pitta-Pantazi, 2007), which could explain the lack of consensus in the literature about what exactly each conceptualization entails and the ways they interrelate. We use findings from the larger fractions literature to argue that these five conceptualizations do in fact form a coherent learning progression—that they can be viewed as ordered levels of a single construct and that the level of sophistication with which a student can understand fractions and use this understanding to solve mathematical problems increases from one conceptualization to the next. The levels of the expected progression through this construct (from lowest to highest) are described in the following subsections and are also summarized in Table A-1 in the Appendix.

Level 1: Fractions as parts of a whole

The conceptualization that has historically been used to introduce students to fractions is the *part-whole conception* in which a whole is partitioned into equal parts and some of those parts are mentally disembedded (Moss, 2005; Steffe & Olive, 2010). This can take the form of one object being split into multiple parts, as in an area model in which a pizza, for instance, is cut into equal slices. Alternatively, it may consist of a set of objects, some fraction of which are specified (e.g., if there are three people sitting at a table set for four, then three-fourths of the seats are occupied). This conceptualization is representative of the first level in three of the four existing LPs that we identified (Arieli-Attali & Cayton-Hodges, 2014; Wilkins & Norton, 2018; Yulia et al., 2019). Figure 1 is an example of a set model. In this item, there are five smiley

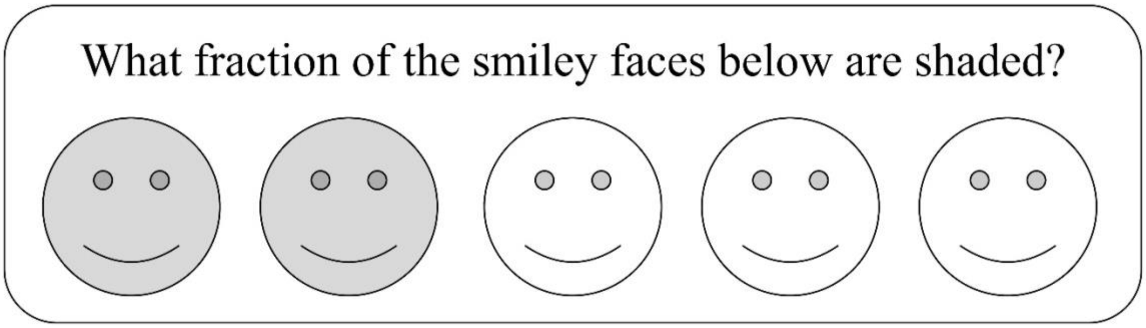

Figure 1. Example of a part-whole item

faces, and the students are asked to identify which fraction of the whole set is shaded. In this case, two of the five faces are shaded, so a student with a part-whole understanding of fractions would give $\frac{2}{5}$ as their answer. This example represents one out of a family of possible items that could be created using different shapes shaded in different combinations to represent different fractional values.

This conceptualization is the first step in our learning progression, as it has typically been considered to be the foundation for the subsequent conceptualizations we introduce (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007). Many in the mathematics education community have since come to see the part-whole view of fractions as a subset of the ratio conceptualization (e.g., Wright, 2014), and the CCSS-M (CCSSI, 2010) does not explicitly include standards relating to factions as parts of a whole, with the focus shifting to equipartioning and iteration (Confrey, Maloney, & Corley, 2014). We chose to include this level in our LP, however, because this is often the beginning of students' informal understandings of fractions (Mack, 1993). This conception also tends to be the easiest for students to grasp since it is the closest to the whole number context that they are accustomed to working in (e.g., students begin developing number sense by counting wholes) (Moss, 2005). Students working within a part-whole framework may maintain their natural number context and apply the counting strategies that have served them well in other situations (Post et al., 1993).

The part-whole conception of fractions is insufficient on its own for students to develop a complete understanding of fractions and the ways that they may be used (Hackenberg & Lee, 2015; Post et al., 1993; Steffe & Olive, 2010). In particular, students who can identify both the total and the specified number of pieces in a fraction may not fully comprehend that all of the pieces must be equal or that the all of the original whole must be used when creating fractional pieces (P. H. Wilson et al., 2012). These errors are demonstrated in Figure 2, which shows responses to an assessment task in which incorrect attempts have been made at creating four equal groups from eight counters. Additionally, although students with this level of understanding may be able to compare fractions with the same denominator because they understand, for example, that $\frac{4}{5}$ means that there are more pieces of the same whole than there are in $\frac{2}{5}$, they may not be able to compare fractions with different denominators because they may not yet realize that the magnitude of the denominator is related to the size of each piece (Steffe & Olive, 2010; Wilson et al., 2012).

The natural number context associated with the part-whole conceptualization may inhibit students' abilities to see fractions as numbers in and of themselves. Students who only see fractions as representing a number of parts may, for example, place $\frac{3}{4}$ at three out of four on a number line rather than three-fourths of the way between zero and one (Kerslake, 1986). Additionally, they may have difficulty recognizing that two fractions can have equivalent numerical values even if they appear different on the surface, which may result in errors such as

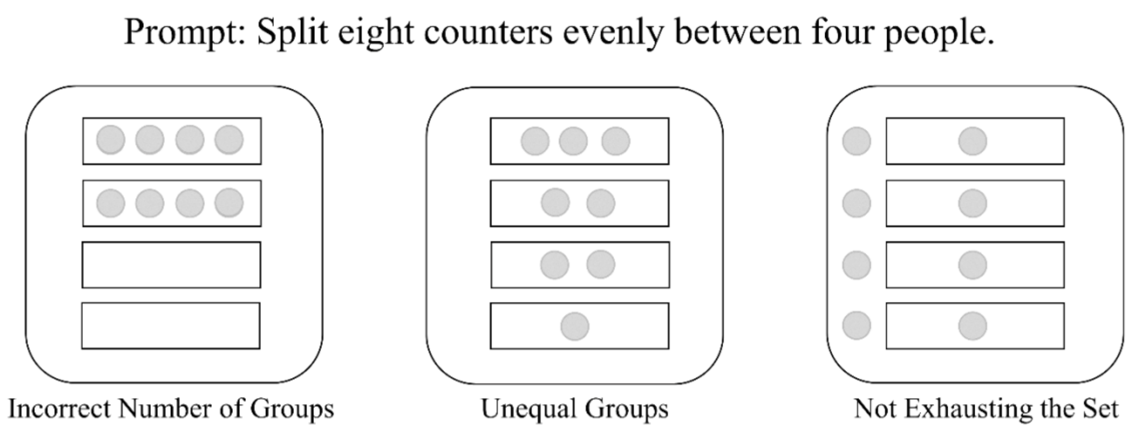

Figure 2. Examples of inaccurate attempts at fair sharing

claiming that $\frac{10}{15} > \frac{2}{3}$ because $10 > 2$ and $15 > 3$ (Hart et al., 1981). These applications of natural-number logic may lead to difficulties adding and subtracting fractions, even those with the same denominator, if students overgeneralize traditional operation procedures (e.g., adding across both the numerators and the denominators) (Newton, 2008; Post et al., 1993; Wu, 2001). In order to move past these obstacles, students must be able to view fractions as representing an equipartitioning process in which a whole is divided into equal parts.

Level 2: Fractions as quotients

The next conceptualization that students need to develop is the often "forgotten notion" (Clarke, 2011) that a fraction is a quotient such that $\frac{a}{b}$ represents the division of a by b. The key to this stage of understanding is the ability to engage in equipartitioning and the creation of "fair shares" (Confrey, Maloney, & Corley, 2014; Confrey, Maloney, Nguyen, et al., 2014; Wilson et al., 2012). In fair sharing, the number of individual units to be shared (a) is divided by the number of shares that are needed (b) such that the size of each share is $\frac{a}{b}$ (a-$b$ths) of one unit (Empson et al., 2006). The foundational example of this process is taking one unit and dividing it into b parts such that each share is $\frac{1}{b}$ (1-bth) of the original. This process may then be expanded to include compound units (Steffe & Olive, 2010). This conceptualization is represented by level (2) in the LP produced by Wilkins and Norton (2018) and by levels (1) and (2) in Wright's (2014) LP.

The concept of fair sharing is exemplified by the item shown in Figure 3. In this item, three friends are equally sharing two chocolate bars. A student with an understanding of the quotient conceptualization will be able to fairly share these chocolate bars so that each person in the problem receives two-thirds of a bar. This partitioning is often accomplished by either

Three friends are sharing two chocolate bars. How could they split the bars so that each friend gets the same amount of chocolate? How many bars worth of chocolate does each friend get?

Figure 3. Example of a quotient item

breaking each bar into thirds and "dealing" out pieces until they have all be used (everyone gets one-third of the first bar and then one-third of the second bar) or by dividing the number of bars (two) by the number of people (three) to arrive at the answer that each person receives two-thirds of a bar (Wilson et al., 2012). This is only one example of a problem that requires the quotient understanding of fractions, and the number and the type of objects to be shared and the number of shares needed can be varied to create new items. This particular example requires students to take fractional parts of multiple objects that constitute a composite unit (Steffe & Olive, 2010), which will be more difficult than if they were to share just one item or if the total number of objects were a multiple of the number of shares needed, as in the coin-sharing problem described in the following paragraph.

The three key features of equipartitioning are creating the correct number of groups, creating groups of the same size, and exhausting the original whole (P. H. Wilson et al., 2012). In developing this ability, students become familiar with the idea that each "share" is one equal piece of the whole, such that if 24 coins were shared among three people, each person would receive eight coins, and they would each have $\frac{1}{3}$ of the total number of coins (P. H. Wilson et al., 2012). As Steffe & Olive (1993) point out, students may also begin to realize that a unit fraction can be iterated to recreate the whole, such that if, for example, someone joined together five equally-size pieces, then each piece is $\frac{1}{5}$ of the resultant whole. The examples Steffe & Olive provide, respectively, demonstrate Kieren's (1993) distinction between what he calls *partitive division* and *quotative division*. While they are both examples of equipartitioning, in partitive division, the focus is on the size of each share, whereas in quotative division, the interest is in the number of shares of a given size. These examples also demonstrate students' abilities to *unitize*, or to conceive of a group of objects (or a partition) as a single unit. In order to understand the connection between these sharing tasks and fractions, students need to view the 24 coins as both 24 individual coins, and one complete set that can be broken up into equally-sized groups.

In creating fair shares between varying numbers of groups, students also come to understand that the more groups that are made from the same whole, the smaller each individual group will be (Cramer et al., 2008; Wilson et al., 2012). This may allow them to begin to make sense of comparisons of fractions with the same numerator and different denominators, because they come to recognize that if there are the same number of pieces, but each piece is relatively smaller, then the overall amount will be less. Students with these understandings, however, may have not yet fully internalized that a fraction represents not only two related whole number values (the numerator and the denominator) but also a numerical value that can be placed at a unique location on the number line (Van Hoof et al., 2018).

Level 3: Fractions as measurements

The third level in our learning progression aligns with the conception of fractions as measurements, which focuses on the magnitude of a fractional value and is closely associated with additive reasoning (Behr et al., 1983). The ability to view a fraction as representing one value rather than as simply a comparison between two values, or even as two unrelated values (see Hecht & Vagi, 2010; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004), allows students to locate a fractional value correctly on a number line (Kerslake, 1986; Siegler et al., 2011; Siegler & Lortie-Forgues, 2015). This is closely associated with the decimal representation of fractions, which requires understanding fractions as quotients because carrying out the division of a by b, either by hand or with a calculator, will produce a decimal that is equivalent to $\frac{a}{b}$. The understanding of unit fractions that students develop as part of the quotient interpretation help them to understand that the relative size of one quantity compared to another is determined by the number of times the first fits into the second (Wilkins & Norton, 2018). They may now be able to use a unit fraction to create a composite fraction beyond simply recreating the whole (Marshall, 1993), such as in Figure 4. In this item, the student needs to recognize that the distance between zero and $\frac{1}{4}$ is iterated three times to produce the desired fraction $\frac{3}{4}$. Other number line items may ask for different fractional values or provide labels for different values. Additionally, the “fair sharing” concept developed in the quotient interpretation may help students place fractions on number lines even if they are not provided with helpful benchmark values and need to create their own (Moss & Case, 1999). If a student, for example, were asked to locate $\frac{3}{8}$ on the number line shown in Figure 4, they would need to split the areas between the one-fourth markings in half to create eights.

A measurement conception of fractions is most important when students need to compare relative magnitudes of fractions with unlike numerators and denominators, usually in a context where they need to determine whether one fractional value is greater than, less than, or equal to another (Steffe & Olive, 2010). If a student does not see a fraction as representing a specific, orderable value, then they will have difficulty making sense of a request to make these

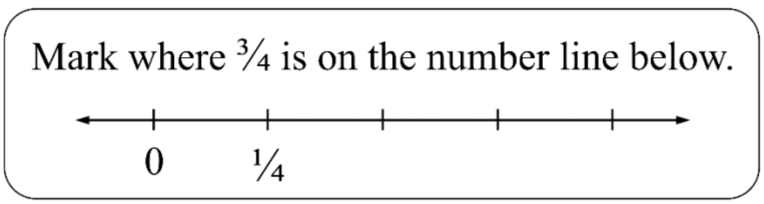

Figure 4. Example of a measurement item

comparisons. They will also have trouble realizing that fractions can be split such that there are infinitely many fractional values that sit between any two fractions on a number line (Lamon, 1999). Furthermore, the ability to recognize and to generate equivalent fractions is an essential prerequisite for adding and subtracting fractions with unlike denominators (Cramer et al., 2008; Wright, 2014). Because the measurement conception covers a fair amount of ground, it is represented by multiple levels in each of the LPs we identified: levels (2) and (3) in Arieli-Attali and Cayton-Hodges (2014), levels (3) and (4) in Wilkins and Norton (2018), Levels (3) and (4) in Wright (2014), and levels (2) and (3) in Yulia and colleagues (2019).

Students need an accurate mental number line (Griffin, 2004; Hamdan & Gunderson, 2017) and a sense of how quantities combine to form new quantities (Jordan et al., 2007) in order to develop fraction addition skills (Keijzer & Terwel, 2003). Having a sense of magnitude and additive properties allows students to reject implausible answers and the procedures that they used to obtain them (Booth & Siegler, 2008; Hiebert & Lefevre, 1986; Siegler et al., 2011). Understanding magnitude and number lines is also essential for comprehending the meaning of improper fractions. If a student were to only make use of a part-whole conception, it would make no sense to have more pieces than there are meant to be in the whole unit (Stafylidou & Vosniadou, 2004). The ability to iterate units, however, can be extended so that students add more unit fractions than there are in the original whole. An example of this may be to extend the number line in Figure 4 and to ask the student to locate $\frac{5}{4}$ by iterating the $\frac{1}{4}$ distance five times. Students who have internalized the part-whole conception of fractions may be able to apply a procedure in which they successfully multiply a fraction by another fraction that is equivalent to one in order to manufacture common denominators for the sake of addition and subtraction. Without a measurement conception of fractions, however, they may have not yet developed the multiplicative understandings that are required for proportional reasoning.

Level 4: Fractions as ratios and rates

A ratio conception of fractions is generally difficult to place in a learning progression because there are several ways that ratios can be interpreted (Clark et al., 2003), and they incorporate aspects of many of the other interpretations (Wright, 2014). The foundational understanding of a ratio is a comparison of two quantities, whether it be part-whole or part-part. This means that ratios are often lumped together with the part-whole conceptualization because they also allow for the use of natural number reasoning, as Kieren (1980) points out in noting that these two conceptualizations are strongly connected. Moss and Case (1999) even suggested that students begin learning about rational numbers by way of percentages rather than the traditional fraction notation, bringing the ratio interpretation to the very start of a learning progression. Other researchers (e.g., Clark et al., 2003; Lovell & Butterworth, 1966; Noelting, 1980; Sowder et al., 1998), however, have argued that ratios are far more than set-set comparisons; they are the basis for multiplicative and proportional reasoning, which are essential precursors to understanding algebraic concepts such as linear relationships (Clark et al., 2003).

The main difference between the part-whole and the ratio interpretation of fractions is the *covariance-invariance property*, which states that the two quantities involved in a ratio vary together such that if both values are multiplied by the same nonzero value, the value of the ratio does not change (Charalambous & Pitta-Pantazi, 2007). This concept clearly builds upon the equivalent-fractions aspect of the measurement conceptualization and can now be applied to ratios in other formats, such as with $1:3 = (1 \times 4):(3 \times 4) = 4:12$. Although some researchers have placed equivalence in the realm of ratios rather than measurement for this reason (e.g., Behr et al., 1983), we argue that equivalence is more closely connected to magnitude concepts and is a prerequisite for understanding the unit rates and proportions involved in the ratio interpretation. None of the LPs that we identified in the research literature had a level for ratios and rates, though Wright (2014) did include ratios and rates as a sub-construct of fractions through which students would gain increasingly sophisticated understanding.

Ratios can be also thought of as *rates*, which are comparisons between two values with different units (Lamon, 1999), particularly when used in a contextualized problem. This is where part-part comparisons come in. Rather than comparing a certain amount of a whole unit (e.g., three out of eight slices of a whole pizza), the part-part comparison allows us to specify how many of one type of unit there is in relation to another type of unit. If a runner completes a 10-kilometer race in 23 minutes, then the ratio of distance to time can be written as $10 \text{ km} : 23 \text{ min}$, as $\frac{10 \text{ km}}{23 \text{ min}}$, or as “10 km to 23 min”. The fraction notation allows for familiar manipulations that emphasize the unit rate interpretation of ratios. In this example, if the numerator and the denominator are both divided by 23, we can see that the new ratio is $\frac{\frac{10}{23}\text{km}}{1 \text{ min}}$, which indicates that the runner covered an average of 10-23rds of a kilometer each minute. This understanding builds upon the quotient interpretation, as students must realize that there is “fair sharing” occurring (Confrey, Maloney, Nguyen, et al., 2014), and is a stepping stone to understanding fractions as operators. Additionally, rates may be seen as reflectively abstracted ratios that emphasize a constant multiplicative comparison (Thompson, 1994). This consistency is needed for students to comprehend the scaling function of fractions when they are used as operators.

Another example is shown in Figure 5, which requires students to consider the relative amounts of orange and pineapple juice in a punch mixture. In this case, Jamie wants her punch to taste more like oranges. This can be accomplished by increasing the ratio of orange juice to

Jamie usually makes punch by mixing 2 cups of orange juice with 3 cups of pineapple juice. This time, she wants it to have a stronger orange flavor. Which of the following ratios of orange juice to pineapple juice could she use?

A) 4:6 B) 2:4 C) 4:3

Figure 5. Example of a ratio item

pineapple juice. The options presented to the students represent three ways to adjust the flavor profile. Option A will yield a punch that tastes the same as the original recipe because both the number of cups of orange juice and the number of cups of pineapple juice have been multiplied by two. The covariance-invariance property states that this results in an equivalent ratio. Option B will result in a punch that has a relatively weaker orange taste than the original because the number of cups of orange juice has stayed the same while more cups of pineapple juice have been added. The final option (C) is the correct one. This option has more cups of orange juice than pineapple juice, making the punch predominantly orange-based. The prompt only said that Jamie wanted the orange taste to be stronger than before, however, so other options that result in relatively more orange juice but that are still less than or equal to half of the mixture could be substituted into the item to increase its difficulty. One such option could be 3:4, which would make the mixture 43% orange juice as opposed to 40% in the original recipe.

Since the part-whole, quotient, and measurement interpretations are all needed to fully make sense of fractions as ratios, we have decided to place them fourth in our progression. Although ratios are not introduced in the CCSS-M until the sixth grade, (CCSSI, 2010), after students have been exposed to concepts surrounding fraction multiplication, we argue that the proportional reasoning associated with ratios and rates are essential prerequisites for students to actually understand what that multiplication means. This is evidenced by the difficulty that students often have deciding whether to multiply or to divide by a rational number in a proportional situation (Bell et al., 1989). As Clark and colleagues (2003) stated:

> most of the time, we don’t think of an operator fraction…as a ratio, even though it is obviously related to a ratio in that the comparison of the resulting length to the starting

length is a consistent multiplicative relationship for a particular stretcher or shrinker. (pp. 306-7)

We, therefore, argue that students must develop an understanding of ratios before they can successfully view fractions as operators and use them to find a proportional value.

Level 5: Fractions as operators

The *operator* interpretation of fractions uses ratios as multipliers to find a proportional amount of an original value (Behr et al., 1993; Behr et al., 1983; Kieren, 1980; Lamon, 1999). The fraction $\frac{a}{b}$, acts as a function that takes an initial value, x_1, and returns a value, x_2, such that $\frac{a}{b} = \frac{x_2}{x_1}$ (Lamon, 1999), which emphasizes the proportional reasoning involved. There are three main ways that fractions can be used as operators. First, a fraction may be interpreted as two consecutive actions: multiplying by the numerator and dividing by the denominator, though not necessarily in that order (Behr et al., 1993). This view is most closely related to the quotient and magnitude interpretations because multiplying a value, c, by $\frac{a}{b}$ can be seen as dividing c by b to find a unit length and then iterating that unit a times by multiplying. This view allows a student to continue using the mental framework of multiplication as repeated addition (Son & Senk, 2010), which has zero as the identity element. The other two options were described by Behr and colleagues (1993) as the *duplicator/partition* and the *stretcher/shrinker* interpretations. In the duplicator/partition view, multiplying an operand by a fractional value results in a new value with a different number of units of the original size. They give the example that multiplying by $\frac{3}{4}$ would replace four units with three equally-sized units. In the stretcher/shrinker view, the number of units remains intact while the size varies, such that when multiplying by $\frac{3}{4}$, for example, if you began with a number of units of size four, you would end up with the same number of units of size three. The operators level of our LP is represented in two of the four existing LPs that we identified (level (4) in both Arieli-Attali & Cayton-Hodges (2014) and Yulia et al. (2019)) and is one of the subconstructs that students are expected to move through in Wright's (2014) LP.

One of the key understandings that students need to master in this view is that their old rules of thumb for shifts in magnitude do not apply to fractions in the ways they do with whole numbers. Students, for example, tend to hold on to their belief that multiplying a number by another number will always result in a value that is larger than the original multiplicand and that dividing always results in a number smaller than the dividend (Alghazo & Alghazo, 2017; Siegler & Lortie-Forgues, 2015). This heuristic serves students well when they need to estimate the magnitude of an operation performed with whole numbers, but it may result in an incorrect estimation when fractional values are involved. This is because scaling a value with multiplication means creating a proportional value, where the base value is a scale-factor of one, which requires a significant intellectual shift away from the base value of zero that is used in additive reasoning schemes. To use Behr and colleagues' terminology, the original value is "shrunk" if it is multiplied by a fraction less than one and is "stretched" if it is multiplied by an improper fraction, which is larger than one.

At the end of Britney's birthday party, ¼ of the cake was left over. The next day, Britney and two friends evenly shared the left-over cake. What fraction of the original cake did each person get?

Figure 6. Example of an operator item

Misunderstanding the *direction of effects* of fraction operations indicates that students may have simply memorized algorithms and do not yet fully understand what fraction multiplication and division mean conceptually. This is likely why students who have not mastered the operator conceptualization of fractions tend to have difficulty selecting procedures when solving proportion problems that are not simply presented as a symbolic missing value, such as $\frac{2}{7} = \frac{6}{x}$. Figure 6 presents one such problem. In this item, three friends are sharing the left-over quarter of the cake such that they each get $\frac{1}{4} * \frac{1}{3} = \frac{1}{12}$ of the original whole. This item requires students to recognize that they need to divide the left-over quarter of the cake into three equal pieces, which is the same as multiplying by $\frac{1}{3}$. Similar items may change the context and the proportions involved. A more difficult item may change the information provided such that the students are told that each person got $\frac{1}{12}$ of the total cake and are asked how many friends shared the left-overs.

Connections to Pre-existing Learning Progressions

Although the pre-existing LPs that we identified earlier have slightly different grain sizes and scopes, the levels, once mapped onto the levels we have just defined, do follow the same ordering that we have put forth. Table 1 presents the LP levels from each source and the fractions-related standards from the CCSS-M mapped onto our LP levels. The only ordering discrepancies that we see are in the CCSS-M standards. Specifically, students are exposed to ratios and rates for the first time in sixth grade, but they encounter some operator concepts in fifth grade. Additionally, there is a quotient-level standard in fifth grade although the students began learning measurement concepts in fourth grade.

While all the existing LPs have levels that fit well within our defined levels, they are either missing one or more of the conceptualizations Kieren identified or are too focused on individual skills for our purposes. The LP proposed by Arieli-Attali and Cayton-Hodges (2014) does have conceptually-based levels that cover the big ideas of a unit, fractions as numbers, and the additive and multiplicative structure of fractions, but it does not include levels that address the Quotient or Ratios understandings. Yulia et al.'s (2019) LP has similar levels. Wright's (2014) LP also has conceptual levels (unit forming, unit coordination, equivalence, and comparison), but these levels only cover the Quotient and Measurement conceptualizations of fractions. The last LP (Wilkins & Norton, 2018) focus heavily on individual skills rather than on big-picture ideas (e.g., "Reproduce a whole by partitioning and iterating a proper fraction").

Table 1. Mapping our LP onto existing LPs

LP Level	CCSS-M (2010)	A-A & C-H (2014)	Wright (2014)	W & N (2018)	Yulia et al. (2019)
Operator	6.NS.1 5.NF.4-7	(4) Multiplicative structure			(4) Operations of fractions
Ratio	6.RP.1-3				
Measurement	5.NF.1-2 4.NF.1-6 3.NF.2-3	(3) Additive structure (2) Fraction as number	(4) Comparison (3) Equivalence	(4) Reproduce a whole by partitioning and iterating an improper fraction (3) Reproduce a whole by partitioning and iterating a proper fraction	(3) Comparing fractions (2) Determining fractions with the same values
Quotient	5.NF.3 3.NF.1		(2) Unit coordination (1) Unit forming	(2) Use iteration to determine the fractional size of a unit fraction	
Part-Whole		(1) Fractional unit		(1) Produce any proper fraction by disembedding parts from the whole	(1) Fraction as a part-whole relationship

Situating our learning progression

The learning progression that we have defined uses a combination of the "cognitive levels" and "disciplinary logic and curricular coherence" approaches to LP design, as described by Lobato and Walters (2017). Our approach is cognitive in the sense that we identify a series of increasingly sophisticated conceptions and in the conjecture that students develop these more sophisticated conceptions about fractions over time. This type of LP typically uses cross-sectional data across several grade levels to help validate this conjecture, which is also the approach we take in our use of the *i-Ready Diagnostic* assessment data. Our approach is also based upon disciplinary logic and curricular coherence in that it was "*informed by research* versus being the *product of research*" (emphasis in original, Lobato & Walters, 2017, p. 87). The five conceptualizations of fractions described in the proceeding sections reflect current understandings of how students learn about fractions over the span of several grades and are highly aligned with the curricular ordering of the CCSS-M.

In addition to detailing approaches to the development of LPs, Lobato and Walters also describe common methods of validation. We use a construct map approach (M. Wilson, 2004; see Table A-1 in the Appendix) in which we attempt to use the results from an item response theory model—specifically, the Rasch model (Rasch, 1960)—to support claims about which types of items students are more or less likely to answer correctly. In this approach, we investigate the extent to which students with stronger overall mathematics performance on the *i-Ready Diagnostic* assessment will be more likely to correctly answer items that invoke a more advanced conceptualization of fractions. In our LP, the five levels that we have identified are ordered to reflect increasingly sophisticated levels of conceptual understanding, which should be evident when the difficulty estimates of items written to correspond to each level are compared.

We model a student's conceptual understanding of fractions as a location on a latent continuum that is assumed to remain relatively stable as a student is completing assessment items on any given occasion, and argue that a person's location on this continuum can be inferred (with some amount of measurement error) through an analysis of their pattern of item responses.

Validation Approach

In order to empirically test our hypothesized learning progression, we examined the relationship between our ordering of the levels of fractional knowledge described in the previous sections and the difficulty estimates of the *i-Ready Diagnostic* items associated with each level. Two key features of the *i-Ready Diagnostic* assessment system allowed us to make these comparisons: the vertical scale on which the items written for students at different grade levels have been calibrated and the "claims" that Curriculum Associates uses to designate the content each item is meant to assess.

The vertical scale

When different items are administered to students in different grades, their difficulty cannot be compared directly with proportions of correct responses, or even with the result of calibrating separate measurement models, because the groups of students being compared are at different levels of development in mathematics. Yet, in order to empirically validate this learning progression, which spans several grade levels, we needed to determine the difficulty of items that were developed for students in different grades in an absolute sense. The *i-Ready* vertical scale facilitates such comparisons. The *i-Ready Diagnostic* uses Item Response Theory to determine student scores and item difficulties such that a student with a scale score of 350, for example, would have a 50% chance of giving a correct answer to an item with a difficulty rating of 350. More specifically, Curriculum Associates uses a vertical scale in which scores from assessments administered to non-equivalent groups of examinees, such as students in different grades, are placed onto a single scale so that, for example, a third grader and a fourth grader with the same numeric score on the *i-Ready Diagnostic* are interpreted as having demonstrated the same level of absolute proficiency despite having taken different diagnostic test items (for an overview of vertical scaling, see Tong & Kolen, 2010). For the *i-Ready* tests, this was done by administering common items across grade levels, and then using the information about student performance on these items to establish a "vertical" scale across grades. Comparisons based upon these vertically-scaled item difficulties form the basis of the empirical support outlined below. Without vertical scaling, it would not be possible to compare items that appear in successive grades by the amount that they differ in their difficulty.

Curricular ordering and associated item difficulties

The instructional sequence plays a critical role in any learning progression (Confrey, Maloney, & Corley, 2014). In mathematics, ideas tend to build upon themselves, and fractions concepts are no exception. While this does not necessarily mean that students must have completely mastered all previous concepts and procedures before moving on to more complex topics, it does imply that later-learned topics are likely to be more sophisticated and more complex than are those that are learned earlier (Confrey, Maloney, & Corley, 2014). Tasks that reflect these later concepts are, therefore, likely to be more difficult. In order to confirm this empirically, we began by inspecting the relationship between the grade-level ordering of the fractions-related lessons in the Ready curriculum, which is highly aligned with the CCSS-M, and the estimated difficulties of the *i-Ready Diagnostic* items that assess the content covered in those lessons.

Content experts at Curriculum Associates coded the items in the *i-Ready Diagnostic* assessment system according to the specific content knowledge and skills that students are presumed to require to accurately complete the problem. Curriculum Associates refers to these codes as anchor claims. We identified 406 fractions items associated with 107 anchor claims. The first and third authors independently coded each anchor claim based on which of the five fraction conceptualizations from our LP would be most important to understand in order to correctly answer an item in that group. Initial agreement was very high, with matches on 101 of the 107 (94%) anchor claims, and we discussed the six mis-matched anchor claims until we agreed upon a code for each. We had access to the actual bank of items used in the *i-Ready Diagnostic* and confirmed that the items faithfully represented the associated anchor claim.

If our hypothesized learning progression holds, then we would expect to see students being exposed to the less complex conceptualizations in the curriculum associated with lower grades and moving through the levels in the order we have specified as they advance into upper grades. We examined the lessons in the Ready curriculum and recorded the lesson in which students first encountered each anchor claim (see Table 2 for illustrative examples). We then labeled the 32

Table 2. Illustrative anchor claims and associated Ready Lessons[3]

LP Level	i-Ready Claim	Associated Ready Lesson	# of Items
5. Operator	Student represents the division of two fractions as a division expression or equation when a verbal description or model is provided, or as a multiplication expression or equation when a division expression or equation is provided.	6.7 Divide with Fractions	5
	Student multiplies or represents the multiplication of a fraction less than 1 by a fraction less than 1 or a whole number, presented without a real-world context and/or with the aid of a visual model.	5.13 Understand Products of Fractions	14
Ratio	Student expresses a ratio presented in the form a:b, a/b, described verbally, or represented on a visual model in a different form	6.1 Ratios	7
Measurement	Student adds and subtracts fractions and mixed numbers with like denominators) without composing or decomposing wholes or uses visual models to represent these problems.	4.16 Add and Subtract Fractions	11
	Student recognizes fractions equivalent to a named fraction, using a visual model showing two or more equivalent wholes partitioned into different numbers of parts (e.g., area models, fraction strips, labeled number lines).	3.16 Understand Equivalent Fractions	8
Quotient	Student expresses a fraction, a/b, as a division expression, a ÷ b, or a division expression, a ÷ b, as a fraction, a/b, where a and b are represented symbolically, or represented numerically with b > a.	5.12 Fractions as Division	4
Part-Whole	Student names part of a whole using a fraction (denominator of 2, 3, or 4). (All models show equal parts. Area is not mentioned for unit fractions.)	3.14 Understand What a Fraction Is	12

[3] The claims presented in this table are a sample of the claims associated with each of our LP levels. The numerical code in the Associated Ready Lesson column is in the format (grade).(lesson number), such that code 6.7 indicates the seventh mathematics lesson in the Ready curriculum for grade six.

lessons that we identified in chronological order such that the very first fractions-related lesson in second grade was labeled as lesson #1 and the last fractions lesson that a student would experience in sixth grade was lesson #32. We then analyzed the association between the lesson ordering and the LP levels.

Results

We find evidence that our learning progression is supported by the ordering of the fraction-related lessons in the Ready curriculum and by the difficulties of the items associated with each LP level. The scatterplot in Figure 7 shows the LP levels associated with each anchor claim in the order in which that content appears in the Ready lessons, with the points scaled to indicate the number of anchor claims covered in a given lesson. It is clear that the first fraction lessons to appear in the Ready curriculum were associated with the Part-Whole conceptualization, followed

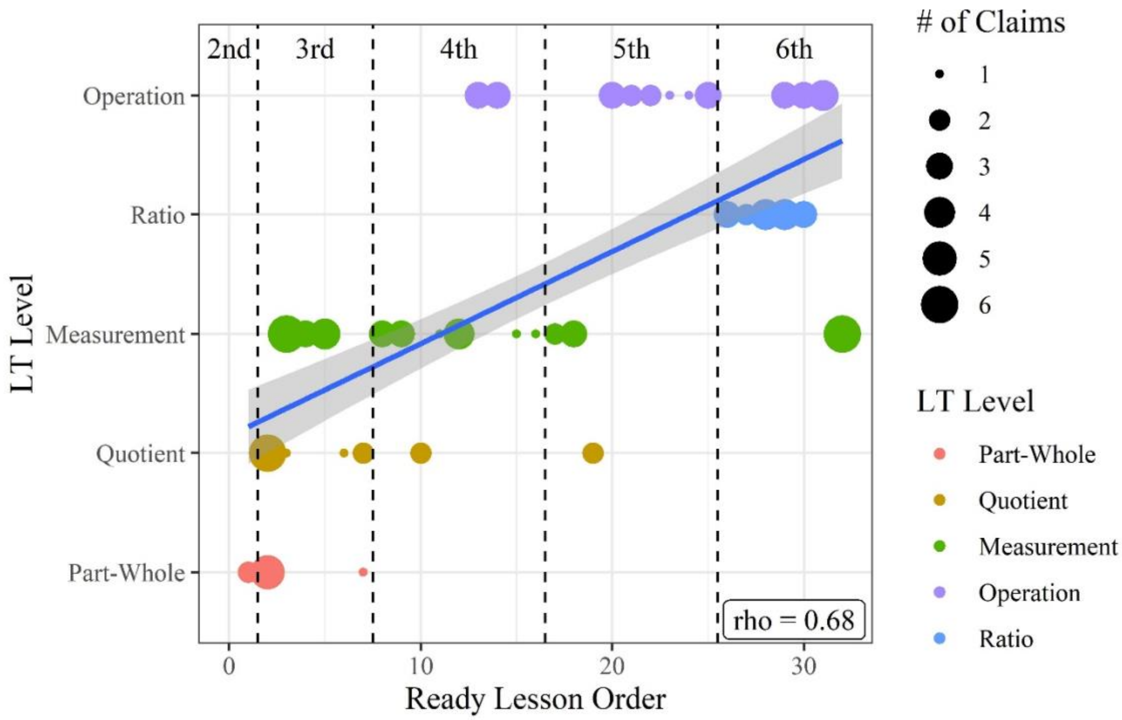

Figure 7. Anchor claim LP levels by the order in which they appear in the Ready lessons

by the Quotient, Measurement, Operator, and Ratio conceptualizations. The conceptual focus also shifts between the grades such that second grade is Part-Whole only and third grade introduces Quotient and Measurement concepts. Fourth grade is mostly Measurement and introduces Operator ideas. By fifth grade, students are mostly learning Operator concepts, and Ratios are introduced in sixth grade, with Operator ideas still strongly featured. There is an upward trend in the regression line shown in Figure 7, which was generated by coding the LP levels from 1-5 (part-whole = 1 and operator = 5). Because of the ordinal nature of our analysis, we used Spearman's ρ to examine the association between the LP levels and the lesson ordering using this numerical coding scheme. The value of ρ was 0.68. This indicates a fairly strong relationship between the levels of fraction understanding that we have defined and the curricular ordering of the Ready curriculum. The correspondence would likely be even stronger if Ratios were introduced sooner, as our evidence points to Ratio items being easier than Operation items in general despite being introduced later.

We can see in Table 3 and Figure 8 that the difficulties of the *i-Ready* assessment items associated with anchor claims tend to increase when the conceptualizations are placed in the order in which they appear in our LP, with a Spearman's ρ value of 0.55. The difficulties of the items associated with Ratios tend to be much more dispersed relative to those of the adjacent Measurement and Operator conceptualizations. It is unsurprising that Ratios would have a wider

Table 3. Descriptive statistics for the items associated with each LP level

	Min	Q1	Median	Q3	Max	Mean	SD
Part-Whole	328.0	398.3	419.5	433.5	489.0	418.0	33.3
Quotient	379.0	431.3	449.5	468.0	531.0	449.3	34.1
Measurement	415.0	457.0	467.0	481.0	524.0	469.1	21.6
Ratio	396.0	452.8	479.0	507.0	566.0	476.3	41.0
Operator	394.0	482.0	498.0	510.0	576.0	493.9	30.8

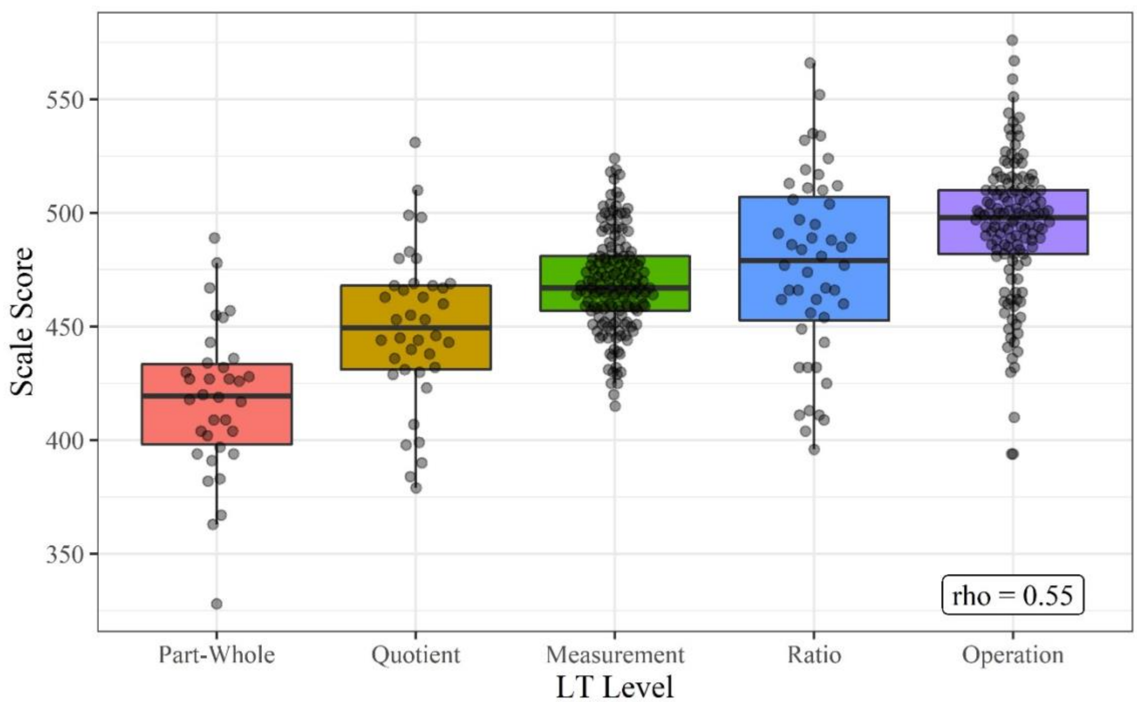

Figure 8. Item difficulties by LP level

range of difficulties because of the Part-Whole aspects of this conceptualization, which are more similar to the lower-level conceptualizations. It is when the items move into the part-part rate ideas that the difficulties increase. This finding also aligns with previous research that has shown that students tend to begin building informal understandings of ratios and proportion fairly early, but it often takes quite a while for the formal ideas to develop (Bruner et al., 1966).

We ran an ANOVA to compare the means of the items in the LP level groups, and found that the differences in means were statistically significant [$F(4, 401) = 52.27$, $p < .001$]. We then used Tukey's HSD to examine the differences in item means between each pairwise combination of LP levels (Table 4). While the means of the Measurement and Ratio items are too close to establish a statistically-significant difference, all other pairs of LP levels have statistically distinguishable differences between their item means. The very low-difficulty Ratio items

Table 4. Pairwise differences in the item means by LP level

	Part-Whole	Quotient	Measurement	Ratio	Operation
Part-Whole	0	31.3***	51.1***	58.4***	75.9***
Quotient		0	19.8**	27***	44.6***
Measurement			0	7.2	24.8***
Ratio				0	17.6**
Operation					0

** $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$*

associated with Part-Whole understandings likely decreased the mean of the Ratio group of items enough to make it indistinguishable from the Measurement mean. Given that the Ratio level spans such a wide range of difficulties and is not distinguishable from other levels, we decided to remove it from the LP.

Using a Learning Progression for Formative Assessment

Our analyses support the existence of a five-level, course-grained learning progression for fractions that, except for the Ratios conceptualization, aligns with the curricular ordering found in the CCSS-M (CCSSI, 2010) and in the *Ready* and the *i-Ready* curricular programs. This suggests that it is possible to use evidence from the large-scale diagnostic assessments that teachers are often required to administer to develop an LP that is consistent with existing LPs in the research literature that were developed using more locally developed assessments.

LPs have the potential to serve as powerful assessment tools, as they may be used formatively by teachers during their informal interactions with students and in more formal classroom assessments that they may create (Clements et al., 2011; Clements & Sarama, 2008; Edgington, 2014; Furtak et al., 2014). Our intention in using a large grain-size for the LP was to allow teachers to more easily internalize a handful of levels that could serve as guideposts for monitoring their students' progress. Teachers could use the levels of our LP to help them select tasks to use during instruction or on informal assessments, as we have provided sample items for each LP level and described how item features can be altered to increase or decrease the difficulty. This may involve beginning with a more straightforward task and altering task features as a lesson progresses to make it increasingly difficult. By using this systematic approach, teachers may be able to match what they hear their students saying and what they see in students' written work to the "is able to" and "common errors" sections of Table A-1 in order to identify which levels of understanding their students currently hold and which misconceptions may be preventing them from moving to the next LP level. These identifications may help teachers target future instruction more effectively.

There are, however, some conditions that must be met for LPs to be used effectively. First, the LP must accurately represent the typical ordering that students experience when learning a new concept. In this study we have provided some evidence that this tends to be the case for the fractions LP that we have proposed. Second, teachers must be open to learning about the LP and how to use it in their classrooms. LPs represent a shift in thinking away from discrete facts and procedures and towards a coherent conceptual system, and this often requires teachers to “relearn” concepts, which may require substantial amounts of time and energy on their part (Furtak et al., 2014; Suh & Seshaiyer, 2015). Third, in order for teachers to effectively use an LP, it must be supported by the context in which they work. Furtak and Heredia (2014), for example, found that when an externally-developed LP was brought into a school, the teachers had a hard time determining how to use it within their school’s accountability system. Finally, once teachers have made sense of the LP and become committed to using it, they must understand how to take the formative information that they get from their students and use the LP to help guide future instruction (Furtak et al., 2014). It is clear that using an LP formatively to promote student learning may require a good deal of support in the form of training, time, and materials.

Conclusion

We have put forward a theoretical LP based upon Kieren’s (1976, 1980) five conceptualizations of fractions, which begins with viewing fractions as Part-Whole relationship and moves on to seeing them as Quotients that equipartition a whole into unit fractions. Next, students may see fractions as Measurements that represent a magnitude, which allows them to order fractional values and to understand the equivalent fractions that allow for the addition of fractions with unlike denominators. Next, students should come to interpret fractions as Operators that take a value and produce a new value that is proportional to the original. This progression has been borne out empirically by the steadily increasing difficulty of the items associated with these conceptualizations in the vertical scale for the *i-Ready Diagnostic* exam. We had hypothesized a level having to do with ratios and rates, but these types of test items spanned a wide range of difficulties and could not be distinguished from adjacent levels. While researchers have used IRT with a specific sample of students and items to validate LPs in the past (see Lobato & Walters, 2017), our use of an existing vertical scale to investigate relative item difficulties across LP levels is a novel approach to this type of work.

Our proposed LP is meant to provide a high-level overview of the ways that students develop fraction understandings that teachers can use to create and interpret assessments of student learning. We also intend to use this LP in the reporting of *i-Ready Diagnostic* scores by mapping the difficulty ranges of each level to student scores on the vertical scale. This way, teachers can use students' scores on a required interim assessment to get an approximation of which level of understanding their students likely hold rather than taking extra time to create and administer their own more fine-grained assessment. We wish to be clear, however, that this information is meant to serve as a starting point for further investigation, and Curriculum Associates would provide suggested follow-up activities that teachers could use for this purpose. The more fine-grained learning progressions that already exist in the literature (e.g., Confrey et al., 2019; Confrey, Maloney, & Corley, 2014; Nizar et al., 2017) may also help teachers to get a closer look at which aspects of a given conceptualization their students do and do not yet understand.

Prior research has found that teachers do tend to use LPs as a rough starting point and then use more detailed tasks to get more information (Alonzo & Elby, 2019), and this type of triangulation is important because "trajectories need to be interpreted stochastically not deterministically" (Wright, 2014, p. 651). Confrey (2019) makes this clear with her distinction between intra- and inter-level variation. While we expect differences in student performance between LP levels (inter-level variation), there is also a fair amount of variation in tasks within a given level (intra-level variation). We saw this represented in Figure 8, as each level had a range of item difficulties, and there was overlap in difficulties across levels. This means that just because a student misses an item associated with the quotient interpretation, that does not mean that they will not be able to answer a measurement item correctly. Similarly, answering an item at one level correctly does not guarantee that a student will be able to answer all items at the level. This is a function of relative item difficulties and the situated nature of learning, which makes it so that students may miss one item and then answer another item at the same LP level correctly if they are presented in different contexts (Alonzo & Steedle, 2008).

Limitations and Future Directions

While or analyses of the *Ready* curriculum and the *i-Ready Diagnostic* items appear to support a four-level LP for fractions, there are some potential limitations. One possible cause for uncertainty in our findings is that 372 of the 406 fractions items (92%) were presented in a multiple-choice format. This represents a threat to the validity of our interpretations, as it is possible that students could use a lower-level strategy than the one we assigned because these items do not require students to explain the reasoning they used to arrive at an answer. Furthermore, the items used in the *i-Ready Diagnostic* were not written with these levels in mind. We simply mapped the content of the items to the descriptions of the levels in our LP.

This means that there may be items that were classified as representing aspects of a given level that also included other constructs. This could have caused construct-irrelevant variance and skewed the difficulties of some items.

Another potential concern is whether our selected grain-size is useful to teachers in the way we expect it to be. Our team will be conducting interviews with teachers who use the *i-Ready Diagnostic*, and we intend to ask them whether these four levels make sense to them and would be helpful to guide their thinking and classroom practice.

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Appendix

Table A-1. Fraction conceptualizations learning progression

Interpretation	Student Characteristics	Item Responses
Operator	**Understands that:** • Multiplying a value by a fraction $\frac{a}{b}$ results in a value that is a-$b$ths of the original value • Understands the difference between multiplying and dividing fractions	**Is able to:** • Use multiplication to find a portion of a value • Determine that multiplying a value by a fraction with magnitude less than 1 will result in a value with smaller magnitude and multiplying by an improper fraction will result in a value with larger magnitude, and vice versa for division, without performing the calculations • Divide a value by a fraction
Ratio	**Understands that:** • Ratios may be expressed in various forms $\left(\frac{a}{b}, a{:}b, \text{ verbal description, or diagram}\right)$ • Ratios may represent either part-whole or part-part relationships • Ratios may represent rates • Equivalent ratios may be created by multiplying both parts by the same value **May not yet understand that:** • Multiplying a rate by a value can provide information about the overall situation (e.g., if a driver goes $\frac{65 \text{ miles}}{\text{hour}}$ for 3 hours, they have gone $\frac{65 \text{ miles}}{\text{hour}} \times 3 \text{ hours} = 195 \text{ miles}$) • The direction of effects for fraction operations are not the same as they are for whole numbers	**Is able to:** • Identify part-whole and part-part relationships • Move between the various representational forms for ratios and rates **Common Errors:** • Selecting the wrong operation when solving problems involving proportional reasoning • Indicating that multiplication always results in a larger value and that division always results in a smaller value

Measurement

Understands that:

- Fractions represent unique numerical values
- Two fractions are equivalent if they represent the same numerical value
- Fractional values can be converted to decimals or percentages while maintaining their numerical value
- Improper fractions may be rewritten as mixed numbers and vice versa
- Fractions with different denominators may be compared or added if they are put into the same units

May not yet understand that:

- Fractions may be written as ratios and may represent part-part relationships or rates

Is able to:

- Create and identify equivalent fractions, including converting between improper fractions and mixed numbers
- Order fractions and mixed numbers with different numerators and different denominators
- Add and subtract fractions and mixed numbers with different denominators

Common Errors:

- Treating all ratios as part-whole
- Treating rates as two independent values with different units

Quotient

Understands that:

- Fractional parts must be equal ("fair shares") but may not appear the same
- The fraction $\frac{a}{b}$ represents the division of a by b
- Unit fractions can be iterated to reproduce the original whole or part of the whole
- Dividing the same whole into more parts (larger denominator) results in smaller unit pieces

May not yet understand that:

- A fraction has its own specific value that can be uniquely placed on a number line.
- The same fractional value may be represented in multiple ways

Is able to:

- "Share" a whole between a specified number of groups
- Identify unit fractions
- Use unit fractions $\left(\frac{1}{b}\right)$ to reproduce composite fractions $\left(\frac{a}{b}\right)$, including the whole $\left(\frac{b}{b}\right)$
- Compare fractions with the same numerator and different denominators
- Add and subtract composite fractions with the same denominator

Common Errors:

- Misplacing a fraction on a number line

		• Incorrectly comparing two fractions with different numerators and different denominators • Not recognizing improper fractions as valid
Part-Whole	**Understands that:** • A fraction represents a specified number of parts out of the total number of parts **May not yet understand that:** • A whole must be partitioned equally • All parts of the whole must be used when partitioning	**Is able to:** • Identify the number of specified and total parts in an area model or in a described situation. • Compare fractions with the same denominator and different numerators **Common Errors:** • Making unequal parts or fail to exhaust the whole when attempting an equipartitioning task • Treating the numerator and denominator of a fraction as unrelated values