APPM 2460
Chaotic Dynamics

1 Introduction

Today we'll continue our exploration of systems of ordinary differential equations, focusing in particular upon systems that exhibit a type of strange behavior known as "chaos." We will first look at simple pendulum systems, then we will work our way up to some of the classical chaotic systems.

2 Attractors and Limit Cycles

- Recall that the general equation for a mass-spring system is

  $$mx'' + bx' + kx = f(t)$$

  Consider such a system that is unforced and undamped, with mass and spring constant $m = k = 1$. We start the mass at $x(0) = 1$ and $x'(0) = 0$. The IVP for this system is

  $$\begin{cases} x'' + x = 0 \\ x(0) = 1,\ x'(0) = 0 \end{cases}$$

  As we have done previously, we could rewrite this second order problem as a system of first order equations. Let $y = x'$. Then $y' = x''$ and the original equation can be written as $y' = -x$. The system of equations is

  $$\begin{cases} x' = y \\ y' = -x \\ x(0) = 1,\ y(0) = 0 \end{cases}$$

  We can also represent the solution to this system as a vector, $\vec{v}(t) = [x(t), y(t)]^T$. The solution would then be

  $$\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

  If we plot this vector in *phase space* – pairs of (x, y) points – then we see a circle traced out by our system. This is just the parametric path $(x, y) = (\cos(t), -\sin(t))$.

  This circle is an example of a *limit cycle*, or a cycle that the system repeats endlessly.

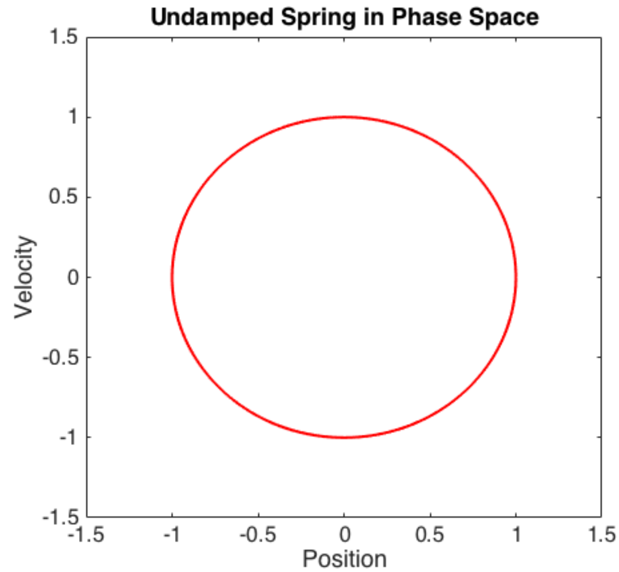

Figure 1: An undamped oscillator in phase space. Note that it traces out a circular cycle. It endlessly loops around this circle.

- Let's take a look at a more complicated system. Suppose that we have a system that is *both* damped and driven. An example of such a system would be

$$\begin{cases} x'' + x' + x = \cos(t) \\ x(0) = 1.5,\ x'(0) = 0 \end{cases}$$

  Loosely speaking, this system *really wants* to oscillate like $x(t) = \sin(t)$ (note that this solves the differential equation, but doesn't quite satisfy the initial conditions). Note that this would mean $x'(t) = \cos(t)$, and so we would be back on our circular limit cycle. However, when we start it at $x(0) = 1.5$, it takes some time to settle down to it's comfortable oscillation.

  Can we show this behavior using Matlab?

  - First, define the system in Matlab as an `m`-file function like we learned before. Remember that the input is a *vector* $\vec{v}(t) = [x, y]^T$ and the output is a *vector* $\vec{v}\,' = [x', y']^T$. When we write the system of equations in this function, let's make it a bit easier to keep track of the "x" and "y" terms by defining `x = v(1)` and `y = v(2)` *before* defining the system of equations. Open a new script and type:

```
function vprime = damped_spring(t,v)

% v is a vector input where v = [x,y]^T
    x = v(1);    y = v(2);

% vprime is a vector of derivatives where vprime = [x', y']^T
    vprime = zeros(2,1);
    vprime(1) = y;
    vprime(2) = - y - x + cos(t);

end
```

– Then, we can run a script that uses `ode45` to solve the IVP over the interval $t \in [0, 20]$ and to plot the phase space:

```
[t,v] = ode45( @damped_spring, [0, 20], [1.5, 0]);
plot(v(:,1), v(:,2))
```

When we run this script, we see the following result.

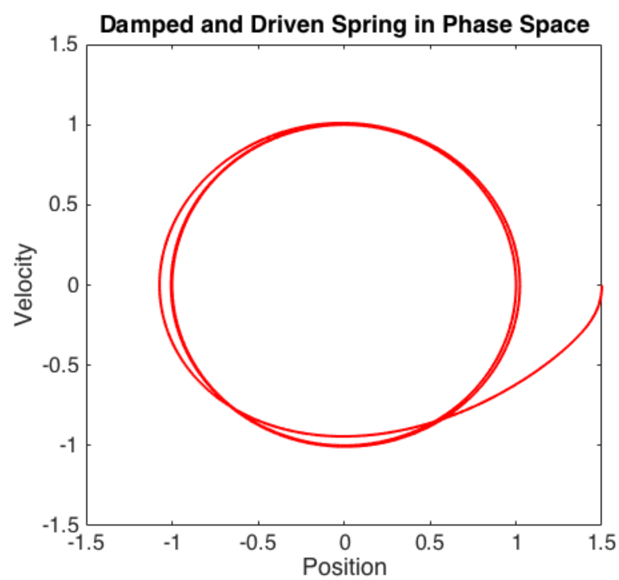

Figure 2: A damped and driven oscillator in phase space. The oscillator tries to oscillate at its natural frequency and amplitude of 1, but it takes time to return to that state since it begins at $(x, y) = (1.5, 0)$.

Here we clearly see the limit cycle behavior: the oscillator starts far from the curve $(x, y) = (\sin(t), \cos(t))$, but it eventually settles into the circular limit cycle.

This limit cycle is a nice circle. The fact that it's such a simple curve indicates that the system settles into predictable, repeating behavior after some amount of time. What about a more complicated system?

3 Chaotic Oscillators

Recall in a past homework assignment we saw that if we use the modified van der Pol system

$$x(t)'' - \mu(1 - x^2(t))x'(t) + x(t) - A\sin(\omega t) = 0$$

(with $\mu = 8.53$, $A = 1.2$, and $\omega = 2\pi/10$ and $x(0) = 2$ and $x'(0) = 0$) we get very strange behavior.

In particular, we get a pattern that does not quite repeat itself. If we plot position against time, as in Figure 3, we can see this quite clearly:

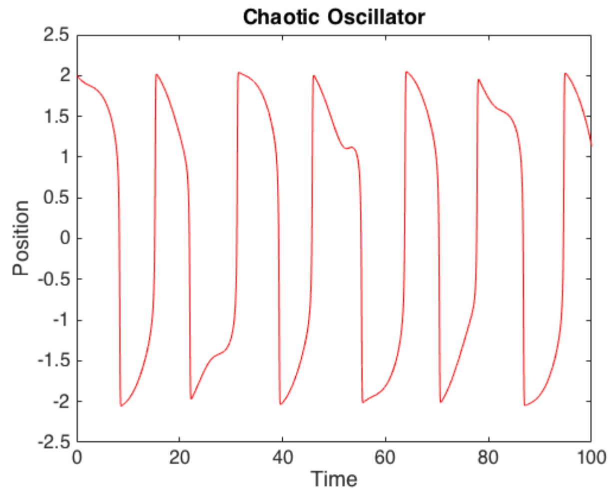

Figure 3: A chaotic van der Pol oscillator. Note how, unlike a regular oscillator, it almost **but never quite** repeats itself.

Figure 4 shows this oscillator in phase space, and we see that it almost settles on a nice curve, but not quite. In Figure 5 we zoom in on that weird region on the top of the curve. We observe something that

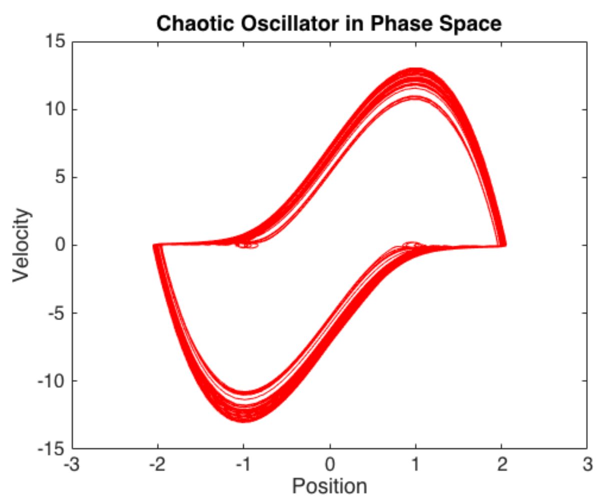

Figure 4: A chaotic van der Pol oscillator in phase space.

is very strange. The system is not settling down onto a simple curve like a circle. In fact, we are always *almost* repeating ourselves, but never quite. This is called a **strange attractor**, and is the hallmark of chaotic dynamics.

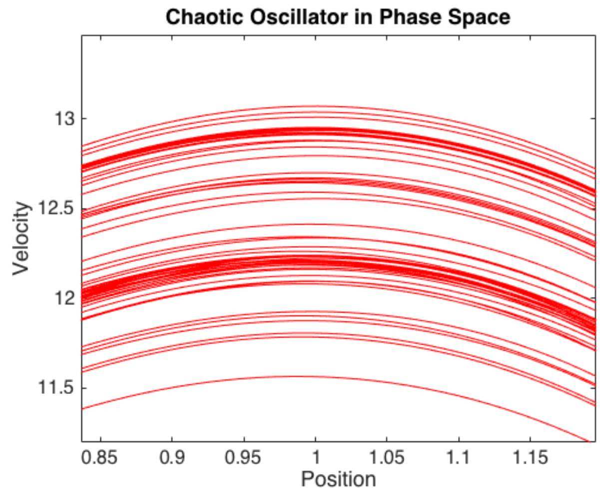

Figure 5: Zooming in on the top of the phase-space curve of the van der Pol oscillator.

4 Chaotic Oscillators in 3-D

Now let's look at something in three dimensions. We'll examine a famous system of equations known as the Lorenz attractor. Consider the following system:

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= x(\rho - z) - y \\ z' &= xy - \beta z \end{aligned}$$

with parameters $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$. This system is named for Edward Lorenz, who studied it extensively.

- As before, we could write an m-file function as follows:

```
function vprime = lorenz(t,v)
    % v is a vector input where v = [x,y,z]^T
        x = v(1);    y = v(2);    z = v(3);
    % constants
        sigma = 10; rho = 28;    beta = 8/3;
    % vprime is a vector of derivatuves where vprime = [x', y', z']^T
        vprime = zeros(3,1);
        vprime(1) = sigma*(y - x);
        vprime(2) = x*(rho - z) - y;
        vprime(3) = x*y - beta*z;
end
```

  As before, we could solve the system of equations with ode45 over the range $t \in [0, 100]$ by calling the function in either the command window or a script:

```
[t,v] = ode45(@lorenz, [0,100], [1,1,1]);
plot3(v(:,1), v(:,2), v(:,3))
```

  Note that we have three variables, so we use plot3 to plot in 3 dimensions. The resulting plot is shown in Figure 6.

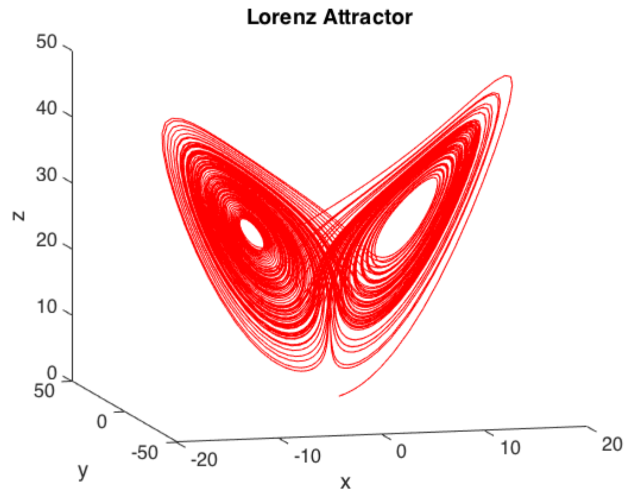

Figure 6: The famous Lorenz attractor.

APPM 2460 Homework Week 14

Submit a published pdf of your script and any other supporting code needed to solve the following problem to Canvas by Monday, April 29 at 11:59 p.m. See the 2460 webpage for formatting guidelines.

Consider the system known as the Rossler system

$$
\begin{aligned}
x' &= -y - z \\
y' &= x + ay \\
z' &= b + z(x - c)
\end{aligned}
$$

(a) Write a script that finds a numerical solution to the Rossler system of equations.

- Use parameters $a = b = 0.1$, and $c = 14$
- Use the initial conditions $x(0) = 15$, $y(0) = 1$, and $z(0) = 0.1$.
- Use the interval $t \in [0, 500]$.

(b) Plot the full xyz-phase space dynamics using `plot3`.

(c) Plot the phase space solution for the x-variable against the z-variable. (That is, plot all sets of points $(x(t), z(t))$. This is a projection of the solution in the xz-plane.)

(d) Plot the phase space solution for the y-variable against the z-variable. (That is, plot all sets of points $(y(t), z(t))$. This is a projection of the solution in the yz-plane.)

(e) Plot the component curve solution for $y(t)$. (That is, plot all sets of points $(t, y(t))$.)

(f) Plot the component curve solution for $z(t)$. (That is, plot all sets of points $(t, z(t))$.)