Chapter 4

DERIVATION AND ANALYSIS OF SOME WAVE EQUATIONS

Wave phenomena are ubiquitous in nature. Examples include water waves, sound waves, electromagnetic waves (radio waves, light, X-rays, gamma rays etc.), the waves that in quantum mechanics are found to be an alternative (and often better) description of particles, etc. Some features are common for most waves, e.g. that they in cases of small amplitude can be well approximated by a simple trigonometric wave function (Section 4.1) Other features differ. In some cases, all waves travel with the same speed (e.g. sound waves or light in vacuum) whereas in other cases, the speed depends strongly on the wave length (e.g. water waves or quantum mechanical particle waves). In most cases, one can start from basic physical principles and from these derive partial differential equations (PDEs) that govern the waves. In Section 4.2 we will do this for transverse waves on a tight string, and for Maxwell's equations describing electromagnetic waves. In both of these cases, we obtain linear PDEs that can quite easily be solved numerically. In other cases, such as water waves, discussed in Section 4.3, the full governing equations are too complex to give here, and we need to restrict ourselves to a number of general observations. In still other cases, such as the Schrödinger equation for quantum wave functions, a quite simple set of PDEs are well known and extremely accurate (often said to describe *all* of chemistry!) but these are prohibitively difficult to solve in all but the simplest special cases. We note in Section 4.4 that some important nonlinear wave equations can be formulated as systems of first order PDEs. Not only are these systems usually very well suited for numerical solution, they also allow a quite simple analysis regarding various features, such as types of waves they support and their speeds. In some cases, discussed in Section 4.5, we find some closed-form analytic solutions. We arrive in Section 4.6 to Hamilton's equations. These are fundamental in many applications, such as mechanical and dynamical systems, and the study of chaotic motions. In the context of this book, their key application is to provide the governing equations for the freak wave phenomenon that is discussed in Chapter ??.

4.1 Wave Function

A progressive wave may at one instant in time look like what is shown in Figure 1. The variable that is displayed here need not correspond to sideways deflections. For sound waves, it is pressure, for light it can be the strength of electric or magnetic fields, etc. At least when the amplitude is

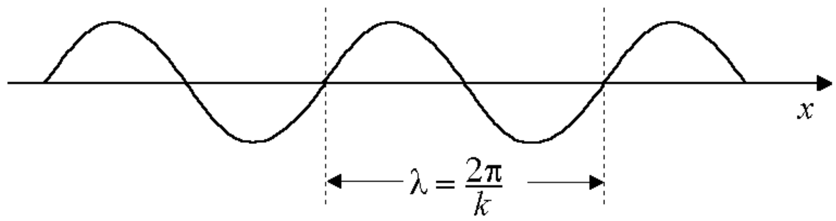

Figure 4.1: Snapshot of a progressive sinusoidal wave

small, such a progressive wave can be well approximated by a single trigonometric mode

$$\phi(x,t) = \phi_0 \; \cos(k\,x - \omega\,t)\ . \tag{4.1}$$

In terms of the *wave number* k and *angular time frequency* ω, we find

$$\begin{array}{lcl} \textit{wave length} \;\; \lambda & = & 2\pi/k \\ \textit{frequency} \;\; \nu & = & \omega/(2\pi) \end{array}$$

If t is increased by one and x by ω/k, the argument of the cosine in (4.1) is unchanged. Hence, the wave in (4.1) travels with

$$\textit{phase speed} \qquad c_p \;=\; \omega/k \quad . \tag{4.2}$$

We will later come across another speed, *group speed*, c_g. If only 'wave speed' is mentioned, or no subscript for c is given, the phase speed c_p is assumed.

For almost all waves, ω is not a constant, but a function of k. In some cases, this relation takes the form $\omega = c \cdot k$, e.g. for sound waves and for light in vacuum. In such cases the wave speed (according to (4.2)), becomes independent of the wave number. In other cases, the *dispersion relation* $\omega = \omega(k)$ takes different forms. For waves on deep water, the leading order approximation (when the wave amplitude is small) can be shown [...] to be

$$\omega = \sqrt{g\,k} \tag{4.3}$$

where g denotes the acceleration of gravity.

It is often very convenient to write the wave function as

$$\phi(x,t) = \phi_0 \;\text{Re}\; e^{i\,(k\,x - \,\omega\,t)},$$

which is mathematically equivalent to (4.1). Many trigonometric manipulations become very much easier if one uses the complex exponential function rather than manipulate sines and cosines directly (cf. Appendix ??). It is even more convenient not to write "*Re*" all the time, but instead let that be implicitly understood. Hence, we will in the following sometimes write the 1-D wave function simply as

$$\phi(x,t) = \phi_0 \; e^{i\,(k\,x - \,\omega\,t)}. \tag{4.4}$$

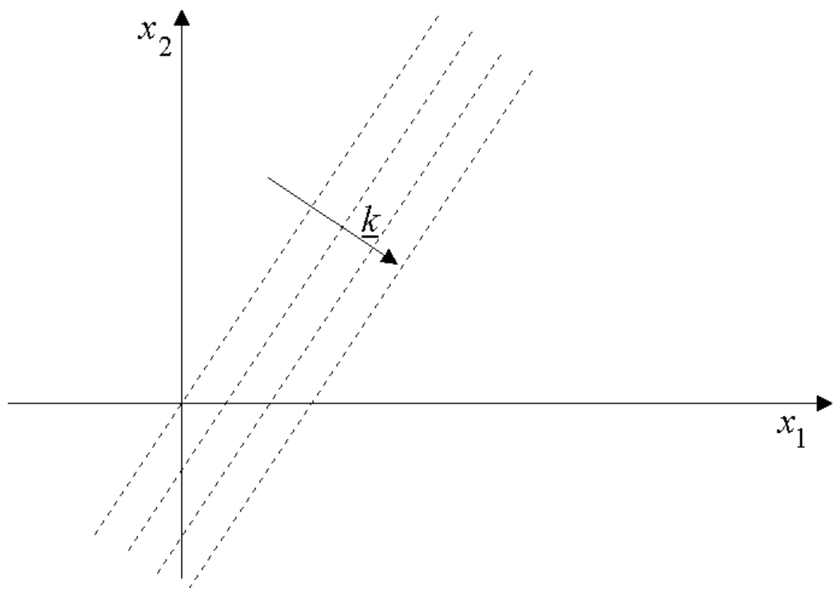

Figure 4.2: 2-D progressing wave. Wave crests are marked with dotted lines; waves progress in the direction of the $\underline{k}$-vector.

Not all wave forms are sinusoidal. However, by Fourier analysis (cf. Chapter ??), any other shape (for a linear wave equation) can be viewed as a superposition of sinusoidal waves of different wave numbers k. Together with knowledge of the dispersion relation $\omega = \omega(k)$, we can analyze how an initial wave form evolves in time.

The 2-D counterpart to (4.4) is

$$\phi(\underline{x}, t) = \phi_0 \; e^{i \; (\underline{k} \cdot \underline{x} - \; \omega \; t)} \tag{4.5}$$

where $\underline{x} = (x_1, x_2)$ and $\underline{k} = (k_1, k_2)$ are two-component vectors. The wave $\phi(\underline{x}, t)$ given by (4.5) clearly reduces to (4.4) in case we introduce a (scalar) x-direction parallel to the $\underline{k}$-vector. We can also note that $\phi(\underline{x}, t)$ is unchanged if $\underline{x}$ moves along any direction orthogonal to $\underline{k}$. From the fist observation follows that the wave length $\lambda = 2\pi \; / \; |\underline{k}|$ and the phase speed $c_p = \omega \; / \; |\underline{k}|$.

4.2 Two Examples of Derivations of Wave Equations

The two cases we will consider are waves traveling along a string under tension, and Maxwell's equations for electromagnetic waves in 3-D. The methods of derivation are rather different, and they illustrate two of the main approaches for obtaining governing equations in many other situations (In Section 4.6, we will come across a third approach - via Hamilton's equations).

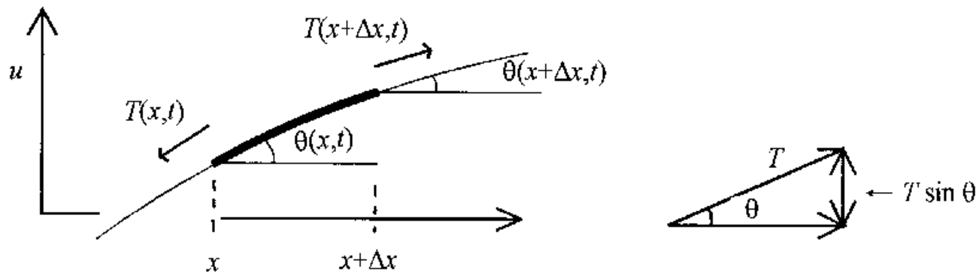

Figure 4.3: Illustration of an infinitesimal section of a transversally vibrating string

4.2.1 Transverse waves in a string under tension

Two kinds of waves will travel along a string under tension - transverse and longitudinal. Their speeds are typically vastly different. Transverse (sideways) oscillations are usually fairly slow, and visible, whereas longitudinal (lengthwise) waves would travel with the speed of sound in the material, maybe of the order of 1 km/s, while causing no visible deflections. In a loosely stretched 'slinky', both wave types can be seen traveling at about 10 m/s. The transverse waves in a string is the simplest case to obtain an equation for, and we will do that next.

A string, with density ρ per unit length, is stretched in the x-direction with a tension force T (cf. Figure 4.3). At any time, the vertical forces on the small string segment must balance. They are

$$\underbrace{\rho \cdot \Delta x}_{\text{mass}} \cdot \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\substack{\text{accele-}\\\text{ration}}} = \underbrace{T(x+\Delta x, t)\ \sin\theta(x+\Delta x, t) - T(x,t)\ \sin\theta(x,t)}_{\text{difference between vertical tension forces at the two ends}}$$

Assuming the deflection angles $\theta(x,t)$ are small, $\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x}$. Dividing both sides by Δx and letting $\Delta x \to 0$ then gives

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\,\frac{\partial u}{\partial x}\right).$$

Still assuming that the deflection is small, the tension T becomes approximately a constant, and can be factored out. Introducing $c^2 = T/\rho$, we arrive at the 1-D wave equation in its standard form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.6}$$

We will soon see that this equation supports waves traveling with the velocity c to either left or right.

Wave type	Cause	Period	Velocity
Sound	Sea life, ships	$10^{-1} - 10^{-5}$ s	1.52 km/s
Capillary ripples	Wind	$< 10^{-1}$s	0.2 - 0.5 m/s
Gravity waves	Wind	1 - 25 s	2 - 40 m/s
Sieches	Earthquakes, storms	minutes to hours	standing waves
Storm surges	Low pressure areas	1 - 10 h	$\sim$ 100 m/s
Tsunami	Earthquakes, slides	10 min - 2 h	$<$ 800 km/h
Internal waves	Stratification instabilities, tides	2 min - 10 h	$<$ 5 m/s
Tides	Moon and sun	12 - 24 h	$<$ 1700 m/h
Planetary waves	Earth rotation	$\sim$ 100 days	1 - 10 km/h

Table 4.1: Types of water waves in lakes and oceans

4.2.2 Maxwell's equations in 3-D

.. To be filled in ....Will start by basic laws of electromagnetics, and then use vector calculus to obtain:

$$\left\{\begin{aligned} \frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \\ \frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \\ \frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \end{aligned}\right. , \quad \left\{\begin{aligned} \frac{\partial H_x}{\partial t} &= -\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) \\ \frac{\partial H_y}{\partial t} &= -\frac{1}{\mu}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) \\ \frac{\partial H_z}{\partial t} &= -\frac{1}{\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \end{aligned}\right. \tag{4.7}$$

Here

$E_x,\ E_y,\ E_z$	the components of the electric field
$H_x,\ H_y,\ H_z$	the components of the magnetic field
μ	permeability
ε	permettivity

For lossy media, we need to subtract $\sigma E_x,\ \sigma E_y,\ \sigma E_z,\ \rho H_x,\ \rho H_y,\ \rho H_z$, resp. from the six RHSs (with σ and ρ denoting conductivity and magnetic resistivity).

4.3 Water Waves

A great variety of different wave phenomena occur in bodies of water in nature. Table 1 summarizes some that can be found in lakes and oceans. We will here describe only the most obvious ones - gravity waves (so called because gravity is the restoring force which strives to keep the surface level). To get started with some analysis of steady translating waves, we make a number of simplifying assumptions:

- no surface tension or wind forces,
- infinitely deep water,
- very small wave amplitude a,

For waves on a string, we could write down a PDE which describes how any initial state evolves forward in time. In contrast to this, for deep water gravity waves, there is no single PDE which describes the evolution of a surface disturbance.

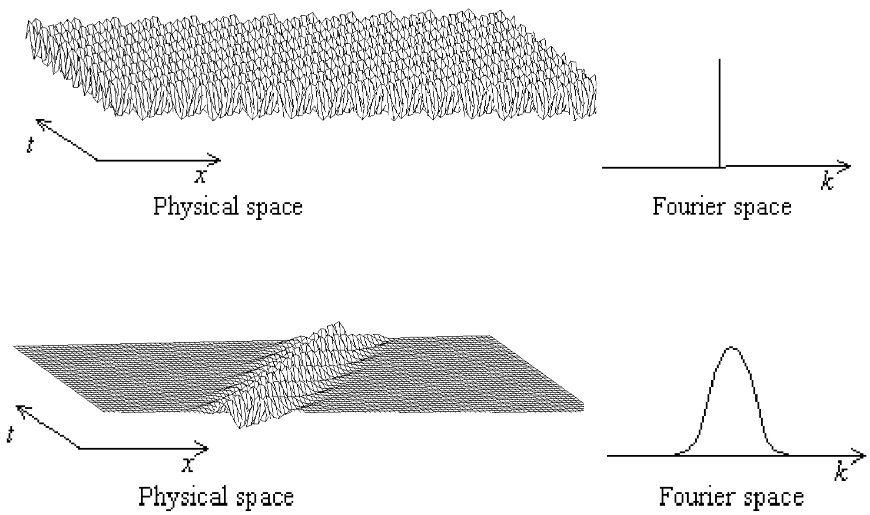

Figure 4.4: Physical and Fourier space representation of a uniform wave train and of a wave packet.

4.3.1 Dispersion relation for deep water

To make some progress, we need to make still one more simplifying assumption, namely that the surface elevation $\phi(x,t)$ for small amplitude becomes approximately sinusoidal:

$$\phi(x,t) = a\ \cos(kx - \omega t) \tag{4.8}$$

As noted earlier (4.3), the dispersion relation becomes

$$\omega = \sqrt{g\,k}$$

where the acceleration of gravity $g \approx 9.8$ m/s^2. Denoting the wave length by $\lambda = 2\pi/k$, the velocity of this wave (cf. (4.2)) becomes

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}, \tag{4.9}$$

and the time period $T = 2\pi/\omega$. The fact that c_p grows proportionally with $\sqrt{\lambda}$ leads to many notable features of water waves. Some of these are mentioned in Section ....

4.3.2 Group speed vs. phase speed

The upper part of Figure 4.4 illustrates a wave train described by (4.8) when just one single frequency k is present. In Fourier space, the wave becomes a delta function. In many situations, waves instead travel in wave packets, as sketched in the lower part of the figure. In Fourier space, the wave packet is a superposition of waves with very similar frequencies. If the peak (in Fourier space) is getting narrower, the packet becomes wider in physical space. We can clearly see two different velocities associated with the wave packet, both of which can be expressed in terms of the dispersion relation:

phase speed	speed of individual crests	$c_p = \dfrac{\omega(k)}{k}$
group speed	speed of the whole group	$c_g = \dfrac{d\omega(k)}{dk}$

Derivation of the formula for c_g : The wave function for a single wave number k_0 can be written

$$\phi(x,t) = \widehat{\phi}_0 \; e^{i(k_0 x - \omega(k_0)\; t)} \tag{4.10}$$

A wave packet with the same main wave number is similarly a superposition of different waves

$$\phi(x,t) = \int_{-\infty}^{\infty} \widehat{\phi}_0 \; e^{i(k\; x - \omega(k)\; t)} \; dk$$

where $\widehat{\phi}(k)$ is very near zero everywhere but has a sharp peak at $k = k_0$. In that small neighborhood of k_0 we have (by Taylor expansion)

$$\omega(k) = \omega(k_0) + (k - k_0)\;\alpha \qquad \text{where} \qquad \alpha = \left.\frac{d\omega}{dk}\right|_{k=k_0}.$$

Hence

$$\begin{aligned}\phi(x,t) &\approx \int_{-\infty}^{\infty} \widehat{\phi}(k)\; e^{i(k\; x - (\omega(k_0) + (k-k_0)\alpha)\; t)}\; dk \\ &= \underbrace{e^{i(k_0\; x - \omega(k_0) t)}}_{\text{Pure harmonic of wave number } k_0} \cdot \underbrace{\int_{-\infty}^{\infty} \widehat{\phi}(k)\; e^{i(k-k_0)(x - \alpha t)}\; dk}_{\text{Factor providing the envelope of the wave packet}}\end{aligned}$$

We notice that in the second factor, the variables x and t appear only in the combination $x - \alpha t$, showing that this expression translates with the speed $\alpha = \frac{d\omega}{dk}\big|_{k=k_0}$, i.e. this quantity is equal to the group speed.

Table 4.2 summarizes the dispersion relations and the two speeds c_p and c_g for some different types of waves (in the case of surface waves, the liquid is assumed to be water, with density $\rho = 1$). The results in this table have some notable implications:

- Since $c_p > c_g$ for gravity waves, a surfer gets the longest ride is he/she can catch a wave at the end of a wave packet
- Compared to the wave lengths of tsunami waves (hundreds of kilometers), all oceans are shallow. Timing of the departure and arrivals of such waves offered the first means (in the 19th century) of estimating average ocean depths.

Type of wave	Dispersion relation $\omega =$	$c_p = \omega/k$	$c_g = \partial\omega/\partial k$	c_g/c_p	Comment
Gravity wave, deep water	$\sqrt{g\,k}$	$\sqrt{\frac{g}{k}}$	$\frac{1}{2}\sqrt{\frac{g}{k}}$	$\frac{1}{2}$	g = acceleration of gravity
Gravity wave, shallow water	$\sqrt{g\,k\,\tanh\,kh}$	$\sqrt{\frac{g}{k}\tanh\,kh}$	$c_p \cdot (c_g/c_p)$	$\frac{1}{2} + \frac{kh}{\sinh(2hk)}$	h = water depth
Capillary wave	$\sqrt{T\,k^3}$	$\sqrt{T\,k}$	$\frac{3\sqrt{T\,k}}{2}$	$\frac{3}{2}$	T = surface tension
Quantum mechanical particle wave	$\frac{hk^2}{4\pi m}$	$\frac{hk}{4\pi m}$	$\frac{hk}{2\pi m}$	2	h = Planck's constant m = particle mass c_g = particle velocity
Light in vacuum	ck	c	c	1	c = 299,792,458 m/s
Light in a transparent medium	$\frac{ck}{n(k)}$	$\frac{c}{n(k)}$	$c_p\left(1 - \frac{kn'(k)}{n(k)}\right)$	$1 - \frac{kn'(k)}{n(k)}$	$n(k)$ = index of refraction in the medium

Table 4.2: Dispersion relations and wave speeds for some different types of waves

- If a twig sticks up in a stream, the phase angle of waves gets locked at it. Since capillary waves have $c_g > c_p$, they can be seen as small stationary ripples upstream of the twig. Gravity waves will instead appear in some relatively narrow sector downstream of the twig.

- The ratio $c_g/c_p = \frac{1}{2}$ for gravity waves can be shown to imply that the distinct V-shaped wake left behind a ship will always form the angle $\arcsin \frac{1}{3} \approx 19.5^o$ to each side of the center line of he wake *independently of the speed of the ship* (intuition might wrongly suggest that the wake should get narrower with increased speeds, like the case is for shocks generated by a fast-moving object).

- In quantum mechanics, a particle's position is undetermined within the width Δx of its wave packet. Fourier analysis will show that Δx and its spread in Fourier space are related by $\Delta x \cdot \Delta k > 1$ (or $\Delta x \cdot \Delta k >$ constant; there is some arbitrariness in how wide one regards a Gaussian pulse to be). De Broglie's relation $k = 2\pi m\nu/h$ relates wave number k to Planck's constant h; here m is the particle mass and ν ($= c_g$) its velocity. From this, we obtain Heisenberg's uncertainty relation $\Delta x \cdot \Delta \nu > h/(2\pi m)$. In other words, the product of the uncertainties in a particle's position and velocity must always exceed $h/(2\pi m)$.

4.3.3 Stokes' waves

These are periodic translating solutions $\eta(x,t) = \eta(x-ct)$ for the surface elevation in the case of infinite depth and no surface tension; gravity is assumed to be the only restoring force. However, we no longer consider only infinitesimal amplitudes, so the approximation (4.8) amounts only to the first term in an expansion for small a. Including terms up to a^3 can be shown to give

$$\eta(x,t) = a\,\cos(kx-\omega t) + \frac{1}{2}ka^2\cos 2(kx-\omega t) + \frac{3}{8}k^2a^3\cos 3(kx-\omega t) + \ldots \tag{4.11}$$

where $\omega \approx \left(1 + \frac{1}{2}k^2a^2\right)\sqrt{gk}$ is the next order approximation beyond (**??**) for the dispersion relation. Some observations:

- The governing equations for surface water waves turn out to be nonlinear. Apart for infinitesimal waves, solutions can therefore not be linearly superposed (or multiplied by scalars) to give other solutions.

- In the limit of large amplitude, already Stokes (1880) showed that the wave would have a top angle of 120^o. However, it was noted later that the local wave structure near the top - in this limit - will feature a complicated fine structure.

- The expansion (4.11), if continued to more terms, will diverge before the highest Stokes' wave is reached. This is related to the fact that a Taylor expansion will fail to converge at a radius determined by the nearest singularity - in this case there will arise unphysical complex singularity points for the real variable a.

- When including further terms in (4.11), the coefficients for the individual modes will no longer be pure powers of a, but will turn into power series expansions in a.

- In real water, high Stokes' waves are never seen. Such a wave is (near the top) unstable to oscillatory disturbances. Another (physically more significant) instability arises already at low amplitudes. The *Benjamin-Feir instability* causes uniform periodic wave trains to loose their periodicity. Wave trains will always exhibit irregularities in amplitude between the individual waves.

4.4 First Order System Formulations for some Linear wave Equations

It is often practical to rewrite higher order ODEs into systems of first order ones. The main theme of this section is to extend that idea to linear PDEs which describe different types of waves. These first order formulations are usually very well suited for numerical calculations. We will here also use them to analytically determine the different types of translating waves that the equations can feature.

4.4.1 High order ODEs as first order systems

One example suffices to illustrate the general procedure. If we, for example, consider the ODE

$$y''' + \frac{y'' \sin y'}{1+t^2 y'} - t\, y + 11 \arctan \frac{y}{1+t} = 0 \tag{4.12}$$

we can introduce the new variables

$$\begin{cases} u_1 &= y \\ u_2 &= y' \\ u_3 &= y'' \end{cases} .$$

The ODE then becomes

$$\begin{cases} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= -\dfrac{u_3 \sin\ u_2}{1+t^2 u_1} + t u_1 - 11\ \arctan \dfrac{u_1}{1+t} \end{cases}$$

The initial conditions for y, y' and y'' become the initial values for the three variables u_1, u_2 and u_3. This system, like the more general first order system

$$\begin{cases} u_1' &= f_1(u_1,\ldots,u_n,t) \\ u_2' &= f_2(u_1,\ldots,u_n,t) \\ \cdots & \\ u_n' &= f_n(u_1,\ldots,u_n,t) \end{cases},$$

can be solved accurately with almost any of the standard numerical techniques for ODEs (such as Runge-Kutta or linear multistep methods).

Analytical solutions of systems of ODEs are rarely available. However, linear systems

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

where A is a matrix with constant coefficients, form a notable exception. If the f-vector is absent, and A has distinct eigenvalues $\lambda_1,\ldots,\lambda_n$ with corresponding eigenvectors $\underline{v}_1,\ldots,\underline{v}_n$, the general solution becomes

$$\begin{bmatrix} \\ \underline{u} \\ \\ \end{bmatrix} = c_1 e^{\lambda_1 t}\begin{bmatrix} \\ \underline{v}_1 \\ \\ \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} \\ \underline{v}_2 \\ \\ \end{bmatrix} + \ldots + c_n e^{\lambda_n t}\begin{bmatrix} \\ \underline{v}_n \\ \\ \end{bmatrix}. \tag{4.13}$$

The coefficients $c_1,\ c_2,\ldots,c_n$ will follow from the initial conditions. Standard ODE text books will discuss the minor modifications that will need to be done to the form of (4.13) in the (usually rare) cases of multiple eigenvalues or missing eigenvectors. If the f-vector is present, *variation of parameters* can be used to obtain a general solution.

4.4.2 High order PDEs as first order systems

A similar approach to the one for ODEs can be used to transform higher order PDEs into lower order ones. This time, success is not guaranteed, but it still works out in the constant coefficient cases that are of most interest in our present context.

1-D acoustic wave equation

The equation describes for example the vibrations of a string (4.6) or acoustic waves in 1-D

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

While keeping u, we can introduce an additional variable v defined by $v_t = c\ u_x$. It then follows that

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{4.14}$$

2-D acoustic wave equation

The governing equation in this case is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

This time, we similarly introduce v and w by $v_t = c\, u_x$ and $w_t = c\, u_y$, and obtain

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = c \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

2-D elastic wave equation

If a 2-D thin plate also possesses elastic properties, the most straightforward derivation of the governing equations (for motions within the $x, y-$plane) leaves them in the form

$$\begin{cases} \frac{\partial u}{\partial t} &= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) / \rho \\ \frac{\partial v}{\partial t} &= \left(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} \right) / \rho \\ \frac{\partial f}{\partial t} &= (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} \\ \frac{\partial g}{\partial t} &= \mu \frac{\partial v}{\partial x} + \mu \frac{\partial u}{\partial y} \\ \frac{\partial h}{\partial t} &= \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu) \frac{\partial v}{\partial y} \end{cases} . \tag{4.15}$$

Here,

$u,\ v$	local displacements in x-and y-directions
$f,\ g,\ h$	local x-compression, shear, and y-compression respectively
$\lambda,\ \mu$	density, and elastic constants (wrt. compression and shear)

The equations (4.15) are directly obtained in first order system form, and it is therefore immediate to express them in terms of matrices

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/\rho & 0 & 0 \\ 0 & 0 & 0 & 1/\rho & 0 \\ \lambda + 2\mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \\ f \\ g \\ h \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1/\rho & 0 \\ 0 & 0 & 0 & 0 & 1/\rho \\ 0 & \lambda & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \\ 0 & \lambda + 2\mu & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u \\ v \\ f \\ g \\ h \end{bmatrix}.$$

3-D Maxwell's equations

We obtained these equations directly in the form of a first order system (4.7). In the case that ε and μ are constants, we can re-arrange them into higher order equations for each individual field component (However, this is seldom a useful thing to do since, in most applications, material interfaces, conductors, etc. are present. The main task is often to understand the influence these will have. Mainly out of mathematical interest, we will do this re-arrangement in two different ways.

Differentiation of the governing equations Differentiation of the first and the two last equations in (??) with respect to t , z and y respectively gives $(E_x)_{tt} = \frac{1}{\varepsilon}\left((H_z)_{ty} - (H_y)_{tz}\right) = \frac{1}{\varepsilon\mu}\left((E_x)_{yy} + (E_x)_{zz}\right) - \frac{1}{\mu\varepsilon}\left((E_y)_{yx} + (E_z)_{zx}\right)$. Since the electrical field is divergence free $0 = div\ E = (E_x)_x + (E_y)_y + (E_z)_z$, we also have $0 = \frac{\partial}{\partial x} div\ E = (E_x)_{xx} + (E_y)_{yx} + (E_z)_{zx}$. With the last result, the expression for $(E_x)_{tt}$ simplifies to

$$(E_x)_{tt} = \frac{1}{\mu\varepsilon}\left((E_x)_{xx} + (E_x)_{yy} + (E_x)_{zz}\right).$$

Similarly, all the other components of the electric and magnetic fields will each satisfy the 3-D acoustic wave equation (to repeat ourselves, on the very restrictive assumption that ε and μ are constants).

By vector algebra Writing (??) as

$$\frac{\partial}{\partial t}\underline{E} = \frac{1}{\varepsilon}\left(\nabla \times \underline{H}\right) \quad , \quad \frac{\partial}{\partial t}\underline{H} = -\frac{1}{\mu}\left(\nabla \times \underline{E}\right) \tag{4.16}$$

we get

$$\begin{aligned}
\frac{\partial^2 \underline{E}}{\partial t^2} &= \frac{1}{\varepsilon}\left(\nabla \times \frac{\partial \underline{H}}{\partial t}\right) && \text{by (4.16)}\\
&= -\frac{1}{\varepsilon\mu}\left(\nabla \times (\nabla \times \underline{E})\right) && \text{by (4.16)}\\
&= -\frac{1}{\varepsilon\mu}\left(\nabla(\nabla \underline{E}) - \nabla^2 \underline{E}\right) && \text{by the vector identity (...)}\\
&= \frac{1}{\varepsilon\mu}\nabla^2 \underline{E} && \text{since } \nabla \underline{E} = 0
\end{aligned}$$

Similarly,

$$\frac{\partial^2 \underline{H}}{\partial t^2} = -\frac{1}{\varepsilon\mu}\nabla^2 \underline{H},$$

implying again that each component of both fields will satisfy the 3-D acoustic wave equation.

4.4.3 Determination of wave types and speeds

First order formulations are particularly well suited for numerical solution. They also provide a good approach to some analytical work. We next re-visit the examples in the previous section in order to determine what kinds of waves they support.

1-D acoustic wave equation

We look for a translating solution to (4.14) of the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u(t-\alpha x) \\ v(t-\alpha x) \end{bmatrix},$$

i.e. for waves with the speed $1/\alpha$. Then $\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}'$ and $\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} = -\alpha \begin{bmatrix} u \\ v \end{bmatrix}'$. Substituting into (4.14) gives

$$\begin{bmatrix} u \\ v \end{bmatrix}' = -c\alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}', \tag{4.17}$$

which we can write as

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}' = \frac{-1}{c\alpha} \begin{bmatrix} u \\ v \end{bmatrix}'. \tag{4.18}$$

We recognize this as an eigenvalue problem. Since the matrix has eigenvalues ± 1, we get $\alpha = \pm\frac{1}{c}$ and we can conclude that (4.14) admits translating solutions with speeds $1/\alpha_{1,2} = \pm c$

When we next turn to more space dimensions, (4.18) will generalize in a way that no longer allows this interpretation as an eigenvalue problem. That difficulty can be avoided by noting that (4.17) alternatively can be written as

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system has a non-trivial (non-zero) solution if and only if $\begin{bmatrix} 1 & c\alpha \\ c\alpha & 1 \end{bmatrix}$ is singular. From

$$0 = \det \begin{bmatrix} 1 & c\alpha \\ c\alpha & 1 \end{bmatrix} = 1 - c^2\alpha^2$$

follows again $\alpha_{1,2} = \pm 1/c$, etc.

2-D acoustic wave equation

We confine ourselves again to look for translating solutions, now of the form

$$\begin{bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \end{bmatrix} = \begin{bmatrix} u(t-\alpha x-\beta y) \\ v(t-\alpha x-\beta y) \\ w(t-\alpha x-\beta y) \end{bmatrix}.$$

This solution moves in the $(\alpha,\beta)-$direction with the velocity $\frac{1}{\sqrt{\alpha^2+\beta^2}}$. From

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}', \quad \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = -c\alpha \begin{bmatrix} u \\ v \\ w \end{bmatrix}', \quad \frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = -c\beta \begin{bmatrix} u \\ v \\ w \end{bmatrix}'$$

follows

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}' = -c\alpha \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}' - c\beta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}'$$

with non-trivial solutions only if

$$0 = \det \begin{bmatrix} 1 & c\alpha & c\beta \\ c\alpha & 1 & 0 \\ c\beta & 0 & 1 \end{bmatrix} = 1 - c^2(\alpha^2+\beta^2).$$

This shows that there is no 'preferred direction' in the $(x,y)-$plane. There are solutions which translate with speed c in any direction.

2-D elastic wave equation

Analogously to the previous case, looking for translating solutions leads us to consider

$$
\begin{aligned}
0 = \det \begin{bmatrix} 1 & 0 & \alpha/\rho & \beta/\rho & 0 \\ 0 & 1 & 0 & \alpha/\rho & \beta/\rho \\ (\lambda+2\mu)\alpha & \lambda\beta & 1 & 0 & 0 \\ \mu\beta & \mu\alpha & 0 & 1 & 0 \\ \lambda\alpha & (\lambda+2\mu)\beta & 0 & 0 & 1 \end{bmatrix} \\
= \left[\mu(\alpha^2+\beta^2)-\rho\right]\cdot\left[(\lambda+2\mu)(\alpha^2+\beta^2)-\rho\right]/\rho^2 \,.
\end{aligned}
$$

There are now two types of possible waves, both with velocities that are direction independent:

$$
\begin{array}{ll} P \text{ (pressure or primary) wave:} & c_p = \left[(\lambda+2\mu)/\rho\right]^{1/2} \\ S \text{ (shear or secondary) wave:} & c_s = \left[\mu/\rho\right]^{1/2} \end{array} \,.
$$

In many materials (such as for seismic waves in the earth), the two material constants λ and μ are of similar size. It then follows that $c_p/c_s \approx \sqrt{3}$.

3-D Maxwell's equations

Omitting the details, the same approach as above shows that there is exactly one type of wave - again direction independent - and that it travels with the velocity $c = 1/\sqrt{\varepsilon\mu}$. Since ε and μ can be found independently by experiments, this relation offers one possibility for determining the speed of light.

4.5 Analytic solutions of the acoustic wave equation

We have already come across the acoustic wave equation in 1-D, 2-D and 3-D. In n-D it takes the form

$$
\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}\right). \tag{4.19}
$$

This can be written more compactly as $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$ where $u(\underline{x},t)$ is a scalar function of $\underline{x}= (x_1,x_2,\ldots,x_n)$ and, as before, $\nabla = (\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_n})$ i.e. $\nabla^2 = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}$.

Since (4.19) has a second derivative in time, we will need two initial conditions to get a solution started:

$$
\left\{ \begin{array}{ll} u(\underline{x},0) & = f(\underline{x}) \\ u_t(\underline{x},0) & = g(\underline{x}) \end{array} \right. . \tag{4.20}
$$

In the three subsections below, we discuss briefly the general solution to (4.19), (4.20) in 1-D, in n-D, and finally, we show how (4.19) simplifies in the case of radial symmetry. The analytic solutions all assume constant material properties, and they cannot easily be extended to irregular domains. They can still be useful for obtaining general insights about the character of wave equations. However, in most practical situations, numerical solutions are needed.

4.5.1 Acoustic wave equation in 1-D; d'Alembert's solution.

The general solution to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.21}$$

can be written down as

$$u(x,t) = F(x-ct) + G(x+ct) \tag{4.22}$$

where F and G are completely arbitrary functions. Verification that (4.22) satisfies (4.21) is straightforward:

$$u_{xx} = \quad F''(x-ct) + \quad G''(x+ct)$$

and

$$u_{tt} = c^2 F''(x-ct) + c^2 G''(x+ct) = c^2\ u_{xx}$$

Hence (4.21) holds. An easy way to arrive at (4.22) (rather than just verifying it) is by Fourier analysis - cf. the exercises in Part V.

For the common situation when

$$\begin{cases} u(x,0) & = f(x) \\ u_t(x,0) & = g(x) \end{cases} \tag{4.23}$$

are given as initial conditions (rather than $F(x)$ and $G(x)$), we need a way to 'translate' $f(x)$, $g(x)$ over into the functions $F(x)$ and $G(x)$. This is achieved by *d'Alembert's formula*:

The unique solution to (4.21), (4.23) is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(\xi)\ d\xi \right]. \tag{4.24}$$

Derivation of (4.24): As just indicated, the idea for the proof is to convert $f(x)$ and $g(x)$ into $F(x)$ and $G(x)$. From (4.23) and (4.22) follow

$$\begin{cases} u(x,0) = f(x) & = \quad F(x) + \quad G(x) \\ u_t(x,0) = g(x) & = -cF'(x) + c\,G'(x) \end{cases} \tag{4.25}$$

After differentiating the top equation (4.25), we can solve the system for $F'(x)$ and $G'(x)$ in terms of $f(x)$ and $g(x)$:

$$\begin{cases} F'(x) & = \frac{1}{2}\left[f'(x) - \frac{1}{c}\ g(x)\right] \\ G'(x) & = \frac{1}{2}\left[f'(x) + \frac{1}{c}\ g(x)\right] \end{cases}$$

Integrating these give $F(x)$ and $G(x)$

$$\begin{cases} F(x) & = \frac{1}{2}\left[f(x) - \frac{1}{c}\ \int_0^x g(\xi)d\xi\right] + c_1 \\ G(x) & = \frac{1}{2}\left[f(x) + \frac{1}{c}\ \int_0^x g(\xi)d\xi\right] + c_2 \end{cases}$$

and by (4.22)

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) = \\ &= \frac{1}{2}\left[f(x-ct) - \frac{1}{c}\ \int_0^{x-ct} g(\xi)\ d\xi\right] + \frac{1}{2}\left[f(x-ct) + \frac{1}{c}\ \int_0^{x+ct} g(\xi)\ d\xi\right] + c_3 = \\ &= \frac{1}{2}\left[f(x-ct) + f(x+ct) + \frac{1}{c}\ \int_{x-ct}^{x+ct} g(\xi)\ d\xi\right] + c_3 \end{aligned}$$

It remains only to note that setting $t = 0$ shows that the constant $c_3 = 0$.

4.5.2 Acoustic wave equation in n-D.

One can write down formulas in n-D that, completely analogously to d'Alembert's formula in 1-D, give the $u(\underline{x}, t)$ which satisfies (4.19) in terms of the initial conditions (4.20). Rather than doing that, we will here just summarize a few things that these formulas tell us:

1-D: The solution at a point $u(x,t)$ depends on $f(x-ct)$, $f(x+ct)$ and on all the g-values in-between $x-ct$ and $x+ct$. As a consequence, the general solution describes two very different kinds of outgoing wave motions:

1. Cleanly translating pulses (if started by $f(x) \neq 0$ within some small area only, and $g(x) = 0$ everywhere). Such waves leave a zone of perfect silence behind them,

2. Disturbances which spread out throughout a complete interval (whenever $g(x) \neq 0$ initially).

Both of these situations can arise on a tight string: Case 1 if the string is locally deformed to feature a small 'hump' but is otherwise straight, and is then released, and Case 2 if a straight string is lightly hit at one point.

2-D: (and also higher *even* dimensions): The solution $u(\underline{x}, t)$ depends both on f and g everywhere within a distance ct from the point $\underline{x}$. Both solution types (with $f(\underline{x}) \neq 0,\ g(\underline{x}) = 0$ and $f(\underline{x}) = 0,\ g(\underline{x}) \neq 0$ resp.) fall in the second category above, i.e. outgoing signals never leave any zone of silence behind.

3-D: (and also higher *odd* dimensions): The solution $u(\underline{x}, t)$ at the point $\underline{x}$ depends on the values of f and g only on *the surface of a sphere* centered at $\underline{x}$ and with radius ct. As a consequence, however we initiate a sound signal at a point in 3-D, it will leave perfect silence behind itself as it travels out. This very remarkable property makes speech possible in 3-D. These outgoing sound signals attenuate with the distance traveled, but undergo no other changes. The conclusion is that 'clean speech' is possible only in odd dimensions from three and up.

4.5.3 Form of the wave equation in radial symmetry

In the special case of signals emerging from one point and propagating out with equal strength in all directions, the wave equation can be simplified from (4.19) to a form that depends on t and $r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ only:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} \right) \tag{4.26}$$

Derivation of (4.26): The chain rule gives

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\partial u}{\partial r}\, x_i\, \frac{1}{r} \qquad \text{(noting that } r^2 = x_1^2 + x_2^2 + ... + x_n^2 \text{ implies } 2\, r\, \frac{\partial r}{\partial x_i} = 2x_i\text{),}$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\frac{x_i}{r}\right)\frac{x_i}{r} = \frac{\partial^2 u}{\partial r^2}\left(\frac{x_i}{r}\right)^2 + \frac{\partial u}{\partial r}\frac{1}{r} - \frac{\partial u}{\partial r}\frac{x_i^2}{r^3}$$

Summation over i now gives $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r}$, from which (4.26) follows.

4.6 Scalar first order wave equations in n-D; Hamilton's equations.

Analogously to the 1-D acoustic (two-way wave) equation (4.21) with its general solution (4.22), the 1-D one-way wave equation

$$\frac{\partial u}{\partial t} = c\,\frac{\partial u}{\partial x} \tag{4.27}$$

has as its general solution

$$u(x,t) = F(x+ct).$$

This solution remains constant along any straight line $x + ct = const$ in the x,t-plane. In this section, we first generalize (4.27) to several space dimensions

$$\frac{\partial u}{\partial t} = a_1\frac{\partial u}{\partial x_1} + a_2\frac{\partial u}{\partial x_2} + \ldots + a_n\frac{\partial u}{\partial x_n}\ . \tag{4.28}$$

The solution is also in this case constant along a set of straight lines. In the following subsection, we expand on this observation by generalizing the equation further to allow arbitrary nonlinearities. Out of this effort - largely a big exercise in using the chain rule - emerges Hamilton's equations. These equations are of tremendous importance in modeling of a vast number of phenomena in many branches of the sciences. In the context of the present book, they leads us to the equations for modeling freak water waves.

4.6.1 Linear scalar wave equation in n-D

Mathematically, there is little in (4.28) that distinguishes t from any of the space variables $x_1, x_2, \ldots, x_n$ - it enters in almost the same way. Therefore, we can take away the $\frac{\partial u}{\partial t}$ term (and, if we so wish, let its role be assumed by a $\frac{\partial u}{\partial x_k}$ - term for some k). Also permitting the coefficients to be functions of $\underline{x}$ leads us to consider the equation

$$a_1(\underline{x})\frac{\partial u}{\partial x_1} + a_2(\underline{x})\frac{\partial u}{\partial x_2} + \ldots + a_n(\underline{x})\frac{\partial u}{\partial x_n} = 0. \tag{4.29}$$

The solution u to (4.29) stays constant along *characteristic curves* (or *characteristic paths*) which can be described in parameter form $x_1(s),\ x_2(s), ..., x_n(s)$ as follows: Starting at some point $x_1(0),\ x_2(0),\ \ldots\ , x_n(0)$, we let $x_1(s),\ x_2(s),\ \ldots\ , x_n(s)$ evolve according to the coupled systems of ODEs

$$\frac{dx_1}{ds} = a_1(\underline{x}),\ \frac{dx_2}{ds} = a_2(\underline{x}),\ \ldots\ , \frac{dx_n}{ds} = a_n(\underline{x}). \tag{4.30}$$

To verify that the solution indeed is constant along the particular path that (4.30) traces out in $(x_1, x_2, \ldots, x_n)$-space, we evaluate $\dfrac{du}{ds}$ as follows:

$$\begin{aligned}
\frac{du}{ds} &= \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{ds} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{ds} + \ldots + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{ds} = && \text{by chain rule}\\
&= \frac{\partial u}{\partial x_1}\cdot a_1(\underline{x}) + \frac{\partial u}{\partial x_2}\cdot a_2(\underline{x}) + \ldots + \frac{\partial u}{\partial x_n}\cdot a_n(\underline{x}) = && \text{by path definition (4.30)}\\
&= 0 && \text{by the ODE (4.29)}
\end{aligned}$$

4.6.2 Nonlinear scalar wave equation in n-D

We replace next (4.29) with a completely general (differentiable) relationship involving u and $x_i,\ \frac{\partial u}{\partial x_i},\ i = 1, 2, \ldots, n$:

$$F(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}) = 0$$

or

$$F(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) = 0 \tag{4.31}$$

where we have used $p_1, p_2, \ldots, p_n$ to denote $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}$ respectively. The equations for the characteristic curves (generalizing how (4.30) was obtained from (4.29)) become now

$$\frac{\partial x_i}{\partial s} = \frac{\partial F}{\partial p_i}, \quad i = 1, 2, \ldots, n. \tag{4.32}$$

Along these paths, $\dfrac{du}{ds}$ will no longer simplify all the way to zero, but we will nevertheless get a quite simple ODE for the evolution of u:

$$\frac{du}{ds} = \sum_{j=1}^{n} p_j \, \frac{\partial F}{\partial p_j} \quad . \tag{4.33}$$

Along the path, we need this time also to know how $p_1, p_2, \ldots, p_n$ change. There turns out to be simple ODEs for that as well:

$$\frac{\partial p_i}{\partial s} = -\,\frac{\partial F}{\partial u}\, p_i - \frac{\partial F}{\partial x_i}, \quad i = 1, 2, \ldots, n. \tag{4.34}$$

The equations (4.32)-(4.34) form a set of $2n + 1$ coupled ODEs with s as the independent variable. They can be solved with most standard numerical ODE solvers, providing paths through $(x_1, x_2, \ldots, x_n)$-space and, along these paths, values for u and also for its derivatives $p_1, p_2, \ldots, p_n$.

Equation (4.32) is a definition, but equations (4.33) and (4.34) need to be proven. This can be done as follows for (4.33):

$$\begin{aligned}
\frac{du}{ds} &= \textstyle\sum_{j=1}^{n} \dfrac{\partial u}{\partial x_j}\, \dfrac{\partial x_j}{\partial s} = && \text{by chain rule on } u \\
&= \textstyle\sum_{j=1}^{n} p_j \, \dfrac{\partial F}{\partial p_j} && \text{by the definition of the path (4.32) and of } p_j = \tfrac{\partial u}{\partial x_j}.
\end{aligned}$$

and for (4.34):

$$\begin{aligned}
0 &= \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x_i} + \textstyle\sum_{j=1}^{n} \dfrac{\partial F}{\partial p_j}\dfrac{\partial p_j}{\partial x_i} = && \text{differentiate } F \text{ with respect to } x_i \text{, apply the chain rule} \\
&= \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u}\, p_i + \textstyle\sum_{j=1}^{n} \dfrac{\partial x_j}{\partial s}\dfrac{\partial^2 u}{\partial x_i \partial x_j} = && \text{by (4.32)} \\
&= \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u}\, p_i + \textstyle\sum_{j=1}^{n} \dfrac{\partial x_j}{\partial s}\dfrac{\partial p_i}{\partial x_j} = && \text{swap order in second derivative terms and use again } p_j = \tfrac{\partial u}{\partial x_j} \\
&= \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u}\, p_i + \frac{\partial p_i}{\partial s} && \text{by chain rule on } p_i.
\end{aligned}$$

4.6.3 Hamilton's equations

A particularly important special case of the set of equations (4.32) and (4.33) arises when (4.31) does not depend on u. Employing a slightly different notation (to be consistent with what is more commonly used in this case), we write (4.31)

$$H(\underline{x}, \underline{p}) = 0 \tag{4.35}$$

where $\underline{p}$ is a gradient of some scalar function $\phi(\underline{x})$:

$$\underline{p} = \nabla\phi(\underline{x}) \tag{4.36}$$

Calling the parameter t instead of s (since it in applications often turns out to correspond to physical time), the equations (4.32) and (4.34) show that $x(\underline{t})$ and $p(\underline{t})$ satisfies the *Hamilton's* system of equations

$$\left\{ \begin{aligned} \frac{\partial x_i}{\partial t} &= \frac{\partial H}{\partial p_i} \\ \frac{\partial p_i}{\partial t} &= -\frac{\partial H}{\partial x_i} \end{aligned} \right. \quad , \ i = 1, 2, \ldots, n. \tag{4.37}$$

Since $H(\underline{x}, \underline{p})$ is assumed to be a known function, the RHSs in (4.37) are explicitly known, and these equations amount to a system of ODEs.

Relations of the form (4.35), (4.36) arise very frequently in areas such as classical mechanics, nonlinear wave motion, and chaos. Equation (4.35) then express a conservation law, such as conservation of energy. Assuming we have managed to find a formulation such that (4.36) also holds, then the equations (4.37) become available, and can be solved to provide the evolution of the $\underline{x}$ and $\underline{p}$ - variables.

The procedure of moving from (4.35) and (4.36) over to the system (4.37) offers a very important approach for obtaining equations that are suitable for numerical solution. We will next give several examples of this.

Example 1: ...

Example 2: ...

Example 3: ...

Example 4: ...