Wallis' Formula and Stirling's Formula

In class we used Stirling's Formula

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}.$$

Here, "$\sim$" means that the ratio of the left and right hand sides will go to 1 as $n \to \infty$.

We will prove Stirling's Formula via the Wallis Product Formula.

Wallis' Product Formula

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2}$$

Proof of Wallis Product Formula:

- Define $c_n = \int_0^{\pi/2} (\sin x)^n \, dx$. Note that $c_0 = \pi/2$ and $c_1 = 1$.
- Integrating by parts using $u = (\sin x)^n$ and $dv = \sin x \, dx$ gives

$$\begin{aligned}
c_n &= -(\sin x)^{n-1} \cos x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos^2 x \, dx \\
&= \underbrace{-(\sin x)^{n-1} \cos x \Big|_0^{\pi/2}}_{0} + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) \, dx \\
&= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) \, dx \\
&= (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \, dx - (n-1) \int_0^{\pi/2} (\sin x)^n \, dx
\end{aligned}$$

  So, we have that

$$c_n = (n-1)\, c_{n-2} - (n-1)\, c_n,$$

  which implies that

$$c_n = \frac{n-1}{n} c_{n-2}.$$

- Recalling that $c_0 = \pi/2$ and $c_1 = 1$, this recursion gives us

$$c_{2n} = \frac{\pi}{2} \prod_{i=1}^{n} \frac{2k-1}{2k}$$

  and

$$c_{2n+1} = \prod_{i=1}^{n} \frac{2k}{2k+1}.$$

- Looking back at the original definition of c_n, we see that, since $0 \leq \sin x \leq 1$ on the interval of integration, $\{c_n\}$ is a decreasing sequence.
- Thus, we have that

$$1 \leq \frac{c_{2n}}{c_{2n+1}} \leq \frac{c_{2n-1}}{c_{2n+1}} = \frac{c_{2(n-1)+1}}{c_{2n+1}} = \frac{\prod_{k=1}^{n-1} \frac{2k}{2k+1}}{\prod_{k=1}^{n} \frac{2k}{2k+1}} = \frac{2n+1}{2n} = 1 + \frac{1}{2n}.$$

- So, we have that

$$1 = \lim_n \frac{c_{2n}}{c_{2n+1}} = \lim_{n\to\infty} \frac{\frac{\pi}{2}\prod_{i=1}^{n} \frac{2k-1}{2k}}{\prod_{i=1}^{n} \frac{2k}{2k+1}} = \frac{\pi}{2} \lim_{n\to\infty} \prod_{i=1}^{n} \frac{2k-1}{2k} \cdot \frac{2k+1}{2k}$$

which implies that

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \lim_{n\to\infty} \prod_{i=1}^{n} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} = \frac{\pi}{2}$$

as desired.

Stirling's Formula

We want to show that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} = 1. \tag{1}$$

Part A: First, we will show that the left-hand side of (1) converges to something without worrying about what it is converging to. For this, we can ignore the $\sqrt{2\pi}$.

Define

$$a_n := \frac{n!}{n^{n+1/2} e^{-n}}.$$

Define

$$b_n := \ln a_n.$$

We are going to show that that $\{b_n\}$ is a decreasing sequence that is bounded below. This implies that $\lim b_n$ exists and therefore that $\lim a_n$ exists.

Note that

$$b_n - b_{n+1} = \cdots = \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1.$$

We want to show that this is positive. To this end, it will be helpful for us to write a Taylor series expansion for $\ln\left(\frac{n+1}{n}\right)$. A "direct approach" will result in a series with alternating signs and it will not be clear whether it converges to something positive or negative. So, we will instead try to be "clever".

Consider, for $|t| < 1$, the Taylor series expansions

$$\begin{aligned} \ln(1+t) &= t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 + \cdots \\ \ln(1-t) &= -t - \tfrac{1}{2}t^2 - \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 - \cdots \end{aligned}$$

Now,

$$\begin{aligned} \ln\left(\tfrac{1+t}{1-t}\right) &= \ln(1+t) - \ln(1-t) \\ &= 2t + \tfrac{2}{3}t^3 + \tfrac{2}{5}t^5 + \cdots \\ &= 2\left[t + \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + \cdots\right] \\ &= 2\textstyle\sum_{k=0}^{\infty} \tfrac{1}{2k+1} t^{2k+1}. \end{aligned}$$

Note that we can get $(1+t)/(1-t)$ to equal $(n+1)/n$ if we take $t = 1/(2n+1)$. So, we have

$$\ln\left(\frac{n+1}{n}\right) = 2\sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k+1}.$$

Now we have

$$\begin{aligned} b_n - b_{n+1} &= \left(n + \tfrac{1}{2}\right)\ln\left(\tfrac{n+1}{n}\right) - 1 \\ &= (2n+1)\textstyle\sum_{k=0}^{\infty} \tfrac{1}{2k+1}\left(\tfrac{1}{2n+1}\right)^{2k+1} - 1 \\ &= \textstyle\sum_{k=0}^{\infty} \tfrac{1}{2k+1}\left(\tfrac{1}{2n+1}\right)^{2k} - 1 \\ &= \textstyle\sum_{k=1}^{\infty} \tfrac{1}{2k+1}\left(\tfrac{1}{2n+1}\right)^{2k} \end{aligned}$$

This is clearly positive. So, we have shown that

$$b_n > b_{n+1}$$

which implies that $\{b_n\}$ is a decreasing sequence.

We will now show that the sequence is bounded below. Note that

$$\begin{aligned} b_n - b_{n+1} &= \textstyle\sum_{k=1}^{\infty} \tfrac{1}{2k+1}\left(\tfrac{1}{2n+1}\right)^{2k} \\ &< \textstyle\sum_{k=1}^{\infty} \left(\tfrac{1}{2n+1}\right)^{2k} \\ &= \textstyle\sum_{k=1}^{\infty} \left(\tfrac{1}{(2n+1)^2}\right)^{k} \\ &= \tfrac{1}{(2n+1)^2}\,\tfrac{1}{1-\frac{1}{(2n+1)^2}} \quad \text{(geometric sum)} \\ &= \tfrac{1}{(2n+1)^2-1} = \tfrac{1}{4n(n+1)}. \end{aligned}$$

Now

$$\begin{aligned} b_1 - b_n &= (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_{n-1} - b_n) \\ &< \tfrac{1}{4}\textstyle\sum_{k=1}^{n-1} \frac{1}{k(k+1)} < \tfrac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \tfrac{1}{4}. \end{aligned}$$

(To do that last sum write $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ to get a nice telescoping sum.)

We now have that

$$b_n > b_1 - \frac{1}{4}.$$

By definition of a_n we have that $a_1 = e$ and therefore that $b_n = \ln e = 1$.

So,

$$b_n > \frac{3}{4} \quad \forall\, n \geq 1.$$

Since $\{b_n\}$ is decreasing and bounded below, $\lim\limits_{n\to\infty} b_n$ exists, which implies that $\lim\limits_{n\to\infty} a_n$ exists.

Part B: We now know that a_n converges to some constant. Therefore $\frac{1}{\sqrt{2}}a_n$ converges to some constant. Let's call it c.

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2}n^{n+1/2}e^{-n}} = c$$

for some constant c, or, equivalently,

$$\lim_{n\to\infty} \frac{n!}{c\sqrt{2}n^{n+1/2}e^{-n}} = 1.$$

In other "words"

$$n! \sim c\sqrt{2}n^{n+1/2}e^{-n}.$$

If we can show that $c = \sqrt{\pi}$, we are done.

Wallis' Formula may be rewritten as

$$\lim_{n\to} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2}.$$

Now

$$\frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} \sim \frac{2^{4n}(c\sqrt{2}n^{n+1/2}e^{-n})^4}{(c\sqrt{2}(2n)^{2n+1/2}e^{-2n})^2(2n+1)}.$$

So,

$$\begin{aligned} \frac{\pi}{2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} &= \lim_{n\to\infty} \frac{2^{4n}(c\sqrt{2}n^{n+1/2}e^{-n})^4}{(c\sqrt{2}(2n)^{2n+1/2}e^{-2n})^2(2n+1)} \\ &= \cdots c^2 \lim_{n\to\infty} \frac{n^2}{n(2n+1)} = \frac{c^2}{2} \end{aligned}$$

which implies that $c = \sqrt{\pi}$, as desired.