Wallis’ Formula and Stirling’s Formula

In class we used Stirling’s Formula

\[ n! \sim \sqrt{2\pi n^{n+1/2}} e^{-n}. \]

Here, “\( \sim \)” means that the ratio of the left and right hand sides will go to 1 as \( n \to \infty \).

We will prove Stirling’s Formula via the Wallis Product Formula.

Wallis’ Product Formula

\[
\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2}
\]

Proof of Wallis Product Formula:

- Define \( c_n = \int_0^{\pi/2} (\sin x)^n \, dx \). Note that \( c_0 = \pi/2 \) and \( c_1 = 1 \).
- Integrating by parts using \( u = (\sin x)^n \) and \( dv = \sin x \, dx \) gives
  \[
  c_n = -(\sin x)^{n-1} \cos x \bigg|_0^{\pi/2} + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos^2 x \, dx \\
  = -(\sin x)^{n-1} \cos x \bigg|_0^{\pi/2} + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) \, dx \\
  = (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) \, dx \\
  = (n-1) \int_0^{\pi/2} (\sin x)^{n-2} dx - (n-1) \int_0^{\pi/2} (\sin x)^n \, dx
  \]

So, we have that

\[ c_n = (n-1) c_{n-2} - (n-1) c_n, \]

which implies that

\[ c_n = \frac{n-1}{n} c_{n-2}. \]

- Recalling that \( c_0 = \pi/2 \) and \( c_1 = 1 \), this recursion gives us
  \[ c_{2n} = \frac{\pi}{2} \prod_{i=1}^{n} \frac{2k-1}{2k} \]

and
  \[ c_{2n+1} = \prod_{i=1}^{n} \frac{2k}{2k+1}. \]
• Looking back at the original definition of $c_n$, we see that, since $0 \leq \sin x \leq 1$ on the interval of integration, $\{c_n\}$ is a decreasing sequence.

• Thus, we have that

$$1 \leq \frac{c_{2n}}{c_{2n+1}} \leq \frac{c_{2n-1}}{c_{2n+1}} = \frac{c_{2(n-1)+1}}{c_{2n+1}} = \prod_{k=1}^{n-1} \frac{2k}{2k+1} = \frac{2n+1}{2n} = 1 + \frac{1}{2n}.$$ 

• So, we have that

$$1 = \lim_{n \to \infty} \frac{c_{2n}}{c_{2n+1}} = \lim_{n \to \infty} \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k+1} = \frac{\pi}{2} \lim_{n \to \infty} \prod_{k=1}^{n} \frac{2k-1}{2k+1} \cdot \frac{2k+1}{2k}$$

which implies that

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n+2}{2n+1} = \lim_{n \to \infty} \prod_{k=1}^{n} \frac{2k-1}{2k+1} \cdot \frac{2k+1}{2k} = \frac{\pi}{2}$$

as desired.

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**Stirling’s Formula**

We want to show that

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n^{n+1/2}e^{-n}}} = 1. \quad (1)$$

**Part A:** First, we will show that the left-hand side of (1) converges to something without worrying about what it is converging to. For this, we can ignore the $\sqrt{2\pi}$.

Define

$$a_n := \frac{n!}{n^{n+1/2}e^{-n}}.$$ 

Define

$$b_n := \ln a_n.$$ 

We are going to show that that $\{b_n\}$ is a decreasing sequence that is bounded below. This implies that $\lim b_n$ exists and therefore that $\lim a_n$ exists.

Note that

$$b_n - b_{n+1} = \cdots = \left(n + \frac{1}{2}\right) \ln \left(1 + \frac{1}{n}\right) - 1.$$ 

We want to show that this is positive. To this end, it will be helpful for us to write a Taylor series expansion for $\ln \left(1 + \frac{1}{n}\right)$. A “direct approach” will result in a series with alternating signs and it will not be clear whether it converges to something positive or negative. So, we will instead try to be “clever”.

Consider, for $|t| < 1$, the Taylor series expansions

$$\ln(1 + t) = t - \frac{1}{2} t^2 + \frac{1}{3} t^3 - \frac{1}{4} t^4 + \cdots$$

$$\ln(1 - t) = -t - \frac{1}{2} t^2 - \frac{1}{3} t^3 - \frac{1}{4} t^4 - \cdots$$

Now,

$$\ln\left(\frac{1+t}{1-t}\right) = \ln(1 + t) - \ln(1 - t)$$

$$= 2t + \frac{2}{3} t^3 + \frac{2}{5} t^5 + \cdots$$

$$= 2 \left[ t + \frac{1}{3} t^3 + \frac{1}{5} t^5 + \cdots \right]$$

$$= 2 \sum_{k=0}^{\infty} \frac{1}{2k+1} t^{2k+1}.$$ 

Note that we can get $(1 + t)/(1 - t)$ to equal $(n + 1)/n$ if we take $t = 1/(2n + 1)$. So, we have

$$\ln\left(\frac{n+1}{n}\right) = 2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{1}{2n+1}\right)^{2k+1}.$$ 

Now we have

$$b_n - b_{n+1} = \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1$$

$$= (2n + 1) \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{1}{2n+1}\right)^{2k+1} - 1$$

$$= \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{1}{2n+1}\right)^{2k} - 1$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{1}{2n+1}\right)^{2k}$$

This is clearly positive. So, we have shown that

$$b_n > b_{n+1}$$

which implies that \(\{b_n\}\) is a decreasing sequence.

We will now show that the sequence is bounded below. Note that

$$b_n - b_{n+1} = \sum_{k=1}^{\infty} \frac{1}{2k+1} \left(\frac{1}{2n+1}\right)^{2k}$$

$$< \sum_{k=1}^{\infty} \left(\frac{1}{2n+1}\right)^{2k}$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{(2n+1)^2}\right)^k$$

$$= \frac{1}{1-\frac{1}{(2n+1)^2}} - 1 \quad \text{(geometric sum)}$$

$$= \frac{1}{(2n+1)^2 - 1} = \frac{1}{4n(n+1)}.$$
Now
\[ b_1 - b_n = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_{n-1} - b_n) \]
\[ < \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} < \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{4}. \]
(To do that last sum write \( \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \) to get a nice telescoping sum.)

We now have that \( b_n > b_1 - \frac{1}{4} \).

By definition of \( a_n \) we have that \( a_1 = e \) and therefore that \( b_n = \ln e = 1 \).

So, \( b_n > \frac{3}{4} \forall n \geq 1 \).

Since \( \{b_n\} \) is decreasing and bounded below, \( \lim_{n \to \infty} b_n \) exists, which implies that \( \lim_{n \to \infty} a_n \) exists.

**Part B:** We now know that \( a_n \) converges to some constant. Therefore \( \frac{1}{\sqrt{2}} a_n \) converges to some constant. Let’s call it \( c \).

\[ \lim_{n \to \infty} \frac{n!}{\sqrt{2}n^{n+1/2}e^{-n}} = c \]
for some constant \( c \), or, equivalently,

\[ \lim_{n \to \infty} \frac{n!}{c\sqrt{2}n^{n+1/2}e^{-n}} = 1. \]

In other “words”

\[ n! \sim c\sqrt{2}n^{n+1/2}e^{-n}. \]

If we can show that \( c = \sqrt{\pi} \), we are done.

Wallis’ Formula may be rewritten as

\[ \lim_{n \to \infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2}. \]

Now

\[ \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} \sim \frac{2^{4n}(c\sqrt{2}n^{n+1/2}e^{-n})^4}{(c\sqrt{2}(2n)^{2n+1/2}e^{-2n})^2(2n+1)}. \]

So,

\[ \frac{\pi}{2} = \lim_{n \to \infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \lim_{n \to \infty} \frac{2^{4n}(c\sqrt{2}n^{n+1/2}e^{-n})^4}{(c\sqrt{2}(2n)^{2n+1/2}e^{-2n})^2(2n+1)} \]
\[ = \cdots c^2 \lim_{n \to \infty} \frac{n^2}{n(2n+1)} = \frac{c^2}{2} \]
which implies that \( c = \sqrt{\pi} \), as desired.