Brief introduction to topology, supplementary notes
APPM 5440 Fall 2014 Applied Analysis

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Abstract

Version 1, 10/6/2014. The main sources are unpublished notes from László Fejér, 2003. He suggests the book of Munkres for more details. For a readable introduction to topology, including algebraic topology, see Armstrong's *Basic Topology* (Springer-Verlag 1983). Many older books, such as Ward's *Topology* (1972) and Kelley's *General Topology* (1955) are only point-set topology and are really generalized analysis; Kelley sub-titles his book "What every young analyst should know."

1 Description of topology

The idea of topology is to study shapes of things, but it is different from geometry in that there is "no distance." In this sense it is more "fundamental", as the questions seem more basic: do two objects have the same "basic" properties even if they are not exactly the same size? Could you deform one object into another in a "continuous" manner?

Some types of problems that topologists ask:

- The two pancake problem: suppose you have two pancakes on a plate. Find a straight cut halving both.
- The sleeping snake: our snake sleeps in a tubelike bed of the same length as the snake. The snake goes to sleep straight but has nightmares and wakes up all twisted. Show that there is a point on the snake which wakes up at the spot it went to sleep.
- A polynomial of odd order has a root.
- Is there a good map of Earth? If it is one piece, then it is extremely distorted at the poles. Otherwise you need two pieces.
- When can you untie a knot? For a mathematician, a knot has no ends, it is the image of a circle in space without self-intersections.
- How to distinguish knots?
- Can you brush a hedgehog? Is there a nowhere zero vector field on the sphere? Equivalently, is it possible that the wind blows at every point on the surface of the Earth at the same time? (The Earth being a big sphere and the wind always blowing tangentially to the surface)
- The Dirac experiment: if you rotate an object 360° then things (or your hands) get twisted, but if you do it once more the strings get untwisted. See "Dirac scissor experiment" at http://www.youtube.com/watch?v=17Q0tJZcsnY.

Broadly speaking, topology separates into

- **point-set topology**, which studies basic properties like continuity, compactness, etc., much like we covered in chapter 1 on metric spaces, and

- **algebraic topology**, which seeks to classify topological structures into algebraic groups. For example, the famous **Poincaré conjecture** is of this form. It is: "Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere."

2 A little bit of point-set topology

For a set X, we create a set of sets $\mathcal{T}$ (the notation $\mathcal{U}$ is also common) that define the **open sets** of a **topology** on X, as long as the three following rules are satisfied:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ (and if these are the only sets in $\mathcal{T}$, we call it the *trivial* or *indiscrete* topology which is very "coarse"
2. $U, V \in \mathcal{T}$ implies $U \cap V \in \mathcal{T}$.
3. For any (possibly infinite or even uncountable) index set I, if $U_i \in \mathcal{T}$ then $\cup_{i \in I} U_i \in \mathcal{T}$.

The **closed** sets are those whose complements are open. The smallest closed set containing a set A is the **closure** of A, denoted $\bar{A}$, and the largest open set inside A is the interior of A, denoted $\text{int}(A)$. The difference between the closure and interior is the **boundary** of the set, denoted ∂A.

Just as we can have different metrics on the same space X, we can have different topologies, hence we refer to a topological space as the pair $(X, \mathcal{T})$. If we have a metric d on a space, it induces a topology using the open balls, and we denote this topology by $\mathcal{T}_d$. The "largest" (or "finest") possible topology on any space X is the **discrete** topology, where every element in X is open; this is induced by the discrete metric. A topology is **metrizable** if there is some metric that induces it (this means that open sets are defined by sets U such that for all $x \in U$, there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset U$, where the ball is defined by the particular metric). The trivial topology is not metrizable whenever X contains more than 2 points.

We define a function as **continuous** if the inverse image of an open set is open (alternatively, see Definition 4.6 in the textbook). We say two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are **homeomorphic** if there is a *continuous, bijective* function $f : X \to Y$ such that f^{-1} is also continuous (and such a function is a **homeomorphism**). N.B. do not confuse homeomorphism with the general concept of a **homomorphism**.

In metric spaces, we can define open sets using open balls, which is more convenient than just listing *all* open sets! In general we can find a few sets such that we can manipulate these sets to give us all open sets, which is what the notion of a **base** is (N.B. not the same as a **basis** in linear algebra! In our book, they use the terminology "base" though other references may call this a "basis"). We say that a set of subsets of X, called $\mathcal{B} \subset \mathcal{P}(X)$, is a **base** for a topology $\mathcal{T}$ if $\mathcal{T} = \{\cup_{i \in I} B_i \mid B_i \in \mathcal{B}\}$. For example, in a metric space, the collection of all open balls (of all radii, around all points) is a base; the collection of all singleton sets is a base for the discrete topology.

An even smaller set that can still induce the whole topology is known as a *sub-base*, and we say that a set $S \subset \mathbb{P}(X)$ is a sub-base if the set of all possible finite intersections of S forms a base and S covers X.

We may think of something like the **intermediate value theorem** as inherent to metric spaces, but we can actually formulate a version on topological spaces. We say a space $(X, \mathcal{T})$ has the **intermediate value property** (IMV) if for any continuous function $f : X \to \mathbb{R}$, the image of X is a (possibly infinite) interval, e.g., $[a, b]$ or (a, ∞), etc. We can say more about this. First, declare a space $(X, \mathcal{T})$ to be **connected** if for any nonempty $U, V \in \mathcal{T}$ such that $U \cup V = X$, we have $U \cap V \neq \emptyset$. (Equivalently, we can let U, V be closed). For example, if $X = (-\infty, 0) \cup (1, +\infty)$, with the usual topology (i.e., the topology induced by the Euclidean norm), then X is not connected, which agrees with our intuition.

Then we have results like (1) $(X, \mathcal{T})$ is connected iff the only open and closed sets ("clopen" sets) are $\emptyset$ and X, and (2) a topological space has the IMV property iff it is connected, and (3) the continuous image of a connected space is continuous. This reassures us that our formal notions of "connected" and "continuous" agree with our intuition.

We can even define the notion of convergence if a sequence without using a metric! Let (x_n) be a sequence, then we say that $x_n \to x$ or (x_n) **converges** to x, if for every open set U containing x, the set

$\{x_n \mid x_n \notin U\}$ is finite. Under the trivial topology, you can convince yourself that every sequence converges to every point! So in this case, the idea of convergence is not so useful. Since we can speak of convergence, we can define a function to be **sequentially continuous** if $x_n \to x$ implies $f(x_n) \to f(x)$. If a function is continuous, it is sequentially continuous, but the converse may not be true in a general topological space.

Now, topologies can be quite nasty, and not metrizable. To add some order, we can impose various **separation axioms**. We say that a space is $\mathbf{T_0}$ if for any two distinct points, there is an open set containing one of them but not the other, so this precludes the trivial topology. A stronger notion (it implies T_0) is the idea of a $\mathbf{T_1}$ space, which means for any two distinct points x and y, there is an open set contain x but not y, and another open set containing y but not x; a space is T_1 iff every point is closed. And finally we say a space is $\mathbf{T_2}$ (or **Hausdorff**) if for any two distinct points x and y, there are disjoint open sets $U, V \in \mathcal{T}$ with $x \in U$ and $y \in V$.

The good news is that **in a Hausdorff space, a sequence has at most one limit**. Also, every metric space (and its generated topology) is Hausdorff. Non-Hausdorff spaces are rare, but they do occur, as in algebraic geometry. Let $X = \mathbb{R}^n$ and call $F \subset X$ closed if there are polynomials $\{p_i \mid i \in I\}$ of n variables such that $F = \{x \in X \mid p_i(x) = 0 \, \forall i \in I\}$. These sets F are called **algebraic** sets, and this topology is called the **Zariski** topology.

We can add even more structure by adding in some **countability axioms**. We say a space is $\mathbf{M_1}$ or **first countable** if for every $x \in X$ there exits open sets $\{U_n \mid n \in \mathbb{N}\}$ containing x such that for all open sets V containing x, there is some n such that $U_n \subset V$. These spaces are nice since a function defined on a first countable space is **continuous iff sequentially continuous**. It also means that in a first countable space, **closed** sets can be defined as **sets that include their limit points**. Not surprisingly, every metric space is first countable. We also define a space to be $\mathbf{M_2}$ or **second countable** if it has a countable basis. In $\mathbb{R}^n$, the usual Euclidean metric induces a M_2 topology, but the discrete metric only induces a M_1 topology.

Finally, we get around to **compactness**, where a space is compact if every open cover has a finite sub cover. In a Hausdorff space, compact subsets are also closed. In general, we still have that if f is continuous, then it maps compact sets to compact sets, and if $f : X \to \mathbb{R}$ where X is compact, then f is bounded and attains its maximum/minimum. If the space is M_1, then **compact iff sequentially compact**. We define $x \in X$ to be an **accumulation point** of a set A if $A \cap (U \setminus x)$ is non-empty for all open sets U containing x. A set X is compact implies that every infinite set has an accumulation point.

A final result, which uses the axiom of choice in its proof (see the proof of Urysohn's metrization theorem for details) is that if X is compact, Hausdorff and M_2, then it is metrizable. Of course being compact is not necessary for being metrizable in general, but being compact lets us know when we can characterize it as such.

In general, we have **Urysohn's metrization theorem**, which is that every Hausdorff (T1), second-countable (M2), regular space (T3) is metrizable. A *regular* space is another separation condition (if Hausdorff, know as T3). There are a few other standard conditions, such as T4 (*normal* Hausdorff), T5 (*completely normal* Hausdorff), and T6 (*perfectly normal* Hausdorff), as well as Tychonoff spaces.

3 A bit of algebraic topology

We want to discuss the topology of the **space of continuous functions** $C(X, Y)$. One notion is to say that two functions are **homotopic** to each other if there is a "continuous" deformation from one to the other; we will not make this notion precise in these notes. We write $f \sim g$ when there is a homotopy between two functions, and this is an equivalence relation. We will not define here the standard notions of **loops**, **homotopy equivalence classes of loops** (e.g., the first homotopy or **fundamental group** π_1), **concatenations**, **categories** nor **functors**, but we do mention that a classical result of algebraic topology is the **Brouwer fixed point theorem**.

The idea of this machinery is to be able to classify spaces. As all mathematicians quote, a topologist does not distinguish a coffee mug from his donut. So in $\mathbb{R}^3$, just how many possible "unique" shapes are there? We can get some first answers below.

First, define an n-dimensional **manifold** M as a topological space that is M_2 and T_2 and such that for all $m \in M$, there is an open set U containing m such that U is homeomorphic to $\mathbb{R}^n$, i.e., a manifold looks "locally" like Euclidean space. The dimension of a manifold is unique and topologically invariant. Euclidean space itself is of course a manifold. These structures arise in many fields of applied math, so they are worth knowing about.

1. 1-dimensional manifolds are called **curves** and the only **connected, compact** 1-dimensional manifold is $\mathbb{S}^1$, the sphere $\mathbb{S}^1 = \{x \in X \mid \|x\| = 1\} \subset \mathbb{R}^2$ (i.e., the boundary of the unit ball). The only **connected, non-compact** curve is $\mathbb{R}^1$ itself.

2. 2-dimensional manifolds are called **surfaces**. Examples of surfaces are $\mathbb{R}^2$ itself, and the annulus $(0, 1) \times S^1$ (like the curved part of a cylinder, without the ends, known as the open cylinder; or think of the **punctured plane**, or a ring with finite thickness and open boundaries). Compact surfaces are the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and the torus (or donut) $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The complete list of connected compact surfaces is known. For example, here is a theorem from Armstrong:

   Theorem 3.1 (Classification theorem)**.** *Any closed* surface *(i.e., 2-manifold) is homeomorphic to either the sphere, or to the sphere with a finite number of handles added, or to the sphere with a finite number of discs removed and replaced by Möbius strips (and no two of these aforementioned surfaces are homeomorphic).*

   A **Möbius strip** (this also applies to a cylinder) by itself is not a surface since it has a boundary, which is homeomorphic to the upper half-plane and not $\mathbb{R}^2$. The whole plane itself is not included since it is not closed; the theorem in essence requires compact and connected surfaces. The example of a sphere with a disc removed and replaced by a Möbius strip is not possible to imagine in 3D without self-intersections (so not all 2-manifolds can be easily imaged in 3D!), but you can imagine that this is possible since the Möbius strip does have a single circle as its boundary. The result is known as the **projective plane** denoted $\mathbb{P}^2$. If you do this with 2 removed discs, we recover the **Klein bottle**.

   Here are three descriptions of the projective plane:

   - Take the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and partition it into subsets which contain exactly two points, the points being antipodal (at opposite ends of a diameter), and then identifying these antipodal points with each other (think of this as some weird boundary condition).
   - Begin with $\mathbb{R}^3 \setminus \{0\}$ and identify two points iff they lie on the same straight line through the origin.
   - Begin with the unit ball $B \subset \mathbb{R}^3$ (like the sphere, but including the interior) and identify antipodal points of its boundary sphere.

3. For higher dimensions, there is no known complete list, and is an active area of research. Some examples are S^n and T^n, as well as **Stiefel manifolds** which appear in numerical analysis all the time.

A classic result of topology, much like the Brouwer fixed point theorem in its importance, is the **Jordan Curve Theorem** or Jordan Separation Theorem:

Theorem 3.2. *Any simple closed curve in the plane divides the plane into two pieces.*

The theorem is notorious for being "obvious" but extremely difficult to prove rigorously.

There is much more to algebraic topology (you could say that it is "more interesting" than point-set topology, or at least that it has more areas for research), but we stop here.