Partial Information Breeds Systemic Risk

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- **Systemic risk** has been studied widely.
  - *Homogeneous* inter-bank lending and borrowing
    - No control: FOUQUE & SUN (2013)
    - Adding (delayed) controls: CARMONA ET AL. (2015), CARMONA ET AL. (2018), ...
    - More general reserve processes: FOUQUE & ICHIBA (2013), SUN (2018), GARNIER ET AL. (2013, 2013, 2017), ...
  - *Heterogeneity* among banks:
    - Reserve dynamics, costs: FANG ET AL. (2017), SUN (2022), ...
    - Capital requirements: CAPPONI ET AL. (2020), ...
    - Network locations: BIAGINI ET AL. (2019), FEINSTEIN & SOJMARK (2019), ...

The underlying thesis:

Inter-bank transactions trigger systemic risk.

Our Ideas:

1) Systemic risk should be more general than this...

2) Can other transactions trigger systemic risk?

- **In this talk:**
  - Consider an *optimal investment* model for N investors.
    - No inter-bank activity is involved.
  - Present a new cause of systemic risk.

THE MODEL

- $N \in \mathbb{N}$ investors (e.g., fund managers) trading

$$\frac{dS(u)}{S(u)} = \mu du + \sigma dW(u), \quad S(t) = s > 0, \tag{1}$$

  on a finite time horizon $T > 0$.

- Investor i's wealth process:

$$dX_i(u) = rX_i(u) + \pi_i(u)(\mu - r)du + \pi_i(u)\sigma dW(u), \qquad X_i(t) = x_i \in \mathbb{R}. \tag{2}$$

- **Assume:** $\sigma, r > 0$ are known;
  μ is only *partially known*.

THE MODEL

- **Relative performance criterion:**
  - Investor i considers

$$(1-\lambda_i)X_i(T)+\lambda_i(X_i(T)-\overline{X}(T)). \tag{3}$$

    - $\overline{X}(T) := \frac{1}{N}\sum_{i=1}^{N} X_i(T)$.
    - $\lambda_i \in [0,1]$.
  - The resulting **mean-variance objective:**

$$J_i\left(t,\boldsymbol{x},\{\pi_j\}_{j\neq i},\pi_i\right) := \mathbb{E}^{t,\boldsymbol{x}}\left[X_i(T)-\lambda_i^M\overline{X}(T)\right]-\frac{\gamma_i}{2}\mathrm{Var}^{t,\boldsymbol{x}}\left[X_i(T)-\lambda_i^V\overline{X}(T)\right], \tag{4}$$

    - Allow for two λ_i values (i.e., λ_i^M, λ_i^V).
    - $\gamma_i > 0$: risk aversion coefficient.
  - ESPINOSA & TOUZI (2015), LACKER & ZARIPHOPOULOU (2019):
    - Consider (3) under utility maximization.
    - Obtain a Nash equilibrium for the N investors.

The Model

- **Partial information:**
  (a) Investors observe the evolution of S.
  (b) Don't know μ precisely ($\implies$ can only infer it from (a)).
- **Assume:** Investors know μ takes either μ_1 or μ_2 ($\mu_1 > \mu_2$).
  - **Scenario 1:** $\mu \in \mathbb{R}$ is a fixed constant
    - Need to infer true value of μ between μ_1 and μ_2 (e.g., a stock with unreported innovation)
  - **Scenario 2:** μ alternates between μ_1 and μ_2
    - Need to infer recurring changes of μ between μ_1 and μ_2 (e.g., changes between a bull and a bear market)

- **Our Goals:**
  - Find a Nash equilibrium $(\pi_1^*, \pi_2^*, ..., \pi_N^*)$ for the N investors
    - under *full* information;
    - under *partial* information.
  - **Question:**

How does investors' wealth change
from *full* to *partial* information?

As we will see:
Partial information triggers systemic risk.

- **What constitutes a Nash equilibrium $(\pi_1^*, ..., \pi_N^*)$?**
  - *Inter-personally*, investor i selects π_i in response to $\{\pi_j\}_{j\neq i}$.
  - *Intra-personally*, π_i needs to resolve *time inconsistency* among investor i's current and future selves...

Definition

$\boldsymbol{\pi}^* = (\pi_1^*, ..., \pi_N^*)$ is a Nash equilibrium for (4) if, for any $i = 1, ..., N$,

$$\liminf_{h\downarrow 0} \frac{J_i\left(t, \boldsymbol{x}, \{\pi_j^*\}_{j\neq i}, \pi_i^*\right) - J_i\left(t, \boldsymbol{x}, \{\pi_j^*\}_{j\neq i}, \pi \otimes_{t+h} \pi_i^*\right)}{h} \geq 0, \tag{5}$$

for all $(t, \boldsymbol{x}) \in [0, T) \times \mathbb{R}^N$ and π.

- All investors achieve intra-personal equilibrium *simultaneously*
  - —"*soft* inter-personal equilibrium" (Huang & Zhou (2022)).
  - "*Sharp* inter-personal equilibrium" hard to define here...

Scenario 1: Constant μ

Consider

$$\kappa_i := \frac{1}{\gamma_i}\left(1 - \frac{\lambda_i^V}{N}\right)^{-1}\left(1 - \frac{\lambda_i^M}{N}\right) > 0 \quad i = 1, \cdots, N, \tag{6}$$

$$\overline{\kappa} := \frac{1}{N}\sum_{i=1,\dots,N}\kappa_i \quad \text{and} \quad \overline{\lambda}^V := \frac{1}{N}\sum_{i=1,\dots,N}\lambda_i^V. \tag{7}$$

Theorem 1.1 ($\mu \in \mathbb{R}$ is known)

A Nash equilibrium $\boldsymbol{\pi}^* = (\pi_1^*, \dots, \pi_N^*)$ for (4) is given by

$$\pi_i^*(t) = e^{-r(T-t)}\left\{\frac{\mu - r}{\sigma^2}\left(\kappa_i + \frac{\lambda_i^V}{1 - \overline{\lambda}^V}\overline{\kappa}\right)\right\}, \quad \forall i = 1, \dots, N. \tag{8}$$

- If $\lambda_i^M = \lambda_i^V = 0$, becomes $\pi_i^*(t) = e^{-r(T-t)}\frac{\mu - r}{\sigma^2\gamma_i}$.

Theorem 1.1 ($\mu \in \mathbb{R}$ is known)—*continued*

The value function under the Nash equilibrium $\boldsymbol{\pi}^*$ is

$$V_i(t, \boldsymbol{x}) = e^{r(T-t)}\left(x_i - \frac{\lambda_i^M}{N}\overline{x}\right) + (T-t)N_i, \quad \forall i = 1, ..., N. \tag{9}$$

where

$$N_i := \left(\frac{\mu - r}{\sigma}\right)^2 \Bigg\{ \left(\kappa_i + \frac{\lambda_i^V - \lambda_i^M}{1 - \overline{\lambda}^V}\overline{\kappa}\right) - \frac{\gamma_i}{2}\left(\frac{2\lambda_i^V}{1 - \overline{\lambda}^V}\left(1 - \frac{\lambda_i^V}{N}\right)\overline{\kappa} + \left(1 - \frac{2\lambda_i^V}{N}\right)\kappa_i\right)^2 \Bigg\}.$$

Under **partial information**, consider

$$\widehat{\mathfrak{p}}_j(u) := \mathbb{P}\left(\mu = \mu_j \mid \{S(v)\}_{t \le v \le u}\right), \quad j = 1, 2. \tag{10}$$

Lemma 1

Fix $t \ge 0$. Given S in (1), the process $\{\widehat{W}(u)\}_{u \ge t}$ given by

$$\widehat{W}(u) := \frac{1}{\sigma}\left[\log\left(\frac{S(u)}{S(t)}\right) - (\mu_1 - \mu_2)\int_t^u \widehat{\mathfrak{p}}_1(s)ds - \left(\mu_2 - \frac{\sigma^2}{2}\right)(u - t)\right] \tag{11}$$

is a Brownian motion w.r.t. the filtration of S. Moreover, $\{\widehat{\mathfrak{p}}_1(u)\}_{u \ge t}$ is the unique strong solution to

$$dP(u) = \frac{\mu_1 - \mu_2}{\sigma}P(u)(1 - P(u))d\widehat{W}(u), \quad P(t) = \widehat{\mathfrak{p}}_1(t) \in (0, 1), \tag{12}$$

which satisfies $P(u) \in (0, 1)$ for all $u \ge t$ a.s.

- By LIPTSER & SHIRYAEV (2013), WONHAM (1965), Feller's test.

- **Consequences**:
  - By (11), the original dynamics

$$dS(u) = \mu S(u)du + \sigma S(u)dW(u)$$

can be expressed equivalently as

$$dS(u) = \Big((\mu_1 - \mu_2)P(u) + \mu_2\Big)S(u)du + \sigma S(u)d\widehat{W}(u),$$

where P is the unique strong solution to (12).

    - The dynamics is now observable!
    - $\theta(P(u)) := (\mu_1 - \mu_2)P(u) + \mu_2 = \mu_1 P(u) + \mu_2(1 - P(u))$,
      i.e., an estimate of μ based on observations of S.

  - Wealth process (2) now becomes

$$dX_i(u) = rX_i(u) + \pi_i(u)\Big((\mu_1 - \mu_2)P(u) + \mu_2 - r\Big)du + \pi_i(u)\sigma d\widehat{W}(u). \tag{13}$$

- **Mean-variance objective** (under *partial* information):

$$J_i\left(t, \boldsymbol{x}, p, \{\pi_j\}_{j\neq i}, \pi_i\right)$$
$$:= \mathbb{E}^{t,\boldsymbol{x},p}\left[X_i(T) - \lambda_i^M \overline{X}(T)\right] - \frac{\gamma_i}{2}\mathrm{Var}^{t,\boldsymbol{x},p}\left[X_i(T) - \lambda_i^V \overline{X}(T)\right], \quad (14)$$

where X_i satisfies (13).

Definition

$\boldsymbol{\pi}^* = (\pi_1^*, ..., \pi_N^*)$ is a Nash equilibrium for (14) if, for any $i = 1, ..., N$,

$$\liminf_{h\downarrow 0} \frac{J_i\left(t, \boldsymbol{x}, p, \{\pi_j^*\}_{j\neq i}, \pi_i^*\right) - J_i\left(t, \boldsymbol{x}, p, \{\pi_j^*\}_{j\neq i}, \pi \otimes_{t+h} \pi_i^*\right)}{h} \geq 0, \quad (15)$$

for all $(t, \boldsymbol{x}, p) \in [0, T) \times \mathbb{R}^N \times (0, 1)$ and π.

- To find a Nash equilibrium $\boldsymbol{\pi}^* = (\pi_1^*, ..., \pi_N^*)$,
  - Derive and solve **Extended HJB equation**
    - BJÖRK ET AL. (2016): a system of 2 coupled PDEs.
    - Our case: a system of $2N$ coupled PDEs.
  - For each $i = 1, ..., N$, write (14) as

$$J_i(t, \boldsymbol{x}, p, \overline{\pi}_{(-i)}, \pi_i) = \mathbb{E}^{t,\boldsymbol{x},p}\left[F_i(\boldsymbol{X}(T))\right] + \frac{\gamma_i}{2}\mathbb{E}^{t,\boldsymbol{x},p}[H_i(\boldsymbol{X}(T))]^2,$$

where

$$H_i(\boldsymbol{x}) := x_i - \lambda_i^V \bar{x}, \qquad F_i(\boldsymbol{x}) := x_i - \lambda_i^M \bar{x} - \frac{\gamma_i}{2} H_i(\boldsymbol{x})^2.$$

  - Define $\theta, \beta : [0,1] \to \mathbb{R}$ by

$$\theta(p) := (\mu_1 - \mu_2)p + \mu_2, \quad \beta(p) := \frac{\mu_1 - \mu_2}{\sigma} p(1-p). \tag{16}$$

- **Extended HJB equation for $\boldsymbol{\pi}^* = (\pi_1^*, \dots, \pi_N^*)$:** $\forall i = 1, \dots, N,$

$$
\begin{aligned}
\partial_t V_i + \sup_{\pi_i} \Bigg\{ & \sum_{j \neq i} \left(r x_j + (\theta(p) - r) \pi_j^* \right) \partial_{x_j} V_i + \left(r x_i + (\theta(p) - r) \pi_i \right) \partial_{x_i} V_i \\
& + \frac{\sigma^2}{2} \sum_{j \neq i} \sum_{k \neq i} \pi_j^* \pi_k^* \partial_{x_j x_k} V_i + \frac{\sigma^2}{2} \pi_i^2 \partial_{x_i x_i} V_i + \sigma^2 \pi_i \sum_{j \neq i} \pi_j^* \partial_{x_i x_j} V_i \\
& + \eta(p) \partial_p V_i + \frac{\beta(p)^2}{2} \partial_{pp} V_i + \sigma \beta(p) \sum_{j \neq i} \pi_j^* \partial_{x_j p} V_i + \sigma \pi_i \beta(p) \partial_{x_i p} V_i \\
& - \frac{\gamma_i \sigma^2}{2} \sum_{j \neq i} \sum_{k \neq i} \pi_j^* \pi_k^* \partial_{x_j} g_i \partial_{x_k} g_i - \frac{\gamma_i \sigma^2}{2} \pi_i^2 (\partial_{x_i} g_i)^2 - \gamma_i \sigma^2 \pi_i \sum_{j \neq i} \pi_j^* \partial_{x_i} g_i \partial_{x_j} g_i \\
& - \frac{\gamma_i \beta(p)^2}{2} (\partial_p g_i)^2 - \gamma_i \sigma \pi_i \beta(p) \partial_{x_i} g_i \partial_p g_i - \gamma_i \sigma \beta(p) \sum_{j \neq i} \pi_j^* \partial_p g_i \partial_{x_j} g_i \Bigg\} = 0,
\end{aligned}
$$

with $V_i(T, x, p) = x_i - \lambda_i^M \bar{x} = (1 - \lambda_i^M / N) x_i - \lambda_i^M \bar{x}_{(-i)}$.

- $g_i(t, \boldsymbol{x}, p) := \mathbb{E}^{t,\boldsymbol{x},p}[H_i(\boldsymbol{X}^{\boldsymbol{\pi}^*}(T))]$ fulfills

$$\partial_t g_i + \sum_{j=1,\dots,N} \left(r x_j + (\theta(p) - r)\pi_j^*\right) \partial_{x_j} g_i + \frac{\sigma^2}{2} \sum_{j=1,\dots,N} \sum_{k=1,\dots,N} \pi_j^* \pi_k^* \partial_{x_j x_k} g_i$$
$$+ \eta(p)\partial_p g_i + \frac{\beta(p)^2}{2} \partial_{pp} g_i + \sigma\beta(p) \sum_{j=1,\dots,N} \pi_j^* \partial_{x_j p} g_i = 0, \tag{17}$$

with $g_i(T, x, p) = x_i - \lambda_i^V \bar{x} = (1 - \lambda_i^V/N)x_i - \lambda_i^V \bar{x}_{(-i)}$.

- Take the ansatz

$$\begin{aligned} V_i(t, \boldsymbol{x}, p) &= A_i(t)x_i + B_i(t)\overline{x}_{(-i)} + C_i(t, p), \\ g_i(t, \boldsymbol{x}, p) &= a_i(t)x_i + b_i(t)\overline{x}_{(-i)} + c_i(t, p), \end{aligned} \tag{18}$$

with $\overline{x}_{(-i)} := \frac{1}{N} \sum_{j \neq i} x_j$.

- (17) becomes

$$\partial_t a_i x_i + \partial_t b_i \overline{x}_{(-i)} + r a_i x_i + a_i(\theta(p) - r)\pi_i^* + r b_i \overline{x}_{(-i)}$$
$$+ b_i(\theta(p) - r)\overline{\pi}_{(-i)}^* + \partial_t c_i + \eta(p)\partial_p c_i + \frac{\beta(p)^2}{2}\partial_{pp} c_i = 0$$

with $a_i(T) = 1 - \frac{\lambda_i^V}{N}$, $b_i(T) = -\lambda_i^V$, and $c_i(T, y) = 0$.

- **Extended HJB equation** becomes

$$
\begin{aligned}
&\partial_t A_i x_i + r A_i x_i + \partial_t B_i \overline{x}_{(-i)} + r B_i \overline{x}_{(-i)} + \partial_t C_i + \eta(p)\partial_p C_i + \frac{\beta(p)^2}{2}\partial_{pp} C_i \\
&+ B_i(\theta(p) - r)\overline{\pi}^*_{(-i)} - \frac{\gamma_i \sigma^2}{2}\left(\frac{b_i}{N}\right)^2 \sum_{j \neq i}\sum_{k \neq i} \pi^*_j \pi^*_k \\
&- \frac{\gamma_i \beta(p)}{2}(\partial_p c_i)^2 - \gamma \sigma \beta(p) b_i \partial_p c_i \overline{\pi}^*_{(-i)} \\
&+ \sup_{\pi_i}\left\{A_i(\theta(p) - r)\pi_i - \frac{\gamma_i \sigma^2}{2}\pi_i^2 a_i^2 - \gamma_i \sigma^2 a_i b_i \pi_i \overline{\pi}^*_{(-i)} - \gamma_i \sigma \pi_i \beta(p) a_i \partial_p c_i\right\} \\
&= 0,
\end{aligned}
$$

with $A_i(T) = 1 - \frac{\lambda_i^M}{N}$, $B_i(T) = -\lambda_i^M$, and $C_i(T, y) = 0$.

1ST CAUCHY PROBLEM

- Domain $Q := [0, T) \times (0, 1)$.
- Given $i = 1, ..., N$, consider for any $\eta : [0, 1] \to \mathbb{R}$ the Cauchy problem

$$\begin{cases} \partial_t c + \left(\eta(p) - \beta(p)\left(\frac{\theta(p)-r}{\sigma}\right)\right)\partial_p c \\ \qquad + \frac{\beta(p)^2}{2}\partial_{pp} c + \underline{\underline{\kappa_i}}\left(\frac{\theta(p)-r}{\sigma}\right)^2 = 0 & \text{for } (t,p) \in Q, \\ c(T, p) = 0, & \text{for } p \in (0, 1), \end{cases} \tag{19}$$

where $\underline{\underline{\kappa_i > 0}}$ is from (6).

  - **Scenario 1:** $\eta(p) \equiv 0$
  - **Scenario 2:** $\eta(p) = -(q_1 + q_2)p + q_2$

Lemma 2

Assume: for any $t \geq 0$ and $p \in (0, 1)$,

$$dP(u) = \eta(P(u))du + \frac{\mu_1 - \mu_2}{\sigma} P(u)(1 - P(u))dW(u), \quad P(t) = p, \quad (20)$$

has a unique strong solution with $P(u) \in (0, 1)$ for all $u \geq t$ a.s.

Consider: Probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ defined by

$$\mathbb{Q}(A) := \mathbb{E}[1_A Z(T)] \quad \forall A \in \mathcal{F}_T, \quad (21)$$

where

$$Z(u) := \exp\left(-\frac{1}{2}\int_t^u \left(\frac{\theta(P(s)) - r}{\sigma}\right)^2 ds + \int_t^u \frac{\theta(P(s)) - r}{\sigma} dW(s)\right) \quad (22)$$

is a $\mathbb{P}$-martingale. Also consider the $\mathbb{Q}$-Brownian motion

$$W_{\mathbb{Q}}(u) := W(u) - \int_t^u \frac{\theta(P(s)) - r}{\sigma} ds. \quad (23)$$

Lemma 2 (*continued*)

Then, for any $i = 1, \ldots, N$,

(i) (19) has a unique solution $c \in C^{1,2}([0,T) \times (0,1))$ continuous up to $\{T\} \times (0,1)$. Moreover, c is bounded and satisfies

$$c(t,p) = \kappa_i \mathbb{E}_{\mathbb{Q}}^{t,p}\left[\int_t^T \left(\frac{\theta(P(u)) - r}{\sigma}\right)^2 du\right], \quad \forall (t,p) \in [0,T] \times (0,1), \tag{24}$$

- By *elliptic regularization* and *Feynman-Kac-type arguments*.
- **Note:** Under $\mathbb{Q}$, P in (20) becomes

$$dP(u) = \left(\eta(P(u)) - \beta(P(u))\left(\frac{\theta(P(u)) - r}{\sigma}\right)\right)du + \beta(P(u))dW_{\mathbb{Q}}(u), \quad P(t) = p. \tag{25}$$

Lemma 2 (*continued*)

(ii) $\partial_p c$ is bounded and satisfies

$$\boxed{\partial_p c(t,p) = 2\kappa_i \frac{\mu_1 - \mu_2}{\sigma} \mathbb{E}_{\mathbb{Q}}^{t,p}\left[\int_t^T \zeta(u)\left(\frac{\theta\left(P(u)\right) - r}{\sigma}\right) du\right]}, \qquad (26)$$

where ζ is the unique strong solution to

$$d\zeta(u) = \zeta(u)\Gamma(P(u))du + \zeta(u)\Lambda(P(u))dW_{\mathbb{Q}}(u), \quad \zeta(t) = 1, \qquad (27)$$

with P given by (25) and $\Gamma, \Lambda : (0,1) \to \mathbb{R}$ defined as

$$\Gamma(p) := \frac{d}{dp}\left(\eta(p) - \beta(p)\left(\frac{\theta(p) - r}{\sigma}\right)\right), \quad \Lambda(p) := \frac{d}{dp}\beta(p).$$

- **Observe:** for all $u \geq t$,

$$\zeta(u) = \lim_{h \to 0} \frac{P^{t,p+h}(u) - P^{t,p}(u)}{h} \quad \text{in } L^2(\Omega, \mathcal{F}_T, \underline{\underline{\mathbb{Q}}}) \tag{28}$$

$$= \lim_{h \to 0} \frac{P^{t,p}(u+\tau(h)) - P^{t,p}(u)}{h} \quad \text{in } L^2(\Omega, \mathcal{F}_T, \underline{\underline{\mathbb{Q}}}), \tag{29}$$

with $\tau(h) := \inf\{t' \geq 0 : P^{0,p}(t') = p + h\}$.

  - "$=$": by Theorem 5.3 in FRIEDMAN (1975).
  - "$=$": by time-homogeneity, strong uniqueness of P in (25).
  - $\zeta(u)$ measures the *rate of change* of $P^{t,p}(\cdot)$ at time u.

Takeaway:

$$\begin{cases} P^{t,p}(\cdot) \text{ volatile under } \mathbb{Q} \iff \zeta(\cdot) \text{ large} \iff \partial_p c(t,p) \text{ large.} \\ P^{t,p}(\cdot) \text{ stable under } \mathbb{Q} \iff \zeta(\cdot) \text{ small} \iff \partial_p c(t,p) \text{ small.} \end{cases} \tag{30}$$

2ND CAUCHY PROBLEM

- Given solution c_i to (19) for $i = 1, ..., N$, consider the Cauchy problem

$$\begin{cases} \partial_t C + \eta(p)\partial_p C + \frac{\beta(p)^2}{2}\partial_{pp} C \\ \quad + R_i\big(t, p, \partial_p c_1(t,p), \cdots, \partial_p c_N(t,p)\big) = 0 & \text{for } (t,p) \in Q, \\ C(T,p) = 0, & \text{for } p \in (0,1), \end{cases} \quad (31)$$

where

$$R_i\big(t, p, \partial_p c_1(t,p), \cdots, \partial_p c_N(t,p)\big)$$
$$:= (\theta(p) - r)\left\{ \left(\kappa_i \frac{\theta(p) - r}{\sigma^2} - \frac{\beta(p)}{\sigma}\partial_p c_i \right) + \frac{\lambda_i^V - \lambda_i^M}{1 - \overline{\lambda}^V} \left(\overline{\kappa}\frac{\theta(p) - r}{\sigma^2} - \frac{\beta(p)}{\sigma}\overline{\partial_p c} \right) \right\}$$
$$- \frac{\gamma_i \sigma^2}{2} \left\{ \frac{2\lambda_i^V}{1 - \overline{\lambda}^V} \left(1 - \frac{\lambda_i^V}{N} \right) \left(\overline{\kappa}\frac{\theta(p) - r}{\sigma^2} - \frac{\beta(p)}{\sigma}\overline{\partial_p c} \right) \right.$$
$$\left. + \left(1 - \frac{2\lambda_i^V}{N} \right) \left(\kappa_i \frac{\theta(p) - r}{\sigma^2} - \frac{\beta(p)}{\sigma}\partial_p c_i \right) \right\}^2$$
$$- \frac{\gamma_i \beta(p)^2}{2}(\partial_p c_i)^2 - \gamma_i \sigma \beta(p) \partial_p c_i \left(\kappa_i \frac{\theta(p) - r}{\sigma^2} - \frac{\beta(p)}{\sigma}\partial_p c_i \right).$$

Corollary

Let conditions in Lemma 2 hold. Then, (31) has a unique solution $C \in C^{1,2}([0,T) \times (0,1))$ continuous up to $\{T\} \times (0,1)$. Moreover, C is bounded and satisfies

$$C(t,p) = \mathbb{E}^{t,p}\left[\int_t^T R_i\big(u, P(u), \partial_p c_1(u, P(u)), \cdots, \partial_p c_N(u, P(u))\big)du\right],$$

where P is the unique strong solution to (20).

Theorem 1.2 ($\mu \in \mathbb{R}$ is unknown)

A Nash equilibrium $\boldsymbol{\pi}^* = (\pi_1^*, ..., \pi_N^*)$ for (14) is given by

$$\pi_i^*(t,p) = e^{-r(T-t)} \left\{ \frac{\theta(p) - r}{\sigma^2} \left(\kappa_i + \frac{\lambda_i^V}{1 - \overline{\lambda}^V} \overline{\kappa} \right) - \frac{\beta(p)}{\sigma} \left(\partial_p c_i + \frac{\lambda_i^V}{1 - \overline{\lambda}^V} \overline{\partial_p c} \right) \right\}, \quad i = 1, ..., N, \tag{32}$$

where c_i is the unique solution to 1st Cauchy (19) (with $\eta \equiv 0$) and $\overline{\partial_p c} := \frac{1}{N} \sum_{i=1}^N \partial_p c_i$. Moreover, the value function under $\boldsymbol{\pi}^*$ is

$$V_i(t,x,p) = e^{r(T-t)} \left(x_i - \frac{\lambda^M}{N} \bar{x} \right) + C_i(t,p), \quad i = 1, ..., N, \tag{33}$$

where C_i is the unique solution to 2nd Cauchy (31) (with $\eta \equiv 0$).

- 1st term of (32):
  - Identical with (8), except that...

    μ is replaced by the estimate $\theta(p) = p\mu_1 + (1-p)\mu_2$

    - based on *current judgement* "$p = P(t)$."

- 2nd term of (32):
  - Adjusts 1st term, based on

    stability of *judgements* $P(\cdot)$ over time.

  - $P(\cdot)$ is stable under $\mathbb{Q}$ (i.e., stays near $p = P(t)$):
    - $p = P(t)$ is *"reliable"*;
    - $\zeta(\cdot)$ small $\implies$ $\partial_p c_i(t, p)$ small $\implies$ 2nd term of (32) small

  - $P(\cdot)$ is volatile under $\mathbb{Q}$ (i.e., moves away from $p = P(t)$):
    - $p = P(t)$ is *"unreliable"*;
    - $\zeta(\cdot)$ large $\implies$ $\partial_p c_i(t, p)$ large $\implies$ 2nd term of (32) large

- By (6) and (26), rewrite π_i^* in (32) as

$$\pi_i^*(t,p) = e^{-r(T-t)}\left\{\frac{\theta(p)-r}{\sigma^2} - \frac{2\beta(p)}{\sigma}(\mu_1-\mu_2)\mathbb{E}_{\mathbb{Q}}^{t,p}\left[\int_t^T \zeta(u)\frac{\theta(P(u))-r}{\sigma^2}du\right]\right\} \cdot \left(\kappa_i + \frac{\lambda_i^V}{1-\overline{\lambda}^V}\overline{\kappa}\right).$$

  - $\kappa_i := \frac{1}{\gamma_i}\left(1-\frac{\lambda_i^V}{N}\right)^{-1}\left(1-\frac{\lambda_i^M}{N}\right)$, $\lambda_i^V \in [0,1]$: specific to investor i.
  - $\overline{\kappa}$, $\overline{\lambda}^V$: influence from other investors.

- **Observe:**
  - $\lambda_i^V = 0 \implies \pi_i^*$ independent of other investors.
  - *A conservative investor may take large risky positions!*
    - $\gamma_i > 0$ large $\implies \kappa_i > 0$ small,
    - but $\bar{\kappa} > 0$ can still be large.

Scenario 2: Alternating μ

- The stock:

$$dS(u) = \mu(M(u))S(u)du + \sigma S(u)dW(u), \quad S(t) = s, \quad (34)$$

  - M is a two-state continuous-time Markov chain with generator

$$G = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}, \quad q_1, q_2 > 0.$$

  - $\mu(1) = \mu_1$ and $\mu(2) = \mu_2$.
- Investor i's wealth process:

$$dX_i(u) = rX_i(u) + \pi_i(u)(\mu(M(u)) - r)du + \pi_i(u)\sigma dW(u),$$
$$X_i(t) = x_i \in \mathbb{R}. \quad (35)$$

Under *full Information* (M observable),

- **Mean-variance objective:**

$$
\begin{aligned}
&J_i\left(t, \boldsymbol{x}, m, \{\pi_j\}_{j \neq i}, \pi_i\right) \\
&:= \mathbb{E}^{t,\boldsymbol{x},m}\left[X_i(T) - \lambda_i^M \overline{X}(T)\right] - \frac{\gamma_i}{2} \mathrm{Var}^{t,\boldsymbol{x},m}\left[X_i(T) - \lambda_i^V \overline{X}(T)\right], \qquad (36)
\end{aligned}
$$

where X_i satisfies (35).

Definition

$\boldsymbol{\pi}^* = (\pi_1^*, \ldots, \pi_N^*)$ is a Nash equilibrium for (36) if, for any $i = 1, \ldots, N$,

$$
\liminf_{h \downarrow 0} \frac{J_i\left(t, \boldsymbol{x}, m, \{\pi_j^*\}_{j \neq i}, \pi_i^*\right) - J_i\left(t, \boldsymbol{x}, m, \{\pi_j^*\}_{j \neq i}, \pi \otimes_{t+h} \pi_i^*\right)}{h} \geq 0,
$$

for all $(t, \boldsymbol{x}, m) \in [0, T) \times \mathbb{R}^N \times \{1, 2\}$ and π.

Theorem 2.1 (*M* observable)

A Nash equilibrium $\boldsymbol{\pi}^* = (\pi_1^*, ..., \pi_N^*)$ for (36) is given by

$$\pi_i^*(t,m) = e^{-r(T-t)}\left\{\frac{\mu(m) - r}{\sigma^2}\left(\kappa_i + \frac{\lambda_i^V}{1-\overline{\lambda}^V}\overline{\kappa}\right)\right\}, \quad i = 1, ..., N. \tag{37}$$

Theorem 2.1 (<u>M observable</u>)—*continued*

Moreover, the value function under the Nash equilibrium $\boldsymbol{\pi}^*$ is

$$V_i(t,x,m) = e^{r(T-t)}\left(x_i - \frac{\lambda^M}{N}\overline{x}\right) + C_i(t,m), \quad i = 1, ..., N. \tag{38}$$

where $C_i(t,m)$, $m \in \{1,2\}$, is defined as

$$C_i(t,1) := \frac{q_2\widetilde{Q}_i^1 + q_1\widetilde{Q}_i^2}{q_1+q_2}(T-t) + \frac{q_1}{(q_1+q_2)^2}\left(\widetilde{Q}_i^1 - \widetilde{Q}_i^2\right)\left(1 - e^{(q_1+q_2)(T-t)}\right)$$

$$C_i(t,2) := \frac{q_2\widetilde{Q}_i^1 + q_1\widetilde{Q}_i^2}{q_1+q_2}(T-t) - \frac{q_2}{(q_1+q_2)^2}\left(\widetilde{Q}_i^1 - \widetilde{Q}_i^2\right)\left(1 - e^{(q_1+q_2)(T-t)}\right)$$

$$Q_i^m := \left(\frac{\mu(m)-r}{\sigma}\right)^2 \Bigg\{\left(\kappa_i - \frac{\lambda_i^V - \lambda_i^M}{1-\overline{\lambda}^V}\overline{\kappa}\right) - \frac{\gamma_i}{2}\left(\frac{2\lambda_i^V}{1-\overline{\lambda}^V}\left(1-\frac{\lambda_i^V}{N}\right)\overline{\kappa} + \left(1 - \frac{2\lambda_i^V}{N}\right)\kappa_i\right)^2\Bigg\}.$$

Under *partial information* (M unobservable), consider

$$\widetilde{\mathfrak{p}}_j(u) := \mathbb{P}\left(\mu(M(u)) = \mu_j \mid \{S(v)\}_{t \le v \le u}\right), \quad j = 1, 2. \tag{39}$$

Lemma 3

Fix $t \geq 0$. Given S in (34), the process $\{\widetilde{W}(u)\}_{u \geq t}$ given by

$$\widetilde{W}(u) := \frac{1}{\sigma}\left[\log\left(\frac{S(u)}{S(t)}\right) - (\mu_1 - \mu_2)\int_t^u \widetilde{\mathfrak{p}}_1(s)ds - \left(\mu_2 - \frac{\sigma^2}{2}\right)(u-t)\right] \tag{40}$$

is a Brownian motion w.r.t. the filtration of S. Moreover, $\{\widetilde{\mathfrak{p}}_1(u)\}_{u \geq t}$ is the unique strong solution to

$$dP(u) = \Big(-(q_1+q_2)P(u) + q_2\Big)du + \frac{\mu_1 - \mu_2}{\sigma}P(u)(1-P(u))d\widetilde{W}(u),$$

$$P(t) = \widetilde{\mathfrak{p}}_1(t) \in (0,1), \tag{41}$$

which satisfies $P(u) \in (0,1)$ for all $u \geq t$ a.s.

- **Consequences**:
  - By (40), the original dynamics

$$dS(u) = \mu(M(u))S(u)du + \sigma S(u)dW(u)$$

can be expressed equivalently as

$$dS(u) = \Big((\mu_1 - \mu_2)P(u) + \mu_2\Big)S(u)du + \sigma S(u)d\widetilde{W}(u),$$

where P is the unique strong solution to (41).

  - Wealth process (35) now becomes

$$dX_i(u) = rX_i(u) + \pi_i(u)\Big((\mu_1 - \mu_2)P(u) + \mu_2 - r\Big)du + \pi_i(u)\sigma d\widetilde{W}(u).$$

  - **Note:** The dynamics is now observable!

Theorem 2.2 (M unobservable)

A Nash equilibrium $\boldsymbol{\pi}^* = (\pi_1^*, \dots, \pi_N^*)$ for (14) is given by

$$\pi_i^*(t,p) = e^{-r(T-t)} \left\{ \frac{\theta(p) - r}{\sigma^2} \left(\kappa_i + \frac{\lambda_i^V}{1 - \overline{\lambda}^V} \overline{\kappa} \right) - \frac{\beta(p)}{\sigma} \left(\partial_p c_i + \frac{\lambda_i^V}{1 - \overline{\lambda}^V} \overline{\partial_p c} \right) \right\}, \quad i = 1, \dots, N, \quad (42)$$

where c_i is the unique solution to 1st Cauchy (19) with

$$\eta(p) := -(q_1 + q_2)p + q_2, \quad p \in [0,1]. \quad (43)$$

Moreover, the value function under $\boldsymbol{\pi}^*$ is given by (33), where C_i is the unique solution to 2nd Cauchy (31) with η as in (43).

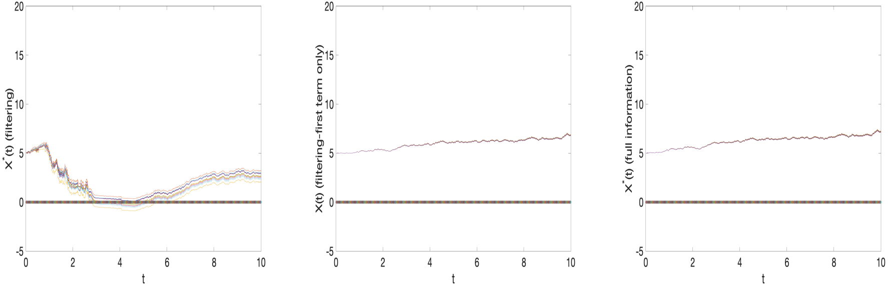
- Same formula as in Scenario 1, with different Cauchy problems.

Numerical Results & Discussions

SCENARIO 1: CONSTANT μ

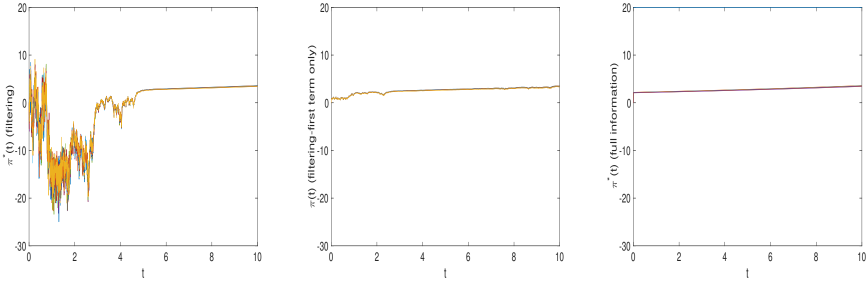
$$T = 10, N = 10, r = 0.05, \mu = \mu_1 = 0.2, \mu_2 = 0.02, \sigma = 0.1,$$
$$\lambda_i^M = \lambda_i^V = 0.5 \text{ and } \gamma_i = 8 + 0.1i \text{ for } i = 1, ..., 10$$

- **Wealth processes** $\{X_i(t)\}_{i=1}^{10}$
  - *Left:* under $\pi_i^*(t)$ in (32) [partial information]
  - *Middle:* under 1st term of (32)
  - *Right:* under $\pi_i^*(t)$ in (8) [full information]



Scenario 1: Constant μ

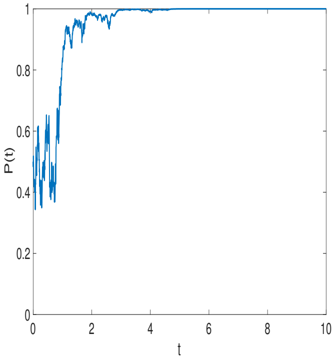
- **Trading strategies** $\{\pi_i^*(t)\}_{i=1}^{10}$**:**
  - *Left:* $\pi_i^*(t)$ in (32) [partial information]
  - *Middle:* 1st term of (32)
  - *Right:* $\pi_i^*(t)$ in (8) [full information]



SCENARIO 1: CONSTANT μ

- **Posterior probability** $P(t) = \widehat{\mathfrak{p}}_1(t)$ satisfies SDE (12):

$$dP(u) = \frac{\mu_1 - \mu_2}{\sigma} P(u)(1 - P(u))d\widehat{W}(u), \quad P(t) = p \in (0,1),$$



- **Observe:**
  1) $P(\cdot)$ oscillates forcefully
  2) $P(\cdot)$ moves in the right direction (i.e., towards 1) quickly
     $\Longrightarrow$ $\theta(P(\cdot))$ moves near $\mu = \mu_1$ quickly

Lemma 4

For SDE $P(\cdot)$ in (12), it holds $\mathbb{P}$-a.s. that

$$\lim_{u \to \infty} P(u) = \begin{cases} 1, & \text{if } \mu = \mu_1, \\ 0, & \text{if } \mu = \mu_2. \end{cases}$$

- It also holds $\mathbb{Q}$-a.s., for $\mathbb{Q}$ in (21).

SCENARIO 1: CONSTANT μ

- Look at π_i^* in (32) more closely:
  - Behaves most radically in $t \in [1.6, 2.3]$.
  - **Financial interpretation:**
    - Over $t \in [0, 1.6]$, investors tend to believe $\mu = \mu_1$ from $P(\cdot)$.
    - Over $t \in [1.6, 2.3]$, we have

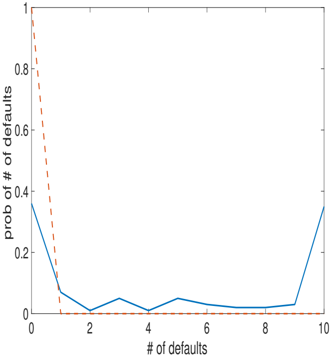
$$p = P(t) \in [0.9, 1) \quad \text{and} \quad \partial_p c_i(t, p) \text{ large } \forall i = 1, ..., N.$$

$\Longrightarrow$ $P(\cdot)$ is volatile under $\mathbb{Q}$ [by (30)]
$\Longrightarrow$ more likely $P(\cdot)$ will move away from 1 under $\mathbb{Q}$
$\Longrightarrow$ more likely $\mu = \mu_1$ is a misbelief [by Lemma 4]
$\Longrightarrow$ more severe change from long to short positions (to make up previous misbelief).

SCENARIO 1: CONSTANT μ

- **Empirical loss distributions:**
  - Computed via 100 simulations of wealth processes.

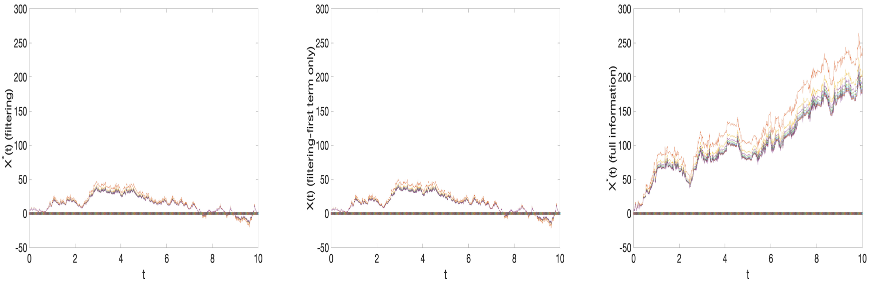
partial information v.s. full information



SCENARIO 2: ALTERNATING μ

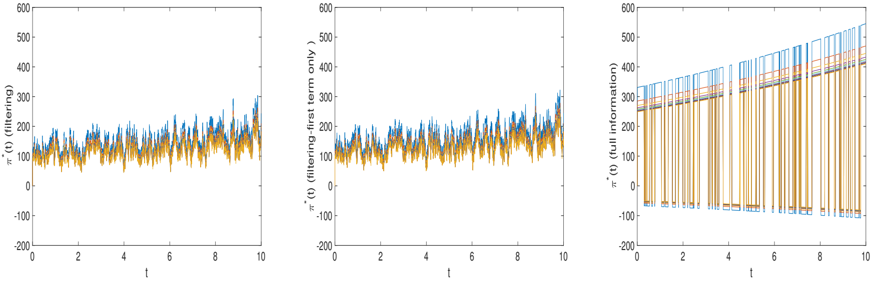
$T = 10, N = 10, r = 0.05,$ μ alternates between $\mu_1 = 0.2$ and $\mu_2 = 0.02$ with $q_1 = q_2 = 10$, $\sigma = 0.1$, $\lambda_i^M = \lambda_i^V = 0.9$ and $\gamma_i = 0.1i$ for $i = 1, \ldots, 10$

- **Wealth processes** $\{X_i(t)\}_{i=1}^{10}$
  - *Left:* under $\pi_i^*(t)$ in (42) [partial information]
  - *Middle:* under 1st term of (42)
  - *Right:* under $\pi_i^*(t)$ in (37) [full information]



Scenario 2: Alternating μ

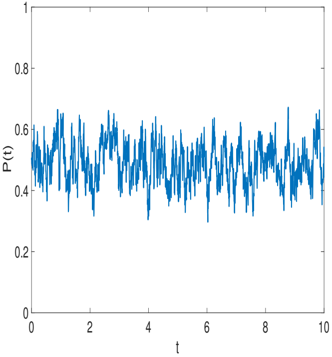
- **Trading strategies** $\{\pi_i^*(t)\}_{i=1}^{10}$**:**
  - *Left:* $\pi_i^*(t)$ in (42) [partial information]
  - *Middle:* 1st term of (42)
  - *Right:* $\pi_i^*(t)$ in (37) [full information]



SCENARIO 2: ALTERNATING μ

- **Posterior probability** $P(t) = \widetilde{\mathfrak{p}}_1(t)$ satisfies SDE (41):

$$dP(u) = \Big(-(q_1+q_2)P(u) + q_2 \Big)du + \frac{\mu_1 - \mu_2}{\sigma} P(u)(1 - P(u))d\widetilde{W}(u),$$
$$P(t) = p \in (0,1).$$



- **Observe:**

  1) $P(\cdot)$ is *mean-reverting*!
  $\implies$ $P(\cdot)$ never gets close to 1 or 0
  $\implies$ $\theta(P(\cdot))$ is never close to $\mu = \mu_1$

  2) Under $\mathbb{Q}$ in (21), the drift of P becomes

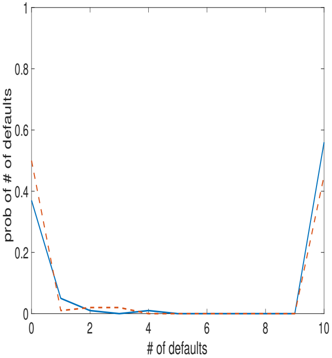
$$\Big(-(q_1+q_2)P(u)+q_2\Big)-\frac{\mu_1-\mu_2}{\sigma}P(u)(1-P(u))\left(\frac{\theta(P(u))-r}{\sigma}\right).$$

  - "*Mean-reverting*" feature remains!
    $\implies$ $P(\cdot)$ under $\mathbb{Q}$ is more stable
    $\implies$ $\partial_p c_i(t,p)$ is smaller $\forall i = 1, ..., N$ [by (30)]
    $\implies$ 2nd term of π_i^* in (32) has less influence

SCENARIO 2: ALTERNATING μ

- **Empirical loss distributions:**
  - Computed via 100 simulations of wealth processes.

partial information v.s. full information



THANK YOU!!

Q & A

Preprint available @ arXiv: 2312.04045

"Partial Information Breeds Systemic Risk"