Existence of the SVD decomposition for an arbitrary matrix A

Idea: 1. Isolate the direction of 'largest action' of A in the $\|\cdot\|_2$-norm.

2. Proceed by induction over the size of A

Recall definition of $\|A\|_2 = \max_{\|\underline{x}\|_2=1} \|A\underline{x}\|_2$.

Given A, let $\underline{v}_1$ be a 'maximizing' vector:

$$\underline{u}_1 = A\underline{v}_1 \qquad \|\underline{v}_1\|_2 = 1$$

call $\|\underline{u}_1\|_2 = \sigma_1$.

Create unitary matrices

$$[U_1] = \left[\underline{u}_1 \,\middle|\, \Xi\right], \quad [V_1] = \left[\underline{v}_1 \,\middle|\, \Xi\right]$$

then $[A][V_1] = \left[\underline{u}_1 \,\middle|\, \Xi\right]$,

$$[U^*][A][V_1] = \left[\begin{array}{c|c} \sigma_1 & \underline{w}^* \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right] = [S]$$

Can we now show that $\underline{w} = \underline{0}$, induction finishes the job

$\|S\|_2 = \|A\|_2 = \sigma_1$ Multiplying by unitary matrices do not change 2-norm.

From previous page:

$$S = \left[\begin{array}{c|c} \sigma_1 & \underline{w}^* \\ \hline \underline{0} & B \end{array}\right], \quad \|S\|_2 = \|A\|_2 = \sigma_1 .$$

Form

$$\underbrace{\left[\begin{array}{c|c} \sigma_1 & \underline{w}^* \\ \hline \underline{0} & B \end{array}\right]\begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix}}_{\infty} = \begin{bmatrix} \sigma_1^2 + \underline{w}^* w \\ \sim \end{bmatrix}$$

$$\| \quad \|_2 = \| \quad \|_2 \geq \sigma_1^2 + \underline{w}^* \underline{w} = (\sigma_1^2 + \underline{w}^* \underline{w})^{1/2} \left\| \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2$$

$$\leq \|S\|_2 \left\| \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2 = \sigma_1 \left\| \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2 .$$

Divide out $\left\| \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2$: Get

$$\sigma_1 \geq (\sigma_1^2 + \underline{w}^* \underline{w})^{1/2} \Rightarrow \underline{w} = \underline{0} .$$

DONE