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Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2015

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<u>Note</u>: Do four of the following five problems. Place an **X** on the line
opposite the number of the problem that you are **NOT** submitting
for grading. Please do not write your name anywhere on this exam.
You will be identified only by your student number, given below and
on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____
Total ____

Student Number _____

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1. At the end of each day, a bridge troll will add up to 1 kg of gold to the pile he is hoarding under his bridge. Specifically, the amount he will add is uniformly distributed between 0 and 1 kg and the amounts for different days are independent. His pile was raided last night by assorted forrest creatures and this morning he has no gold. Such raids (totally cleaning him out) happen as a Poisson process, with the rate of 5 per year.

   (a) What is the expected number of days (from today) that will elapse before he has at least 1 kg of gold?

   (b) What is the expected amount of gold that he will have on the morning after he first reaches 1 kg?

   (c) What is the probability that the troll will never accumulate more than 10 kg of gold in the next year?

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2. Let $X \sim \text{Uniform}(-1, 1)$ and $Z \sim \text{Uniform}(0, 0.1)$ be two independent random variables. Let $Y = X^2 + Z$.

   (a) What is the conditional probability density function of Y given $X = x$?

   (b) What is the joint probability density function of X and Y?

   (c) Are X and Y independent?

   (d) Are X and Y correlated? Compute the correlation coefficient between X and Y (start with the definition for the correlation coefficient).

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3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the continuous distribution with probability density function (pdf)

$$f(x; \theta) = \frac{2\theta(1-x)}{(2x-x^2)^{1-\theta}} I_{(0,1)}(x).$$

Here, $\theta > 0$ and $I_{(0,1)}(x)$ is the indicator function that takes on the value 1 when $0 < x < 1$ and is 0 otherwise.

(a) Find the distribution of $Y_i = -\ln(2X_i - X_i^2)$.

(b) Find the maximum likelihood estimator (MLE) for θ. Show that it is an asymptotically unbiased estimator for θ.

(c) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

(d) Is the UMVUE an efficient estimator of θ? Justify.

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4. Let X_1 and X_2 be two independent and identically Normally distributed random variables, with mean θ and variance 1. Consider the sample mean $\bar{X} = (X_1 + X_2)/2$.

(a) Find the mean and variance of $\bar{X}$. Is $\bar{X}$ an unbiased estimator of θ?

(b) Consider conditioning on X_1, and let $\phi(X_1) = \mathsf{E}(\bar{X} \mid X_1)$. Can the Rao-Blackwell theorem be applied here? Is X_1 a sufficient statistic?

(c) How does $Var(\phi(X_1))$ compare with $Var(\bar{X})$?

(d) Find $\mathsf{E}(\phi(X_1))$. Is $\phi(X_1)$ an unbiased estimator of θ? Why or why not?

(e) Can you say that $\phi(X_1)$ is a better estimator than $\bar{X}$ based on their variances?

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5. Consider the following *random telegraph signal*

$$X(t) = (-1)^{N(t)} Y, \quad t \geq 0,$$

where $\{N(t)\}$ is a homogeneous Poisson process with rate λ and Y is a binary random variable that takes on the values ± 1 with probability $1/2$ each. Assume that Y is independent of $\{N(t)\}$.

(Hint: For the following questions, you may find your work is more streamlined if you recall/use the hyperbolic trigonometric functions $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.)

(a) Let $M(t) = (-1)^{N(t)}$. Find closed form expressions for $P(M(t) = 1)$ and $P(M(t) = -1)$.

(b) Find $\mathsf{E}[M(t)]$.

(c) For $s < t$, let $p_{i,j} = P(M(s) = i, M(t) = j)$. Find $p_{1,1}$, $p_{-1,1}$, $p_{1,-1}$, and $p_{-1,-1}$. You only need to show details of your work for one of them.

(d) Find $\mathsf{E}[M(s)M(t)]$ for any $s, t > 0$. (Hint: Focus first on the case where $s < t$ and then generalize.)

(e) Find $Cov(X(s), X(t))$.