Optimal Stopping under Model Ambiguity

— A Time-Consistent Equilibrium Approach

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Without ambiguity:

- choose $\tau \in \mathcal{T}$ to maximize

$$\mathbb{E}^{\mathbb{P}}[e^{-r\tau} g(X_\tau)]. \tag{1}$$

With ambiguity:

- $\mathcal{P}$: the set of *plausible* probabilities $\mathbb{P}$, i.e. *priors*.
- Worst-case analysis: choose $\tau \in \mathcal{T}$ to maximize

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{-r\tau} g(X_\tau)]. \tag{2}$$

- Best-case analysis: choose $\tau \in \mathcal{T}$ to maximize

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{-r\tau} g(X_\tau)]. \tag{3}$$

- **Worst-case (or best-case) analysis:**

  The dominant approach...

  Riedel (2009), Bayraktar & Yao (2011a, 2011b, 2014, 2017), Cheng and Riedel (2013), Ekren et al. (2014), Nutz & Zhang (2015), ...

- **What is missing?**

An agent's ambiguity attitude.

  - Curley & Yates (1989), Heath & Tversky (1991):

    With the same $\mathcal{P}$, different agents have different levels of ambiguity aversion.

The α-Maxmin Objective

Motivated by the α-maxmin preference in Ghirardato et al. (2004), we propose to maximize

$$\alpha \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{-r\tau}g(X_\tau)] + (1-\alpha) \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{-r\tau}g(X_\tau)].$$

- *Ambiguity* is captured by $\mathcal{P}$.
- *Ambiguity attitude* is captured by $\alpha \in [0,1]$.
  - $\alpha = 1$: worst-case analysis (**purely ambiguity-averse**)
  - $\alpha = 0$: best-case analysis (**purely ambiguity-loving**)

Our goal: optimal stopping under *any* $\alpha \in [0,1]$.

TIME INCONSISTENCY

The problem

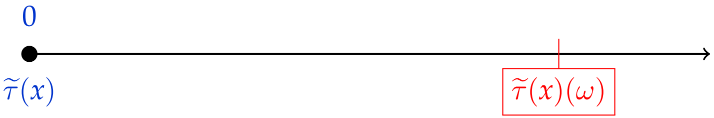
$$\sup_{\tau\in\mathcal{T}}\left(\alpha \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{-r\tau}g(X_\tau)] + (1-\alpha)\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[e^{-r\tau}g(X_\tau)]\right) \quad (4)$$

is **time-inconsistent**!

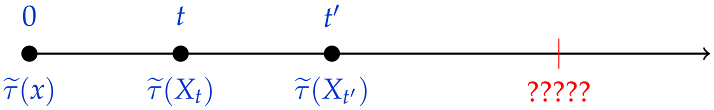
- An α-maxmin preference induces time inconsistency. (Schröder (2011) and Beissner et al. (2016))

Not meaningful to find an optimal stopping time here. (unless one dictates his future selves' behavior).

- **Problem Solved.** *Feeling Good?*



- **The Reality:**



- **Time Inconsistency:**
  - $\widetilde{\tau}(x)$, $\widetilde{\tau}(X_t)$, $\widetilde{\tau}(X_{t'})$ may all be different.

Tower Property

- Time consistency of (1) = tower property of conditional expectations.
- For (2) and (3),
  - **Epstein & Schneider (2003):** tower property holds if

    the set of priors is *rectanguler*
    (stable under pasting conditional probabilities).

    $\Longrightarrow$ Time consistency follows.
  - Nutz and van Handel (2013), Bayraktar and Yao (2014), Ekren et al. (2014), Nutz and Zhang (2015)...
- Under the α-maxmin preference,
  - **Schröder (2011)**, **Beissner et al. (2016)**:
    Tower property fails, even under "stable under pasting".
  - Time inconsistency is a genuine challenge for (4).

How to resolve time inconsistency?

Consistent Planning [Strotz (1955-56)]

- Take into account future selves' behavior.

  Find an *equilibrium* strategy that

  once being enforced over time,
  no future self would want to deviate from.

- How to precisely define and find equilibrium strategies?

ITERATIVE APPROACH

Huang & Nguyen-Huu (2018):
Iterative approach for time-inconsistent stopping problems

- **Equilibrium strategies = fixed points of an operator**
  - find equilibria easily via fixed-point iterations.
- Applications:
  - non-exponential discounting;
  - probability distortion.

  Huang & Nguyen-Huu (2018), Huang & Zhou (2017, 2019), Huang, Nguyen-Huu, and Zhou (2019)

This talk:

- Model ambiguity + *ambiguity attitude*
- A time-inconsistent stopping problem under the α-maxmin preference
- Iterative approach

The Model

- $\Omega := C([0,\infty); R^d)$.
  - $\Omega^x := \{\omega \in \Omega : \omega_0 = x\}, \forall x \in \mathbb{R}^d$.
- B: canonical process.
- $\mathfrak{P}(\Omega)$: the set of probability measures on Ω.
- For any $x \in \mathbb{R}^d$, let

  $$\mathcal{P}(x) \subseteq \{\mathbb{P} \in \mathfrak{P}(\Omega) : \mathbb{P}(\Omega^x) = 1,\ B \text{ is strong Markov under } \mathbb{P}\}$$

  denote the set of *priors* of an agent at $x \in \mathbb{R}^d$.
- $\mathcal{U}(\mathbb{R}^d)$: the set of *universally measurable* subsets of $\mathbb{R}^d$.

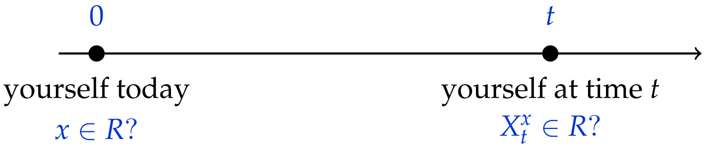
GAME-THEORETIC APPROACH

- Focus on *hitting times* to regions in $\mathbb{R}^d$, i.e.,

$$\tau_R := \inf\{t \geq 0 : B_t \in R\}, \quad R \in \mathcal{U}(\mathbb{R}^d).$$

  - For convenience, we call $R \in \mathcal{U}(\mathbb{R}^d)$ a *stopping policy*.

- Given a stopping policy $R \in \mathcal{U}(\mathbb{R}^d)$,



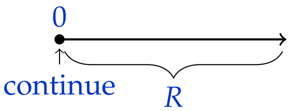
- **Game-theoretic** thinking at time 0:
  Given that every future self will follow R,
  - What is the best stopping strategy at time 0?

BEST STOPPING STRATEGY

The agent at $x \in \mathbb{R}^d$ can either stop or continue.

- If she stops, gets $\boxed{g(x)}$ right away.
- If she continues, she will eventually stop at the moment

$$\rho_R := \inf\{t > 0 \; : \; B_t \in R\}.$$



$\Longrightarrow$ Her α-maxmin expected payoff is then

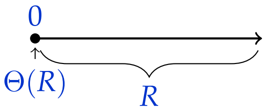
$$J(x,R) := \boxed{\alpha \inf_{\mathbb{P}\in\mathcal{P}(x)} \mathbb{E}^{\mathbb{P}}[e^{-r\rho_R} g(B_{\rho_R})] + (1-\alpha) \sup_{\mathbb{P}\in\mathcal{P}(x)} \mathbb{E}^{\mathbb{P}}[e^{-r\rho_R} g(B_{\rho_R})]}.$$

- *The Best stopping policy* at time 0 is

$$\Theta(R) := S_R \cup (I_R \cap R),$$

where

$$S_R := \{x : g(x) > J(x, R)\},$$
$$I_R := \{x : g(x) = J(x, R)\},$$
$$C_R := \{x : g(x) < J(x, R)\}.$$



- In general, $\Theta(R) \neq R$.
  - Player 0 wants to follow $\Theta(R)$, instead of R.

EQUILIBRIUM

Definition

$R \in \mathcal{U}(\mathbb{R}^d)$ is called an **equilibrium** if $\Theta(R) = R$.

- **Trivial Equilibrium:** Consider $R := \mathbb{R}^d$. Then $I_R = \mathbb{R}^d$, so $\Theta(R) = S_R \cup (I_R \cap R) = R$.
- **In general**, given any $R \in \mathcal{B}(\mathbb{R}^d)$, carry out iteration:

$$R \longrightarrow \Theta(R) \longrightarrow \Theta^2(R) \longrightarrow \cdots \longrightarrow \text{“equilibrium”??}$$

- **To show:**
  (i) $R_* := \lim_{n\to\infty} \Theta^n(R)$ converges (ii) $\Theta(R_*) = R_*$.

STRONG FORMULATION

- $d = 1$ and $\mathbb{P}_0$: the Wiener measure.
- $I = (\ell, r)$, for some given $-\infty \leq \ell < r \leq \infty$. Consider

$$X_t^{x,b,\sigma} = x + \int_0^t b(X_s^{x,b,\sigma})ds + \int_0^t \sigma(X_s^{x,b,\sigma})dB_s, \quad \mathbb{P}_0\text{-a.s.} \quad (5)$$

- $\mathfrak{L}$: the set of all $b, \sigma : I \to \mathbb{R}$ that are

  (i) Lipschitz, grows linearly; (ii) $\sigma^2 > 0$ on I.
- $\mathcal{A}$: the set of all *set-valued* maps $\Pi : I \to 2^{\mathfrak{L}}$.
- $\mathcal{A}^\infty$: the set of all *set-valued* maps $\Pi : I \to 2^{\mathfrak{L}}$ satisfying: for any $x \in I$, $\exists\, K > 0$ such that for *any* $(b, \sigma) \in \Pi(x)$,

$$|b(u) - b(v)| + |\sigma(u) - \sigma(v)| \leq K|u - v|$$
$$|b(u)| + |\sigma(u)| \leq K(1 + |u|), \quad \forall u, v \in I.$$

Strong Formulation

- For each $x \in I$ and $(b, \sigma) \in \mathfrak{L}$, define

$$\mathbb{P}^x_{b,\sigma} := \mathbb{P}_0 \circ (X^{x,b,\sigma})^{-1} \in \mathfrak{P}(\Omega). \tag{6}$$

- Given $\Pi \in \mathcal{A}$, we introduce

$$\mathcal{P}(x) := \{\mathbb{P}^x_{b,\sigma} : (b, \sigma) \in \Pi(x)\}, \quad \forall x \in I. \tag{7}$$

Lemma

Given $x \in I$ *and* $(b,\sigma) \in \mathfrak{L}$, $X^{x,b,\sigma}$ *is a regular diffusion, i.e.,*

$$\text{for any } x \in I, \quad \mathbb{P}_0(T^x_y < \infty) > 0, \quad \forall y \in I.$$

This implies

$$T^x_{(\ell,x)} = T^x_{(x,r)} = 0 \quad \mathbb{P}_0\text{-a.s.}$$
$$\implies \quad \rho_{(\ell,x)} = \rho_{(x,r)} = 0 \quad \mathbb{P}^x_{b,\sigma}\text{-a.s.}$$

Proposition

For any $R \in \mathcal{U}(I)$, $\boxed{R \subseteq \Theta(R)}$. Then,

$$R_* := \lim_{n\to\infty} \Theta^n(R) = \bigcup_{n\in\mathbb{N}} \Theta^n(R).$$

Convergence in Capacity

Standard SDE estimate + proof of Kolmogorov's criterion:

Lemma

For any $\Pi \in \mathcal{A}^\infty$, $\mathcal{P}(x)$ *is relatively compact for all* $x \in I$.

Introduce $R_n := \Theta^n(R),\ \rho^n := \rho_{R_n},\ \rho^* := \rho_{R_*}$.

Lemma

For any $\Pi \in \mathcal{A}^\infty$ *and* $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup_{\mathbb{P}\in\mathcal{P}(x)} \mathbb{P}\left(|\rho^n - \rho^*| \geq \varepsilon\right) = 0, \tag{8}$$

$$\lim_{n\to\infty} \sup_{\mathbb{P}\in\mathcal{P}(x)} \mathbb{P}\left(|B_{\rho^n} - B_{\rho^*}| 1_{\{\rho^n<\infty\}} \geq \varepsilon\right) = 0. \tag{9}$$

Relies crucially on relative compactness of $\mathcal{P}(x)$.

THE MAIN RESULT

Theorem

Fix $\Pi \in \mathcal{A}^\infty$. Let $g : \bar{I} \to \mathbb{R}$ be continuous and

$$\lim_{t\to\infty} e^{-rt} g(X_t^{x,b,\sigma}) = 0 \quad \mathbb{P}_0\text{-}a.s., \quad \forall x \in I,\ (b,\sigma) \in \Pi(x).$$

Then, for any $R \in \mathcal{U}(I)$, R_* is an equilibrium, i.e.

$$\Theta(R_*) = R_*.$$

Consequently,

$$\mathcal{E} = \left\{ \lim_{n\to\infty} \Theta^n(R) : R \in \mathcal{U}(I) \right\}.$$

REAL OPTIONS VALUATION

- Applies financial option pricing techniques to corporate investment decision making.
  - Use risk-neutral pricing to evaluate the right, but not the obligation, to undertake a business plan.
- Suffers *model ambiguity* more severely than pricing a financial option...
  - ... as the underlying can be neither tradable nor fully observable.
- This leads to
  - a set of *plausible* risk-neutral measures
    $\Longrightarrow$ an interval of *plausible* values of a real option.
  - How to deal with these multiple values?
    Unclear in the literature....

EXAMPLE

- $g(x) = (K - x)^+$ for a given $K > 0$.
- The underlying is a GBM:

$$X_t^{x,b,\sigma} = x + \int_0^t bX_s^{x,b,\sigma} ds + \int_0^t \sigma X_s^{x,b,\sigma} dB_s, \quad \mathbb{P}_0\text{-a.s.},$$

  for some *unknown* $b \in \mathbb{R}$ and $\sigma > 0$.
- Riskfree rate $r > 0$ is known.
- $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, for given $0 < \underline{\sigma} < \overline{\sigma}$ (**uncertain volatility model**)

$\Longrightarrow$ The α-maxmin objective:

$$J(x,R) = \alpha \inf_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \mathbb{E}^{\mathbb{P}_0}\left[e^{-rT_R}(K - X_{T_R}^{x,r,\sigma})^+\right] + (1-\alpha) \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \mathbb{E}^{\mathbb{P}_0}\left[e^{-rT_R}(K - X_{T_R}^{x,r,\sigma})^+\right].$$

Define

$$m_1 := \frac{2r}{\underline{\sigma}^2}, \quad m_2 := \frac{2r}{\overline{\sigma}^2}.$$

$$\boxed{a^* := \frac{m_1\alpha + m_2(1-\alpha)}{1 + m_1\alpha + m_2(1-\alpha)}K} \in (0, K).$$

Proposition

- $\mathcal{E} = \{(0, a] : a^* \le a \le K\}$.
- $(0, a^*]$ is *optimal* among $\mathcal{E}$.

"Optimal" in the sense that for any $a^* \le a \le K$,

$$J(x, (0, a^*]) \ge J(x, (0, a]) \quad \forall x \in (0, \infty).$$

- *Optimal equilibrium* in Huang & Zhou (2019, 2017).

Observations

- a^* is increasing in $\alpha \in [0, 1]$.
  - The *larger* α, the *larger* the optimal equilibrium $(0, a^*]$.
  - The *more risk-averse*, the *more eager to stop*—to exit the uncertain environment.
- When $\underline{\sigma} = \overline{\sigma} = \sigma$ (**no ambiguity**),

$$a^* = \frac{2r/\sigma^2}{1 + 2r/\sigma^2} K.$$

This is exactly the optimal stopping threshold for

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}_0}[e^{-r\tau}(K - X_\tau^{x,r,\sigma})^+]$$

(Theorem 2.7.2 in Karatzas and Shreve (1998)).

Summary

- Resolves the time-inconsistent stopping problem under the α-maxmin preference
  - Allow us to go beyond worst-case (best-case) analysis.
- Focuses on ambiguity aversion, a *cause of time inconsistency* only slightly discussed in the literature.
  - relative to *non-exponential discounting*, *probability distortion*,...
- Provides a new approach for real options valuation
  - α-maxmin preference + equilibrium approach
- A new measurable projection theorem.
  - *does not* require specific Borel structure.

THANK YOU!!

- *"Optimal Stopping under Model Ambiguity: a Time-Consistent Equilibrium Approach"*
  (H. and X. Yu), Available @ arXiv:1906.01232.