Set Theory 101
APPM 5450 Spring 2018 Applied Analysis 2

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1. **De Morgan's laws** Let $\overline{A}$ denote the complement of a set A, and let I be an arbitrary index set and A_i arbitrary sets. Then
$$\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}, \quad \text{and} \quad \overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$$

2. Two sets A and B are said to have the **same size**, or cardinality, and denoted by $|A| = |B|$, if there is a bijection between the two sets, i.e., $A \cong B$.

   We write $|A| \leq |B|$ if there is an injective map $f : A \to B$. Write $|A| < |B|$ if $|A| \leq |B|$ and it is not true that $|A| = |B|$.

3. **Cantor-Schröder-Bernstein Theorem**: $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$. Original proof required axiom of choice, but subsequent proofs (e.g., by König) do not need it. The proof is not constructive. This theorem might seem obvious, but it's not (though the proof is not especially difficult). References: p. 100 from "The Book" (3rd ed.), or wikipedia

4. **Cantor's Theorem** Let M be an arbitrary set, and let $\mathcal{P}(M)$ denote its power set. Then $|M| < |\mathcal{P}(M)|$.

   Proof. Suppose there is a bijection $\varphi : M \to \mathcal{P}(M)$. Consider the set
$$U = \{m \in M \mid m \notin \varphi(m)\}.$$
   Then because φ is onto and $U \in \mathcal{P}(M)$, there is some $m \in M$ such that $U = \varphi(m)$. This leads to a contradiction.