Handout on sequences and series
APPM 5440 Fall 2014 Applied Analysis

Stephen Becker

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This is taken from Chapter 9.1 "Sequences and series of numbers" by Patrick Fitzpatrick in *Advanced Calculus* (2nd edition). This is the text used by the undergrad analysis class at CU. You are responsible for knowing this on the analysis prelim. Note that other basic analysis and advanced calculus topics that you are responsible for include:

- Integration of functions of several variables: line and volume integrals in 2D, line, surface, and volume integrals in 3D.
- Differentiation: gradient, curl, divergence, Jacobian. Connection between rotation-free vector fields and potential fields. Partial integration, Green's theorems, Stokes' theorem, Gauss' theorem. The consequences of these theorems for vector fields that are divergence or rotation free.

Basics

All numbers are taken to be real numbers. There are two basic principles behind convergent sequences: the Cauchy criterion (completeness), and the monotone convergence theorem (compactness).

1. **Monotone convergence theorem** aka MCT. A monotone sequence of numbers converges iff it is bounded
2. Proposition: Every convergent sequence is Cauchy
3. Lemma: Every Cauchy sequence is bounded
4. Theorem: a sequence converges iff it is Cauchy

Convergence tests for series

We write $\sum$ to mean $\sum_{n=1}^{\infty}$ or $\sum_{k=1}^{\infty}$.

1. Proposition: Suppose the series $\sum a_n$ converges. Then $\lim_{n\to\infty} a_n = 0$
2. Proposition: If $|r| < 1$ then $\sum r^k = (1-r)^{-1}$.
3. Theorem: if (a_k) is non-negative, then the series $\sum a_k$ converges iff the sequence of partial sums is *bounded*. Trivial proof by MCT.
4. Corollary: **The comparison test**. Suppose (a_k) and (b_k) are sequences such that $0 \le a_k \le b_k$ for all k. Then $\sum a_k$ converges if $\sum b_k$ converges, and $\sum b_k$ diverges if $\sum a_k$ diverges.
5. Corollary: **The integral test**. Let (a_k) be a sequence of *non-negative* numbers, and suppose the function $f : [1, \infty) \to \mathbb{R}$ is continuous and monotonically decreasing and $f(k) = a_k$. Then the series $\sum a_k$ converges iff the sequence of integrals (i_k) is *bounded*, for $i_k = \int_1^k f(x)\,dx$.

6. Example: the series $\sum 1/((k+1)\log(k+1))$ diverges. We show this by integrating the function from 1 to n, which gives $\log(\log(n+1)) - \log(\log(2))$, which is not bounded.

7. Corollary: **The p-test**. For a positive number p, the series $\sum k^{-p}$ converges iff $p > 1$. Prove via the integral test with x^{-p}.

8. Example: the series $\sum k/e^k$ converges. Proof: by Taylor series, for any $b > 0$ we have $e^b > b^3/6$. Hence, for any k, $k/e^k < 6/k^2$, and the series $\sum 1/k^2$ converges.

9. Theorem: **Alternating series test**. Suppose (a_k) is *monotonically decreasing* and converges to 0, then the series $\sum(-1)^{k+1}a_k$ converges. Proof sketch: let s_n be the partial sums. The subsequence (s_{2n}) is monotonically decreasing (hence bounded) and non-negative, so it converges. But s_{2n+1} approaches s_{2n} as $n \to \infty$, so it also converges.

10. Example: $\sum(-1)^{k+1}/k$ converges (in fact, it converges to $\log(2)$).

A few more convergence tests. These are based on the Cauchy criterion instead of the MCT:

1. The series $\sum a_k$ converges iff for every $\epsilon > 0$ there is N such that $|a_{n+1} + \ldots + a_{n+k}| < \epsilon$ for all $n \geq N$ and all $k \in \mathbb{N}$. Proof: definition of convergence and Cauchy sequence, plus Cauchy criterion (i.e., completeness of the reals).

2. Definition: the series $\sum a_k$ is said to **converge absolutely** if $\sum |a_k|$ converges. If the series converges but not absolutely, we say it **converges conditionally**.

3. Corollary: **The absolute convergence tests**. An absolutely convergent sequence converges.

4. Example: $\sum \sin(k)/k^2$ converges (since it converges absolutely).

5. Example: $\sum(-1)^{k+1}/k$ converges conditionally.

6. Theorem: (linear/geometric convergence) suppose there is $r \in [0,1)$ such that $|a_{n+1}| \leq r|a_n|$ for all n sufficiently large. Then $\sum a_k$ is absolutely convergent. Proof: $\sum r^{-n} \leq (1-r)^{-1}$.

7. Corollary: **The ratio test for series**. If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = \ell$, then if $\ell < 1$ the series converges absolutely, and if $\ell > 1$ the series diverges (for $\ell = 1$, the test does not give any information).

We also have (from chapter 2 of Fitzpatrick) some basic criteria for sequences to converge. Suppose (a_n) and (b_n) converge to a and b respectively, then $(a_n + b_n)$ converges to $a + b$. Similarly for multiplication of sequences, as well as scalar multiplication. There are similar results for division.