RUNGE PHENOMENON

Extracts from Chapter 3

A Practical Guide to Pseudospectral Methods

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The two figures to the right illustrate Lagrange kernels for equi-spaced- and Chebyshev-type node distributions. In the equi-spaced case, we clearly ought to expect the Runge phenomenon.

The next two pages give theoretical details for the Runge phenomenon. In particular, we note that for equi-spaced interpolation, one single universal curve (shown in Figure 3.4-2 b) gives the convergence/divergence rate at different parts of the interval. The particular function that we are interpolating will only effect how the universal curve in Figure 3.4-2 a is shifted up/down (i.e. what scalar factor multiplies the function in Figure 3.4-2 b).

Figure 3.3-2. Basis functions $F_i(x)$ in Lagrange's interpolation formula for two different node distributions in the case of $N = 10$. (a) Equi-distributed nodes: $x_i = -1 + 2i/N$, $i = 0, 1, \ldots, N$. (b) Chebyshev-distributed nodes: $x_i = -\cos(i\pi/N)$, $i = 0, 1, \ldots, N$. 

3.4. Rates of convergence and divergence for polynomial interpolation of analytic functions

In a complex \((z = x + iy)\) plane, a Taylor series is well known to converge in the largest circle around the expansion point that is free of singularities. This result generalizes in a straightforward manner to interpolating polynomials, where the nodes are distributed over an interval rather than all lumped at one point.

Although many books on interpolation theory discuss the convergence of polynomial interpolation, it is surprisingly difficult to locate a simple formulation of the following theorem. General references in this area include Turetskii (1968), Walsh (1960), Krylov (1962), Markushevich (1967), Davis (1975), and Gaier (1987); see also Weideman and Trefethen (1988).

**Theorem.** Given a node density function \(\mu(x)\) (on \([-1, 1]\)), we form the potential function

\[
\phi(z) = -\int_{-1}^{1} \mu(x) \ln|z-x| \, dx + \text{(constant)}.
\]

(3.4-1)

Then:

1. **The polynomial** \(p_N(z)\) (interpolating an analytic function \(f(z)\) at \(N\) nodes on \([-1, 1]\)) converges to \(f(z)\) inside the largest equi-potential curve for \(\phi(z)\) that does not enclose any singularity of \(f(z)\), and diverges outside that curve.

Let \(z_0\) denote the limiting singular point of \(f(z)\) (or, for now, any point along this largest equi-potential curve).

2. **The rate of convergence/divergence is exponential, like** \(\alpha(z)^N\) where \(\alpha(z) = e^{\phi(z_0)} - \phi(z)\).

This result is to be understood in the same sense as how a Taylor series around the origin converges/diverges like \(\alpha(z)^N\) with \(\alpha(z) = |z/z_0|\); i.e., the error satisfies \(|R_N(z)|^N \sim \alpha(z)\) as \(N \to \infty\).

3. **PS approximations to any derivative converge (or diverge) in the same fashion as the interpolant does to the function.**

The first two parts of this theorem are demonstrated in Appendix E. The third part is then readily shown by noting that \(p_N(z)\) interpolates \(f'(z)\) at \(N\) points on \([-1, 1]\) separating the \(N+1\) points at which \(p_N(z)\) interpolated \(f(z)\).

**Example.** Determine the convergence rates at different \(x\) positions for equi-spaced interpolation of \(f(x) = 1/(1+16x^2)\), \(x \in [-1, 1]\).

**Figure 3.4-1.** Potential function \(\phi(z)\) for equi-spaced interpolation over \([-1, 1]\). The value of \(\phi(0)\) is set to zero. The contour lines mark the levels \(-2\ln 2\) and \(-2\ln 2\) (i.e., separations corresponding to factors of 2 in the exponential convergence rate \(\alpha(z)\)).

In the case of equi-spaced interpolation, \(\mu(x) = \frac{1}{2}\). The integral (3.4-1) can be evaluated in closed form as follows:

\[
\phi(z) = -\frac{1}{2} \text{Re}((1-z) \ln(1-z) - (-1-z) \ln(-1-z)) + C.
\]

(3.4-2)

This implies \(\phi(0) = \phi(\pm 1) = \ln 2\); i.e., the relation \(\alpha(\pm 1) = 2\alpha(0)\) always holds for equi-spaced interpolation on \([-1, 1]\).

Figure 3.4-1 shows this potential surface (with \(C = 0\)). The heavy (dashed) contour line surrounds the smallest equi-potential domain that includes \([-1, 1]\) (on the imaginary axis, it extends just past \(\pm 0.5255i\)). The function \(f(z)\) must be analytic everywhere within this domain for convergence to occur on \([-1, 1]\). Any singularity within it restricts convergence to a still smaller equi-potential region, leading to the Runge phenomenon: divergence of the interpolant near the ends of the interval.

The function we are interpolating in this example, \(f(x) = 1/(1+16x^2)\), has only two singularities in the complex plane. They are located at \(z = \pm 0.25i\) for \(x \in [-1, 1]\), from (3.4-2) we have

\[
\phi(x) - \phi(\pm 0.25i) = \frac{1}{4}(1-x) \ln(1-x) - \frac{1}{4}(1+x) \ln(1+x)
\]

\[
+ \frac{1}{16} \ln \frac{17}{16} + \frac{1}{4} \arctan(4),
\]
3. Example of a differentiation matrix

shown in Figure 3.4-2(a). Figure 3.4-2(b) shows the corresponding convergence rate $\alpha(x) = \frac{e^{\phi(x)} - \phi(x)}{\phi(x)}$. In particular, $\alpha(0) = 0.6964$ and $\alpha(\pm 1) = 2\alpha(0) = 1.3929$. The crossovers between convergence and divergence occur at $x \approx \pm 0.7942$. Figures 3.4-3(a) and (b) show the results of equi-spaced interpolation using $N = 20$ and $N = 40$, in complete agreement with the predicted rates $\alpha(x)$, and with the (fixed) crossover locations.

Hermite interpolation (requiring not only function values but also first-derivative values to match at the gridpoints) offers no help against the Runge phenomenon. Such interpolation converges and diverges in precisely the same places as does Lagrange interpolation.

Figure 3.4-2. The functions (a) $\phi(x) - \phi(\pm 0.25i)$ and (b) $\alpha(x) = \frac{e^{\phi(x)} - \phi(x)}{\phi(x)}$ displayed on $[-1, 1]$.

Figure 3.4-3. Results of equi-spaced interpolation on $[-1, 1]$ in the case of (a) $N = 20$ and (b) $N = 40$. 