

A collection of explicit Runge-Kutta methods

First order

Forward Euler:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Second order

Explicit midpoint:

$$\begin{array}{c|ccc} 0 & 0 & 0 & \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & \\ & 0 & 1 & \end{array}$$

Heun's method:

$$\begin{array}{c|ccc} 0 & 0 & 0 & \\ \hline 1 & 1 & 0 & \\ & \frac{1}{2} & \frac{1}{2} & \end{array}$$

Generic:

$$\begin{array}{c|ccc} 0 & 0 & 0 & \\ \hline x & x & 0 & \\ & 1 - \frac{1}{2x} & \frac{1}{2x} & \end{array}$$

Third order:

Kutta:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & 0 & \\ 1 & -1 & 2 & 0 & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \end{array}$$

Heun:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & \\ \hline \frac{1}{3} & \frac{1}{3} & 0 & 0 & \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 & \\ & \frac{1}{4} & 0 & \frac{3}{4} & \end{array}$$

Fourth order:

Classic:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 & \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & \\ & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \end{array}$$

3/8 rule:

$$\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & \\ \hline \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \\ \frac{2}{3} & -\frac{1}{3} & 1 & 0 & 0 & \\ 1 & 1 & -1 & 1 & 0 & \\ & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \end{array}$$

Embedded: (Both update and error estimate in single step)

Runge-Kutta-Fehlberg:

$$\begin{array}{c|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \\ \frac{3}{8} & \frac{3}{32} & \frac{9}{32} & 0 & 0 & 0 & 0 & \\ \frac{12}{13} & \frac{1932}{2197} & -\frac{7200}{2197} & \frac{7296}{2197} & 0 & 0 & 0 & \\ 1 & \frac{439}{216} & -8 & \frac{3680}{513} & -\frac{845}{4104} & 0 & 0 & \\ \frac{1}{2} & -\frac{8}{27} & 2 & -\frac{3544}{2565} & \frac{1859}{4104} & -\frac{11}{40} & 0 & \\ \hline & \frac{25}{216} & 0 & \frac{1408}{2565} & \frac{2197}{4104} & -\frac{1}{5} & 0 & \text{Order 4 (use to step forward)} \\ & \frac{16}{135} & 0 & \frac{6656}{12825} & \frac{28561}{56430} & -\frac{9}{50} & \frac{2}{55} & \text{Order 5 (comparison for error check)} \end{array}$$